VARIATION-NORM ESTIMATES FOR THE 
BILINEAR ITERATED FOURIER TRANSFORM

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1. Introduction

In this paper, we consider a variation-norm strengthening of the bilinear iterated Fourier transform

\[ B(f_1, f_2)(x) := \int_{\xi_1 < \xi_2} \hat{f}_1(\xi_1) \hat{f}_2(\xi_2) e^{ix(\xi_1 + \xi_2)} d\xi_1 d\xi_2 \]

It is standard that \( B \) is another form of the bilinear Hilbert transform considered in Lacey–Thiele [11, 12]. For \( r \in (0, \infty) \), let \( T_r \) denote the following operator

\[ \sup_{K, N_0 < \ldots < N_K} \left( \sum_{j=1}^{K} \left| \int_{N_{j-1} < \xi_1 < \xi_2 < N_j} \hat{f}_1(\xi_1) \hat{f}_2(\xi_2) e^{ix(\xi_1 + \xi_2)} d\xi_1 d\xi_2 \right|^{r/2} \right)^{2/r}. \]

\( T_r \) may be thought of as a variation-norm sum of local parts of \( B \), and by definition it controls both \( B \) and the bi-Carleson operator in Muscalu–Tao–Thiele [18].

Given any \( \alpha_1, \alpha_2 \), we could similarly define \( T_r, \alpha_1, \alpha_2 \) by replacing \( e^{ix(\xi_1 + \xi_2)} \) in the definition of \( B \) with \( e^{ix(\alpha_1 \xi_1 + \alpha_2 \xi_2)} \). In other words, \( T_r = T_r,1,1 \).

The main result of this paper is the following theorem:

**Theorem 1.1.** Let \( r > 2 \) and \( \alpha_1 \alpha_2 > 0 \). Then \( T_r, \alpha_1, \alpha_2 \) is bounded from \( L^{p_1} \times L^{p_2} \) to \( L^{p_3} \) for any \( p_1, p_2, p_3 \) such that \( 1/p_3 = 1/p_1 + 1/p_2 \) and

\[ \max(1, \frac{2r}{3r - 4}) < p_1, p_2 < \infty, \quad \max\left( \frac{2}{3}, \frac{r'}{2} \right) < p_3 < \infty. \]

Theorem 1.1 strengthens [11, 12, 18]. It is well-known that the condition \( \alpha_1 + \alpha_2 \neq 0 \) is necessary for the boundedness of the bi-Carleson operator (hence for \( T_r \)), see [20] for counter examples. Theorem 1.1 also implies some range of the \( L^p \) estimates for the \( r \)-variation norm Carleson theorem in [23], where the condition \( r > \max(2, p') \) is necessary.

In the rest of the paper, we will prove Theorem 1.1 for \( \alpha_1 = \alpha_2 = 1 \), and thus \( T_{r,1,1} \equiv T_r \); the general case can be treated similarly. Applications of Theorem 1.1 to ODE with oscillatory driving signals via T. Lyons’ rough path theory [15] will be discussed in [7]. In particular, as a by product of the theory, we’ll obtain the following estimates.
Theorem 1.2. Let $k \geq 3$ and $\alpha_1, \ldots, \alpha_k$ be nonzero real numbers of the same sign, i.e. $\alpha_m \alpha_{m+1} > 0$ for every $1 \leq m \leq k-1$. For any $r > 0$ let $T_{k,r}$ denote

$$
(4) \quad \sup_{K,N_0 < \cdots < N_K} \left( \sum_{j=1}^{K} \int_{N_{j-1} < \xi_1 < \cdots < \xi_k < N_j} \prod_{m=1}^{k} \hat{f}_m(\xi_m) e^{ix_m \xi_m} d\xi_m \right)^{r/k}. \nonumber
$$

Then for every $p \in \left( \frac{3}{2}, \infty \right)$ and $r > \max(2, \frac{p}{r-1})$ it holds that

$$
(5) \quad |T_{k,r}[f_1, \ldots, f_k]|_{p/k} \lesssim \prod_{m=1}^{k} |f_m|_p. \nonumber
$$

The special case $r = \infty$ of (5) was asked in [21]. For more details and a further discussion, see [7]. We'll heuristically explain the constraint (3). By the Holder inequality, the simpler version of $T_r$ where the the bilinear Hilbert symbol $1_{\xi_1 < \xi_2}$ is replaced by 1 is controlled by a product of two variation norm Carleson operators, where we could distribute $2/r = 1/s + 1/t$ and $t, s > 2$ are new variation-norm exponents. Now we need $p_1 > s'$ and $p_2 > t'$ in order to estimate the new variation-norm Carleson operators, and this clearly leads to $p_3 > r'/2$. Since $2/r = 1/t + 1/s$ and $t, s > 2$, it follows that $t', s' > \max(1, 2r/(3r - 4))$, leading to the lower bounds for $p_1, p_2$.

We note that for $r \geq 4$ the range (3) becomes the classical range for the bilinear Hilbert transform (and in particular independent of $r$). Since variation-norm operators are decreasing functions of the variational exponents, we may (and will) assume without loss of generality that $2 < r < 4$ in the rest of the paper.

2. OUTLINE OF THE PROOF

By dualization and monotone convergence, there are measurable functions $K : \mathbb{R} \to \mathbb{Z}_+$ and $N_0(x) \leq N_1(x) < \ldots$ and $d_0(x), d_1(x), \ldots$ such that: (i) $\sum_{j \geq 0} |d_j(x)|^{r/j} = 1$, and (ii) for $j > K(x)$ we have $d_j(x) = 0$ and $N_j = N_K$, and (iii) for every $x \in \mathbb{R}$

$$
|T_r(f_1, f_2)(x)| \lesssim \sum_{j=1}^{K(x)} d_j(x) \left( \int_{N_{j-1} < \xi_1 < \xi_2 < N_j} \prod_{j=1,2} \hat{f}_j(\xi_j) e^{ix_1 \xi_1} d\xi_1 d\xi_2 \right). \nonumber
$$

Let $B_r$ denote the bilinear operator on the right hand side. It suffices to prove the desired estimate $B_r$, provided that the implicit constants depends only on $r$ and the exponents $p_1, p_2, p_3$. We will fix $K$, $(N_j)$, and $(d_j)$ in the rest of the paper. By monotone convergence, we may assume that $K$, $N_j$ are bounded.

For any $M < N$ we'll decompose $1_{M < \xi_1 < \xi_2 < N}$ into three components:

$$
m_{CC}(M, N, \xi_1, \xi_2) + m_{BC}(M, N, \xi_1, \xi_2) + m_{LM}(M, N, \xi_1, \xi_2) \nonumber
$$

- $m_{BC}$ captures the singularity along the line segment from $(M, M)$ to $(N, N)$,
- $m_{CC}$ captures the singularity along the other two edges of the triangle, and
- $m_{LM}$ is an error term that has two singularities at $(M, M)$ and $(N, N)$.

Construction of these symbols are detailed in Section 3 using a hybrid of the arguments in [23] and [18]. Applying this decomposition for $(M, N) = (N_{j-1}, N_j)$ and using the triangle inequality, it follows that $B_r$ is controlled by three corresponding bilinear operators. Via standard arguments (detailed in the Appendices), each of
these three operators is bounded above by a sum of $O(1)$ discrete operators, which will be described in Section 4.

In the rest of the paper, we will prove boundedness of the discrete model operators in the desired $L^p$ ranges. Our proof will use a new $L^p$ theory for outer measures introduced in [6]. It turns out that analogues (for outer measure spaces) of classical singular integral operators arise naturally in our proof, and they are effective tools to handle nested levels of time-frequency analysis. In order to study these operators, we adapted an argument in [6] to prove a Marcinkiewicz interpolation theorem (see Lemma 5.1) for (quasi)sublinear maps between outer measure spaces, which generalizes a simpler interpolation result in [6].

2.1. Notational conventions. By a (standard) dyadic interval we mean $[n2^k, (n+1)2^k)$ for some $n, k \in \mathbb{Z}$. For any $a \in [0, 1)$, by an $a$-shifted dyadic interval we mean an interval of the form $2^k([n, n+1) + (-1)^k a)$.

For any interval $I$ we denote its midpoint by $c(I)$, its left children by $I_-$ and its right children by $I_+$. For any positive $C$, the $C$-enlargement of $I$ is defined to be the $C$-dilation of $I$ from its center and will be denoted by $CI$. Enlargements of cubes are defined similarly. We will define $\tilde{\chi}_I(x) = (1 + \frac{|x - c(I)|}{|I|})^{-2}$.

For any set $A \subset \mathbb{R}$, we define $D_y A := \{2^n \xi : \xi \in A\}$ the dilation of $A$ relative to the origin. For any number $\alpha$ and any interval $I$, let $I + \alpha := \{x + \alpha : x \in I\}$.

We say that $A \lesssim B$ if there is an absolute constant $C \in (0, \infty)$ such that $|A| \leq C|B|$. If $C$ depends on $t_1, t_2, \ldots$ we will say that $A \lesssim_{t_1, t_2, \ldots} B$. (We sometimes suppress some subscripts if the dependence is not important for the relevant discussion.)

If $J$ is an interval we define $M_J f = \sup_{I \supset J} |I|^{-1} \int_I |f(x)| dx$ for every function $f$. We’ll use the following normalizations for inner product and Fourier transforms:

$$\langle f, g \rangle := \int_{\mathbb{R}} f(x) \overline{g(x)} dx, \quad \hat{f}(\xi) := (2\pi)^{-1/2} \int_{\mathbb{R}} f(x) e^{-i\xi x} dx.$$  

3. Decomposition of the triangular symbol

In this section, we will construct the decomposition (6). It suffices to construct $m_{CC}$ and $m_{BC}$. Fix a large integer constant $L_2 \geq 1000$, and let $L_1 = 40L_2$.

3.1. Construction of $m_{CC}$.

3.1.1. Decomposition of $1_{M \subset \xi \subset N}$. Below we adapt a decomposition in [23]. Let $\mathcal{H} = \mathcal{H}(M, N)$ contain all maximal dyadic intervals $J \subset (M, N)$ such that

$$\text{dist}(J, \{M, N\}) \geq L_1 |J|.$$  

Clearly $\mathcal{H}$ forms a partition of $(M, N)$. For every $I \in \mathcal{H}$, the lengths of the left and right neighbors of $I$ in $\mathcal{H}$ are $u(I)|I|$ and $v(I)|I|$ where $u(I), v(I) \in \{1/2, 1, 2\}$. Now, $(u, v)$ depends only on the following details:

- the unique $m \in \mathbb{Z}_+$ such that $M \in I - (m + 1)|I|$;
- the unique $n \in \mathbb{Z}_+$ such that $N \in I + (n + 1)|I|$;
- whether $I$ is the left or right children of its dyadic parent.

Let $A$ denote the set of eligible $(side, m, n)$ for $I \in \mathcal{H}$, where $side \in \{\text{left, right}\}$. For any $\alpha \in A$, let $I_{M,N}(\alpha)$ be the set of matching dyadic $I$. 

Given any $1/2 < c < 5/8$ by elementary arguments we could construct smooth functions $\phi_{u,v}$ indexed by $(u, v) \in \{1/2, 1, 2\}^2$, all supported in $[-c, c]$, such that

$$1_{(M,N)}(\xi) = \sum_{I \in \mathcal{H}(M,N)} \phi_{u(I),v(I)}((\xi - c(I))/|I|).$$

When the details of $I$ matches $\alpha$ we write $\phi_{\alpha,I}(\xi) := \phi_{u,v}(\xi-c(I)|I|^{-1})$, and

$$1_{(M,N)}(\xi) = \sum_{\alpha \in A} \sum_{I \in \mathcal{I}_{M,N}(\alpha)} \phi_{\alpha,I}(\xi). \tag{6}$$

Note that the sum in the right converges absolutely pointwise: every term is nonegative, and at every $\xi$ there are only finitely many nonzero terms in the sum.

3.1.2. Definition of $m_{CC}$. Intuitively, $m_{CC}(M, N, \xi_1, \xi_2)$ is a smooth restriction of $1_{M<\xi_1,\xi_2<N}$ to a neighborhood of the following region

$$\left\{ (\xi_1, \xi_2) \in [M, N]^2 : \min(|\xi_1 - M|, |\xi_2 - N|) \leq |\xi_1 - \xi_2|/200 \right\}, \tag{7}$$

which will be denoted by $R_1(M, N)$. For a more precise statement, see Lemma A.4.

To motivate, note that using (6) we obtain for any $(\xi_1, \xi_2) \in R_1$

$$1_{M<\xi_1,\xi_2<N} = \sum_{\alpha, \beta \in A} \sum_{I \in \mathcal{I}_{M,N}(\alpha)} \sum_{J \in \mathcal{I}_{M,N}(\beta)} \phi_{\alpha,I}(\xi_1)\phi_{\beta,J}(\xi_2) \tag{8}$$

and we’ll construct $m_{CC}$ by removing terms supported far from $R_1$ in the sum.

More specifically, consider the following subsets of $A$, defined by

$$A_1 = \{m \leq n_\alpha/4\} , \quad A_3 = \{m \geq 4n_\alpha\} , \quad A_2 = A - A_1 - A_3.$$

Note that $A_2$ is a finite set.

**Definition 3.1.** Define

$$m_{CC}(M, N, \xi_1, \xi_2) := \sum_{k=1}^{5} m_{CC,k}(M, N, \xi_1, \xi_2)$$

where $m_{CC,k}$ are sub-sums of the right hand side of (8) under some extra constraints. We will always have $I \in \mathcal{I}_{M,N}(\alpha)$ and $J \in \mathcal{I}_{M,N}(\beta)$ and the summands are always $\phi_{\alpha,I}(\xi_1)\phi_{\beta,J}(\xi_2)$, and the following table details the extra constraints:

<table>
<thead>
<tr>
<th>Symbols</th>
<th>Conditions on $\alpha$</th>
<th>Conditions on $\beta$</th>
<th>Extra conditions on $I, J$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m_{CC,1}$</td>
<td>$\alpha \in A_1$</td>
<td>$\beta \in A_1$</td>
<td>$</td>
</tr>
<tr>
<td>$m_{CC,2}$</td>
<td>$\alpha \in A_1$</td>
<td>$\beta \in A_2$</td>
<td>$</td>
</tr>
<tr>
<td>$m_{CC,3}$</td>
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<td>$\beta \in A_3$</td>
<td>None</td>
</tr>
<tr>
<td>$m_{CC,4}$</td>
<td>$\alpha \in A_2$</td>
<td>$\beta \in A_3$</td>
<td>$</td>
</tr>
<tr>
<td>$m_{CC,5}$</td>
<td>$\alpha \in A_3$</td>
<td>$\beta \in A_3$</td>
<td>$</td>
</tr>
</tbody>
</table>
3.2. Construction of $m_{BC}$. In this section, we construct the $m_{BC}$ symbol, which may be viewed as a smooth restriction of $\chi_{M<\xi_1<\xi_2<N}$ to a neighborhood of

$$
\left\{(\xi_1, \xi_2) \in [M, N]^2 : |\xi_1 - \xi_2| < \frac{1}{100} \min \left( \frac{|\xi_1 + \xi_2 - 2M|}{2}, \frac{|\xi_1 + \xi_2 - 2N|}{2} \right) \right\}
$$

which we will denote by $R_2(M, N)$. For a more precise statement, see Lemma A.5.

To motivate the construction, we first note that if $(\xi_1, \xi_2) \in R_2$ then

$$
\chi_{M<\xi_1<\xi_2<N} = \chi_{\xi_1<\xi_2} \chi_{M<|\xi_1 + \xi_2 - 2N|/2<N}.
$$

The construction of $m_{BC}$ is done by writing the product in the right hand side of the last display as a sum of products of wave packets (using a suitable decomposition for each factor), and then removing from this sum essentially those terms that are supported far from $R_2$. We have previously described the decomposition for $\chi_{(M, N)}$, and below we will describe a decomposition for $\chi_{\xi_1<\xi_2}$.

3.2.1. Decomposition of $\chi_{\xi_1<\xi_2}$. We largely follows [12] (see also [25, 16]).

Shifted cubes: For any $b = (b_1, \ldots, b_n) \in [0, 1]^n$ we say that $S = S_1 \times \cdots \times S_n$ is a $b$-shifted dyadic cube if $S_j$ is a $b_j$-shifted dyadic interval, and $|S_1| = \cdots = |S_n|$. In this paper, unless otherwise specified, the coordinates of underlying shifts are assumed to be in $\{0, \frac{1}{3}, \frac{2}{3}\}$.

Whitney decomposition: For each $b \in \{0, \frac{1}{3}, \frac{2}{3}\}^2$ consider the collection $S_b$ of all maximal $b$-shifted squares $S_1 \times S_2$ satisfying

(i) $\text{dist}(S_1 \times S_2, \ell) \geq L_2 |S_1|$, here $\ell$ is the line $\{\xi_1 = \xi_2\}$.

It is clear that for such square it holds that

$$
L_2 |S_1| \leq \text{dist}(S_1 \times S_2, \ell) \leq (2L_2 + 1) |S_1|.
$$

Using a partition of unity argument, we may write

$$
\chi_{\xi_1<\xi_2} = \sum_b \sum_{S_1 \times S_2 \in S_b} \phi_{S_1 \times S_2}(\xi_1, \xi_2)
$$

where $\{\phi_{S_1 \times S_2}\}$ is a family of nonnegative $C^\infty$ bump functions, such that $\phi_{S_1 \times S_2}$ is supported inside $\frac{4}{5} S_1 \times \frac{4}{5} S_2$. Note that if $(\xi_1, \xi_2) \in S$ then $\xi_1 + \xi_2 \in S_1 + S_2$, which could be generously covered by 6 intervals of the form $(3/4) I_j$, $1 \leq j \leq 4$, where $I_j$ are shifted dyadic intervals having the same length as $|S_1|$. Using partitions of unity, we may find nonnegative smooth functions $\phi_{3, I_j}$ with support inside $(4/5) I_j$, such that for every $\xi \in S_1 + S_2$ it holds that $1 = \sum_j \phi_{3, I_j}(\xi)$. We obtain

$$
\chi_{\xi_1<\xi_2} = \sum_{S \in S} \phi_{S_1 \times S_2}(\xi_1, \xi_2) \phi_{3, S_3}(\xi_1 + \xi_2)
$$

where $S$ denote the collection of cubes $S_1 \times S_2 \times S_3$ formed using $S_1 \times S_2 \in \bigcup S_b$ and $S_3$ are the covering intervals $I_j$’s discussed above. Note that every element of $S$ is a shifted dyadic cube, where the underlying shifts are elements of $\{0, \frac{1}{3}, \frac{2}{3}\}^3$. Expanding $\phi_{S_1 \times S_2}(\xi_1, \xi_2)$ into bilinear Fourier series, we obtain

$$
\chi_{\xi_1<\xi_2}(\xi_1, \xi_2) = \sum_{k \in \mathbb{Z}^2} a_k \sum_{S \in S} \phi_{1, S_3, k}(\xi_1) \phi_{2, S_3, k}(\xi_2) \phi_{3, S_3, k}(\xi_1 + \xi_2),
$$

where $(a_k)$ is a rapidly decaying sequence and $\phi_{i, S_3, k}$’s are $C^d$-bump functions uniformly adapted to $S \in S$, and $\phi_{i, S_3}$ is supported inside $\frac{5}{6} S_3$. (In fact $\phi_{3, S_3, k} \equiv \phi_{3, S_3}$ is independent of $k$, but we prefer to use $\phi_{3, S_3}$ for later convenience of notation.) Als, $d$ is a fixed finite constant, but could be chosen arbitrarily large.
3.2.2. Definition of $m_{BC}$. For convenience, let $l(S) = |S_1| = |S_2| = |S_3|$ denote the side length of $S$. Using (6) and (11), it follows that we may write 
\[ \chi_{[\xi_1,\xi_2]}(\xi_1,\xi_2) \] 
over $\chi_{[\xi_1,\xi_2]}(\xi_1,\xi_2)$ as 
\[ \sum_{k,a,S,I} a_k \phi_{a,I}((\xi_1 + \xi_2)/2) \phi_{3,S,k}(\xi_1 + \xi_2) \prod_{j=1}^{2} \phi_{j,S,k}(\xi_j) . \]
Here the summation is over all $k \in \mathbb{Z}^2$, $a \in A$, $S \in S$, and $I \in I_{M,N}(\alpha)$.

**Definition 3.2.** Define

$$m_{BC}(M, N, \xi_1, \xi_2) := \sum_{\substack{k,a,S,I \in \mathbb{Z}^2 \times A \times S \times I \in I_{M,N}(\alpha) \cap |I|}} a_k \phi_{a,I}(\xi_1 + \xi_2)/2 \phi_{3,S,k}(\xi_1 + \xi_2) \prod_{j=1}^{2} \phi_{j,S,k}(\xi_j)$$

4. The Model Operators

4.1. Tiles and wave packets. A (standard) tile $P = I_P \times \omega_P$ is a rectangle of area 1, where the spatial interval $I_P$ is a standard dyadic interval and the frequency interval $\omega_P$ is a standard dyadic interval. We will also use shifted tiles, where $I_P$ is still standard dyadic but $\omega_P$ is a shifted dyadic interval.

Let $P$ be a tile collection. For $1 \leq p \leq \infty$ we say that the collection of functions $\{\phi_P, P \in P\}$ is a $L^p$-normalized wave packet collection if $\hat{\phi}_P \in (5/4)\omega_P$, and

$$\frac{d^n}{dx^n} (e^{-ic(x)\xi_1} \phi_P(x)) \leq c 1 \frac{1}{|I_P|^{n+1/p}} (1 + \frac{|x - c(I_P)|}{|I_P|})^{-n}$$

uniformly over $P \in P$, for all $n \geq 0$. If the estimate holds only for $0 \leq n \leq n_0$ then we say that the collection is of order $n_0$.

4.2. Multi-tiles. We say $Q = (Q_1, \ldots, Q_m)$ is an $m$-tile if the tiles $Q_1, \ldots, Q_m$ share the same spatial interval, denoted by $I_Q$. The cube $\omega_Q := \omega_{Q_1} \times \cdots \times \omega_{Q_m}$ is called the frequency cube of $Q$.

**Definition 4.1** (Sparse). A collection $Q$ of $m$-tiles is sparse relative to a constant $C_0$ if the following holds: for any $Q, R \in Q$ with $|I_R|/C_0 \leq |I_Q| \leq |I_R|$ we must have $|I_Q| = |I_P|$, and furthermore either $\omega_Q = \omega_R$ or $C_0\omega_Q \cap C_0\omega_R = \emptyset$.

**Definition 4.2** (Rank-1). A collection $Q$ of $m$-tiles is of rank 1 relative to $C_1 \geq 1$ if the following holds for any $Q, R \in Q$:

- If there exists $j$ such that $\omega_{Q_j} = \omega_R$, then $\omega_Q = \omega_R$.
- For every $j_0 \in \{1, \ldots, m\}$, if $5\omega_{Q_{j_0}} \subset 5\omega_R$, then $5C_1\omega_Q \subset 5C_1\omega_R$. If furthermore $|I_R| \leq |I_Q|$ then $5\omega_{Q_j} \cap 5\omega_{R_j} = \emptyset$ for every $j \neq j_0$.

4.3. Rigid triples of intervals. We say that $\{([I_{lower}, I, I_{upper}], I \in I)\}$ is a rigid collection of interval triples if there are integers $L_1 < m, n \leq L_1$ such that one of the following situations happens:

(i) For every $I \in I$ we have $I_{lower} = I - m|I|$ and $I_{upper} = I + n|I|$.

(ii) For every $I \in I$ we have $I_{lower} = I - m|I|$ and $I_{upper} = \lceil e(I) + (n-1/2)|I|, \infty \rceil$. 


(iii) For every $I \in \mathbf{I}$ we have $I_{\text{lower}} = (-\infty, \ell(I)) - (m - 1/2)|I|$ and $I_{\text{upper}} = I + n|I|$. 
We say that two collections have the same structure if the same situation (i.e. (i) or (ii) or (iii)) holds for both, with possibly different pairs $(m, n)$.

4.4. Description of the model operators. To bound $T_r$, we will show that it suffices to bound the following four types of operators: $T_{C \times C}$ (product of Carleson operators), $T_{CC}$ (paraproduct of Carleson operators), $T_{BC}$ (composition of bilinear Hilbert transform and Carleson operators), $T_{LM}$ (variational bi-linear Carleson operators). We will define these operators shortly. For convenience of notation, in the following we denote

$$\tilde{\phi}_{1,P,2}(x) = \phi_{1,P}(x) 1_{N_{j-1}(x) \in \omega_{1,P,\text{lower}}} 1_{N_j(x) \in \omega_{1,P,\text{upper}}}$$

and we define $\tilde{\phi}_{2,2,2}$, $\tilde{\phi}_{3,3,2}$, $\tilde{\phi}_{1,1,1}$; ... similarly. As a convention, $P$s will denote tile collections and $Q$s will denote shifted tri-tile collections, and all underlying collection of wave packets are $L^1$-normalized. All interval triples will be rigid, and in type CC we demand that the two underlying rigidity types are the same. Without loss of generality we assume that all tile and tritiles collections are finite and sufficiently sparse (all estimates are uniform over these collections), and for $Q$ we assume that the tri-tiles share the same shift.

$$T_{C \times C}(f_1, f_2) = \sum_{j=1}^K d_j \left( \sum_{P \in \mathbf{P}_1} |I_P| \langle f_1, \phi_{1,P} \rangle \tilde{\phi}_{1,P,2} \right) \left( \sum_{P \in \mathbf{P}_2} |I_P| \langle f_2, \phi_{2,P} \rangle \tilde{\phi}_{2,P,2} \right)$$

$$T_{CC}(f_1, f_2) = \sum_{j=1}^K d_j \left( \sum_{P \in \mathbf{P}_1} |I_P| \langle f_1, \phi_{1,P} \rangle \tilde{\phi}_{1,P,2} \right) \left( \sum_{Q \in \mathbf{Q}} |I_Q| \langle f_2, \phi_{2,Q} \rangle \tilde{\phi}_{2,Q,2} \right)$$

$$T_{BC}(f_1, f_2) = \sum_{j=1}^K d_j \left( \sum_{Q \in \mathbf{Q}} |I_Q| \langle f_1, \phi_{1,Q} \rangle \tilde{\phi}_{1,Q,2} \right) \left( \sum_{P \in \mathbf{P}} |I_P| \langle f_2, \phi_{2,P} \rangle \tilde{\phi}_{2,P,2} \right)$$

$$T_{LM}(f_1, f_2) = \sum_{j=1}^K d_j \left( \sum_{Q \in \mathbf{Q}} |I_Q| \langle f_1, \phi_{1,Q} \rangle \tilde{\phi}_{1,Q,2} \right) \left( \sum_{Q \in \mathbf{Q}} |I_Q| \langle f_2, \phi_{2,Q} \rangle \tilde{\phi}_{2,Q,2} \right) 1_{N_{j-1}, N_j \text{ constraints}}$$

The constraints in $T_{LM}$ read as follow:

- $2N_{j-1} \in \omega_{3, Q, \text{lower}}$, $2N_j \in \omega_{3, Q, \text{upper}}$,
- $N_{j-1} \in \omega_{k, Q, \text{lower}}$ and $N_j \in \omega_{k, Q, \text{upper}}$ for each $k = 1, 2$.

Furthermore, in $T_{LM} Q$ is not shifted and it satisfies the following rigidity constraint: for $m_2, m_3$ fixed bounded integers it holds for every $Q \in \mathbf{Q}$ that

- $\omega_{Q_1}$ is the translation of $\omega_{Q_1}$ by $m_2|\omega_{Q_1}|$,
- $\{x/2 : x \in \omega_{Q_1}\}$ is the translation of the left children of $\omega_{Q_1}$ by $m_3|\omega_{Q_1}|/2$.

4.5. Reduction to model operators. In the Appendix, we will show that $B_r(f_1, f_2)$ is bounded by a finite average of discrete operators of the above types. By the Holder inequality (see also the discussion at the end of the introduction), the desired bounds for $T_{C \times C}$ follow from known $L^p$ estimates for discrete variation-norm Carleson operators \[\text{[23]}\]: in Section \[6\] we will also deduce these estimates (see Theorem \[6.9\]) as a byproduct of several generalized Carleson embedding estimates.
and the $L^p$ theory for outer measure introduced in [6]. The proof of Theorem 6.9 will also serve as a model for the unfortunately more technical treatments for $T_{CC}$, $T_{BC}$, and $T_{LM}$.

5. SOME BACKGROUND ON OUTER MEASURE SPACES

We recall several notions from [6], with some simplifications for the setting of the current paper. An outer measure space $(X, S, \mu)$ consists of:

(i) A countable set $X$. Often we’ll assume $X$ is finite, in that case the underlying estimates are independent of the size of $X$.

(ii) An outer measure $\mu$ generated using countable coverings from a pre-measure on a fixed collection $E$ of non-empty subsets of $X$, which in particular covers $X$.

(iii) A size $S$ which assigns a number in $[0, \infty]$ to each pair $(f, E)$ where $f : X \to \mathbb{C}$ Borel measurable and $E \in E$, such that

$$S(f + g)(E) \leq S(f)(E) + S(g)(E) \quad \text{and} \quad S(\lambda f)(E) = |\lambda| S(f)(E),$$

and if $|f| \leq |g|$ then $S(f)(E) \leq S(g)(E)$ for all $E \in E$.

Assume that

$$\lambda \in \mathbb{R} \quad \text{and} \quad \lambda \mu(S(f) > \lambda) := \inf \{ \mu(F) : \text{outsup}_{X \setminus F} S(f) \leq \lambda \}.$$

Then

$$|f|_{L^{p,\alpha}(X,S,\mu)} = \sup_{\lambda > 0} \lambda^{\alpha} \mu(S(f) > \lambda)^{1/p},$$

and

$$|f|_{L^p(X,S,\mu)} = \left( p \int_0^\infty \lambda^{p-1} \mu(S(f) > \lambda) d\lambda \right)^{1/p}.$$

Many standard properties of classical $L^p$ spaces can be proved for outer $L^p$ spaces, see [6] for details. We summarize several estimates from [6].

**Proposition 5.1** (Outer Radon–Nikodym). Assume that $|f|_{L^\infty(X,S,\mu)} < \infty$, and for some Borel measure $\nu$ on $X$ it holds for every $E \in E$ that

$$\int_E |f| d\nu \leq C_1 \mu(E) S(f)(E).$$

Then it holds that (the implicit constant does not depend on $|f|_{L^\infty(X,S,\mu)}$)

$$\int_X |f| d\nu \leq C_1 |f|_{L^1(X,S,\mu)}.$$

**Proposition 5.2** (Outer Hölder). Suppose that $S(f_1 f_2)(E) \leq \prod_j S_j(f_j)(E)$ for all $E \in E$. Let $p_1, p_2, p_3 \in (0, \infty]$ such that $1/p_1 + 1/p_2 = 1/p_3$. Then

$$|f_1 f_2|_{L^{p_3}(X,S,\mu)} \leq 2 \prod_j |f_j|_{L^{p_j}(X,S,\mu)}.$$

**Proposition 5.3** (Convexity). If $p_1 < p < p_2$ and $1/p = \alpha_1/p_1 + \alpha_2/p_2$ with $\alpha_1, \alpha_2 \in (0, 1)$ and $\alpha_1 + \alpha_2 = 1$, then

$$|f|_{L^p(X,S,\mu)} \leq C \prod_j |f|_{L^{p_j\alpha_j}(X,S,\mu)}^{\alpha_j}.$$
The following Lemma generalizes [6, Proposition 3.5]. Below \((X, S, \mu)\) and \((Y, S', \nu)\) are given outer measure spaces.

**Lemma 5.1.** Let \(K\) be an operator mapping Borel measurable functions on \(X\) to Borel measurable functions on \(Y\), with the following properties:

(i) Scaling invariance: for any \(\lambda > 0\), \(|K(\lambda f)| = |\lambda K(f)|\);
(ii) Quasi sublinear: \(|K(f + g)| \leq C|K(f)| + C|K(g)|\);
(iii) Bounded from \(L^p(X, S, \mu)\) to \(L^{q_j, \infty}(Y, S', \nu)\), i.e. \(|Kf|_{\mathcal{L}^{q_j, \infty}} \leq M_j|f|_{\mathcal{L}^p}\).

Let \(p_0 < p_1\) and \(q_0 < q_1\) such that \(0 < p_j \leq q_j \leq \infty\). For \(\theta \in (0, 1)\) assume that
\[
\frac{1}{p} = \frac{\theta}{p_1} + \frac{1 - \theta}{p_0}, \quad \frac{1}{q} = \frac{\theta}{q_1} + \frac{1 - \theta}{q_0}.
\]
Then \(K\) maps \(L^p(X, S, \mu)\) into \(L^q(Y, S', \nu)\) with norm controlled by \(M_0^{1-\theta}M_1^\theta\).

**Proof.** By scaling invariance, we may assume \(M_0 = M_1 = M = 1\). Let
\[
\alpha := \left(\frac{q_0}{p_0} \right)^{1/q_0 - 1/q_1} \frac{1/p_0 - 1/p_1}{1/p_0 - 1/p_1}.
\]

**Case I:** \(q_1 < \infty\). It follows that \(p_1 < \infty\) since \(p_j \leq q_j\). Without loss of generality, assume that \(|f|_{\mathcal{L}^p(X, S, \mu)} = 1\).

For each \(\lambda > 0\), by definition there exists a set \(U\) such that \(S(f1_U) \leq \lambda^\alpha\) and \(\mu(U) \leq 2\mu(S(f) > \lambda^\alpha)\). Let \(f_{\lambda, l} = f1_U\) and \(f_{\lambda, s} = f1_{U^c}\). Here \(s\) stands for small and \(l\) stands for large.

Since \(f_{\lambda, l}\) is supported on \(U\) and \(|f_{\lambda, l}| \leq \|f\|\) and monotonicity of size, we have
\[
|f_{\lambda, l}|_{\mathcal{L}^q(X, S, \mu)} \lesssim \left(\int_0^{\lambda^\alpha} + \int_{\lambda^\alpha}^{\infty}\right) t^{p_0 - 1} \mu(S(f_{\lambda, l}) > t) dt 
\lesssim \lambda^{p_0\alpha} \mu(S(f) > \lambda^\alpha) + \int_{\lambda^\alpha}^{\infty} t^{p_0 - 1} \mu(S(f) > t) dt.
\]
Using \(|f|_{\mathcal{L}^p} = 1\), it follows in particular that \(|f_{\lambda, l}|_{\mathcal{L}^q} \lesssim \lambda^{1-q/p_0}\), and
\[
\int_0^{\infty} \lambda^{(p-p_0)\alpha - 1} |f_{\lambda, l}|_{\mathcal{L}^q(X, S, \mu)} d\lambda \lesssim \int_0^{\infty} \beta^{p_0 - 1} \mu(S(f) > \beta) d\beta \lesssim 1.
\]
For \(|f_{\lambda, s}|_{p_1}\), using \(S(f_{\lambda, s}) \leq \lambda^\alpha\) and monotonicity of size we similarly obtain
\[
|f_{\lambda, s}|_{\mathcal{L}^q(X, S, \mu)} \lesssim \left(\int_0^{\lambda^\alpha} t^{p_1 - 1} \mu(S(f) > t) dt\right)^{1/p_1} \lesssim \lambda^{1-q/p_1},
\]
\[
\int_0^{\infty} \lambda^{(p-p_1)\alpha - 1} |f_{\lambda, s}|_{\mathcal{L}^q(X, S, \mu)} d\lambda \lesssim \int_0^{\infty} \beta^{p_1 - 1} \mu(S(f) > \beta) d\beta \lesssim 1.
\]
Since \(q_0 \geq p_0\) and \(q_1 \geq p_1\), using quasi linearity and scaling invariance and the given assumption on bounds for \(K\) at the endpoints, it follows that
\[
\nu(S'(K(f)) > \lambda) \lesssim \lambda^{-q_0} |f_{\lambda, l}|_{\mathcal{L}^q(X, S, \mu)}^{q_0} + \lambda^{-q_1} |f_{\lambda, s}|_{\mathcal{L}^q(X, S, \mu)}^{q_1} 
\lesssim \lambda^{(p-p_0)\alpha - q} |f_{\lambda, l}|_{\mathcal{L}^q(X, S, \mu)}^{p_0} + \lambda^{(p-p_1)\alpha - q} |f_{\lambda, s}|_{\mathcal{L}^q(X, S, \mu)}^{p_1},
\]
here we have used the definition of $\alpha$. It follows that
\[
|Kf|^q_{L^q([0,\infty))} \leq \int_0^\infty \lambda^{n-1} \nu(S'(K(f)) > \lambda) d\lambda
\]
\[
\leq \int_0^\infty \lambda^{(p-p_0)\alpha-1} |f_{\lambda,l}|^{p_0}_{L^{p_0}(X,S,\mu)} d\lambda + \int_0^\infty \lambda^{(p-p_0)\alpha-1} |f_{\lambda,l}|^{p_0}_{L^{p_0}(X,S,\mu)} d\lambda \leq 1 .
\]

**Case 1:** $q_1 = \infty$. As before, we decompose $f = f_{\lambda,l} + f_{\lambda,s}$ where $S(f_{\lambda,s}) \leq c\lambda^\alpha$ for $c > 0$ small (chosen later), and $f_{\lambda,l}$ is supported in a set $U$ such that $\mu(U) \leq 2\mu(S(f) > c\lambda^\alpha)$. Since $M_1 = 1$, by choosing $c > 0$ sufficiently small we obtain $\nu(S'(K(f)) > \lambda) \leq \nu(S'(Kf_{\lambda,l}) > \lambda)$. The rest of the proof is similar. □

The following Lemma is a multilinear extension of Lemma 5.1. Below $(X,S,\mu)$ and $(X_j,S_j,\mu_j)$ are outer measure spaces, $j = 1,\ldots,n$. Let $0 < t_j \leq \infty$, $j = 1,\ldots,n$. Let $A$ be the $n$-dimensional rectangle $\{(x_1,\ldots,x_n) : 1/t_j \leq x_j \leq 1/s_j\}$.

**Lemma 5.2.** Let $K$ maps measurable function $(f_1,\ldots,f_n)$ on $X_1 \times \cdots \times X_n$ to measurable functions on $X$. Assume that $K$ has the following properties:

(i) **Scaling invariance:** for any $\lambda \geq 0$ and $1 \leq j \leq n$,
\[
|K(\ldots,\lambda f_j,\ldots)| = |\lambda K(\ldots f_j,\ldots)|
\]

(ii) **Quasi-linearity:** for any $1 \leq j \leq n$
\[
|K(f_1,\ldots,f_j+g_j,\ldots,f_n)| \leq C|K(f_1,\ldots,f_j,f_n)| + C|K(f_1,\ldots,g_j,f_n)|
\]

(iii) For every $(p,p_1,\ldots,p_n)$ such that $1/p = 1/p_1+\cdots+1/p_n$ and $(1/p_1,\ldots,1/p_n)$ is one of the vertices of $A$ it holds that
\[
|Kf|_{L^{p,\infty}p} \leq p_{p_1,\ldots,p_n} \prod_j |f_j|_{L^{p_j}}
\]

Then for every $(p,p_1,\ldots,p_n)$ such that $1/p = \sum 1/p_j$ and $(1/p_1,\ldots,1/p_n)$ is in the interior of $A$ it holds that
\[
|Kf|_{L^{p}p} \leq p_{p_1,\ldots,p_n} \prod_j |f_j|_{L^{p_j}} .
\]

**Proof.** For simplicity we will show the proof for $n = 2$, the general case is similar.

Let $(1/p_1,1/p_2)$ be in the interior of $A$ and $f_j \in L^{p_j}$ for $j = 1,2$, and assume $1/p = 1/p_1 + 1/p_2$. We normalize $|f_j|_{p_j} = 1$ and we’ll show that $|K(f_1,f_2)|_p \leq 1$.

**Case 1:** $t_1,t_2 < \infty$.

For every $\lambda > 0$ consider the decomposition $f_j = f_{j,\lambda,s} + f_{j,\lambda,l}$ where $f_{j,\lambda,l}$ is the restriction of $f_j$ to a some $U_j \subset X_j$ chosen such that $\mu_j(U_j) \leq \lambda$ and $\mu_j(U_j) \leq 2\mu_j(S(f_j) > \lambda)$. From the proof of Lemma 5.1 using $p_j < t_j$ we have
\[
\int_0^\infty \lambda^{p_j-t_j-1} |f_{j,\lambda,s}|^{t_j}_{p_j} d\lambda \leq |f_j|_{p_j} = 1,
\]
\[
\int_0^\infty \lambda^{p_j-t_j-1} |f_{j,\lambda,l}|^{t_j}_{p_j} d\lambda \leq |f_j|_{p_j} = 1
\]

Now, given $\lambda > 0$ we will decompose $K(f_1,f_2)$ by decomposing $f_1 = f_{1,\alpha_1,s} + f_{1,\alpha_1,l}$ and similarly $f_2 = f_{2,\alpha_2,s} + f_{2,\alpha_2,l}$ with $\alpha_1 = \lambda^{p/p_1}$ and $\alpha_2 = \lambda^{p/p_2}$ (clearly $\alpha_1\alpha_2 = \lambda$).

This leads to a decomposition of $K(f_1,f_2)$ into four terms, and we will estimate each of them using the known weaktype estimate at one suitable vertex of $A$. 

We show below the treatment for $K(f_1, f_2)$, for which we'll use the vertex $(t_1, t_2)$. Letting $t$ denote $t_1 t_2 / (t_1 + t_2)$, it follows that

$$
\int_0^\infty \lambda^{p-1} \mu(S(K(f_1, f_2, s_2, s))) > \lambda) d\lambda \leq \int_0^\infty \lambda^{p-t-1} \prod_{j=1,2} |f_{j, \alpha_j, s_j}|_{t_j}^{t_1 t_2 / (t_1 + t_2)} d\lambda
$$

(classical Holder) \leq \prod_{j=1,2} \left( \int_0^\infty \lambda^{p-t_j-1} |f_{j, \alpha_j, s_j}|_{t_j}^{t_1 / (t_1 + t_2)} d\lambda \right)

\begin{align*}
(\text{using } \lambda^{p-1} d\lambda = C \alpha_j^{p_j-1} d\alpha_j) & \leq C \prod_{j=1,2} \left( \int_0^\infty \alpha_j^{p_j-t_j-1} |f_{j, \alpha_j, s_j}|_{t_j}^{t_1 / (t_1 + t_2)} d\lambda \right) \\
& \leq \prod_{j=1,2} |f_j|^{p_j t_j / (t_1 + t_2)} \leq 1 .
\end{align*}

The other terms (in the decomposition for $K(f_1, f_2)$ could be treated similarly, thus by quasilinearity of size it follows immediately that

$$
|K(f_1, f_2)|_{\mathcal{C}(X, \nu, \mu)} = (p \int_0^\infty \lambda^{p-1} \mu(S(K(f_1, f_2)) > \lambda) d\lambda)^{1/p} \leq 1 .
$$

Case 2: Exactly one of $t_1, t_2$ is $\infty$.

Without loss of generality assume that $t_1 < t_2 = \infty$. In this case we still carry out the same decompositions as before. The two terms that does not involve $f_{2, \alpha_2, s}$ could be treated as before. For $K(f_1, f_2, s)$ using the assumed weak-type estimate at $(t_1, t_2 = \infty)$ we have

$$
\mu(S(K(f_1, f_2, s)) > c\lambda) \leq \lambda^{-t_1} |f_{1, \alpha_1, s}|_{t_1} t_1 \leq \alpha_1^{t_1} \lambda^{-t_1} |f_{1, \alpha_1, s}|_{t_1} = \alpha_1^{-t_1} |f_{1, \alpha_1, s}|_{t_1} ,
$$

$$
\int_0^\infty \lambda^{p-1} \mu(S(K(f_1, f_2, s)) > c\lambda) d\lambda \leq \int_0^\infty \alpha_1^{p_1-t_1-1} |f_{1, \alpha_1, s}|_{t_1} d\alpha_1 \leq 1 .
$$

The term $K(f_1, f_2, s)$ could be treated similarly.

Case 2: $t_1 = t_2 = \infty$.

In this case we modify the decompositions slightly so that $S_j(f_j, \alpha_j, s) \leq c \alpha_j$ for both $j = 1, 2$, where $c > 0$ is sufficiently small. It follows from the assumed weak-type estimate at $(t_1, t_2) = (\infty, \infty)$ that

$$
|K(f_1, f_2, s)|_{\infty} \leq O(c^2 \alpha_1 \alpha_2) = O(c^2 \lambda)
$$

therefore by choosing $c$ sufficiently small we obtain, for some $C > 0$ large,

\begin{align*}
\nu(S(K(f_1, f_2)) > \lambda) & \leq \nu(S(K(f_1, f_2, s)) > \lambda / C) + \\
& \quad + \nu(S(K(f_1, f_2, s)) > \lambda / C) + \\
& \quad + \nu(S(K(f_1, f_2, s)) > \lambda / C) .
\end{align*}

The three terms on the right hand side could be treated as before. \(\square\)

6. Generalized Carleson embeddings and outer $L^p$ estimates for discrete variation-norm Carleson operators

Let $\{\phi_P, P \in X\}$ be $L^1$-normalized Fourier wave packets where $P$ is finite sparse, such that $\text{supp} \phi_P \subset (5/4) \omega_P$, and $\{\omega_{P, \text{lower}}, \omega_P, \omega_{P, \text{upper}}\}, P \in P$ is rigid. For
simplicity we assume that $\omega_{P,\text{upper}}$ is finite and $\omega_{P,\text{lower}}$ is a half line (the other settings could be handled similarly). For technical convenience, assume that $1000\omega_P$ is strictly between $\omega_{P,\text{lower}}$ and $\omega_{P,\text{upper}}$ for every $P \in P$.

Let $f$ be a Schwarz function on $\mathbb{R}$, and $\alpha_j : \mathbb{R} \to [0, \infty]$, and define

$$T_1 f(P) := \langle f, \phi_P \rangle, \quad T_2 f(P) := \left( f, \sum_{1 \leq j \leq K} d_j \phi_{P,j} 1_{|P| \leq \alpha_j(x)} \right),$$

where $\phi_{P,j}$ is defined by (13).

In this section, we consider embedding estimates for $T_1$ and $T_2$ from $L^p(\mathbb{R})$ to outer measure spaces on $P$. Following [9] we will refer to these estimates as generalized Carleson embeddings.

6.1. Outer measure spaces. For every $P \in P$, let $\overline{\omega}_P$ be the convex hull of $50\omega_P$ and $50\omega_{P,\text{upper}}$.

Generating subsets: A nonempty $E \subset P$ is a generating set (i.e. $E \subset E$) if there exists a dyadic interval $I_E$ and $\xi_E \in \mathbb{R}$ such that for every $P \in E$ we have

$$I_P \subset I_E, \quad (\xi_E - \frac{1}{2|I_E|}, \xi_E + \frac{1}{2|I_E|}) \subset \overline{\omega}_P$$

We say that $E$ is lacunary if furthermore $\xi_E \in 50\omega_{P,\text{upper}}$ for every $P \in E$, and $E$ is overlapping if $\xi_E \notin \overline{\omega}_P \setminus 50\omega_{P,\text{upper}}$ for every $P \in E$.

Outer measure: The outer measure $\mu$ will be generated from $\mu(E) = \inf \sum_j |I_{E_j}|$, infimum taken over all countable coverings of $E$ by generating sets.

Size: For any $0 < t < \infty$ and any generating set $E \subset P$, let $S_t$ be the size

$$S_t(f)(E) = \sup_{S \subset E} \left( \frac{1}{|S|} \sum_{P \in S} |f(P)|^t |I_P| \right)^{1/t}.$$

If the supremum is taken over only lacunary subsets $S$, we call $S_{t,\text{lac}}$ the corresponding size, and define $S_{t,\text{overlap}}$ similarly. When $t = \infty$ the three sizes agree

$$S_{\infty,\text{lac}}(f)(E) = S_{\infty,\text{overlap}}(f)(E) = S_{\infty}(f)(E) = \sup_{P \in E} |f(P)|.$$ 

It is clear that $S_t$ and $S_{t,\text{lac}}$ and $S_{t,\text{overlap}}$ are decreasing functions of $t$.

Strong disjointness: Let $(E_m)$ be lacunary. We say they are strongly disjoint if for any $m \neq n$:

- If $P_1 \in E_m$ and $P_2 \in E_n$ such that $10\omega_{P_1} \cap 10\omega_{P_2} \neq \emptyset$ and $|I_{P_1}| \leq |I_{P_2}|$ then $I_{P_1} \cap I_{E_n} = \emptyset$. (In particular $I_{P_1} \cap I_{P_2} = \emptyset$.)

The following estimate follows from a standard argument, see e.g. [23] or [12].

Proposition 6.1. Let $(E_m)$ be strongly disjoint lacunary and $P_1 = \bigcup E_m$. Then

$$\left| \sum_{P \in P_1} |f(P)| \phi_P \right|_{L^2(\mathbb{R})} \lesssim \left( \sum_{P \in P_1} |f(P)|^2 \right)^{1/2} + \sup_{P \in P_1} |f(P)| \left( \sum_m |I_{E_m}| \right)^{1/2}.$$

6.2. Embeddings for $T_1$. The following is a discrete version of [6, Theorem 5.1] and its $L^{2,\infty}$ endpoint is reformulation of the well-known size lemma in [12].

Theorem 6.1. For any $2 < p \leq \infty$ it holds that $\|T_1 f\|_{L^p(P,S_{2,\text{lac}})} \lesssim \|f\|_{L^p(\mathbb{R})}$, and the weak-type estimate holds at $p = 2$. 


Proof. Without loss of generality assume $|f|_{L^p(\mathbb{R})} = 1$. The endpoint $p = \infty$ is a consequent of Lemma 6.3, so by interpolation it suffices to consider the weak-type estimate at $p = 2$.

Fix any $\lambda > 0$. We’ll show that there exists $\mathbf{P}' \subset \mathbf{P}$ such that $\mu(\mathbf{P}') \leq \lambda^2$ and $|T_1f|_{L^\infty(\mathbf{P}',\mathcal{S}_{2,\text{lac},\|\cdot\|})} \leq \lambda$. Without loss of generality assume that $|T_1f|_{L^\infty} > 2\lambda$.

We define $\mathbf{P}' = \bigcup_i F_i$, where $F_i$ are selected as follows:

(i) If there is $E \subset \mathbf{P}$ lacunary with $\left(\frac{1}{|I_E|} \sum_{P \in E} |T_1f(P)|^2 |I_P|\right)^{1/2} > \lambda$, we select one such $E_1$ with smallest possible $\xi_E$.

(ii) Let $F_1 = \{P \in \mathbf{P} : (\xi_{E_1} - \frac{1}{4|I_{E_1}|} \xi_{E_1} + \frac{1}{4|I_{E_1}|}) \in \mathring{\mathcal{C}}_P, \ I_P \subset I_{E_1}\}$.

We remove $F_1$ from $\mathbf{P}$ and repeat the above argument and select $E_m$, $F_m$, $m = 2, 3, \ldots$. Since $\mathbf{P}$ is finite the process will stop, and by geometry and sparseness of $\mathbf{P}$ it is clear that $(E_m)_{i \geq 1}$ is strongly disjoint. Let $\mathbf{P}_1 = \bigcup E_m$. It follows that

$$\lambda^2 \sum_m |I_{E_m}| \leq \sum_{P \in \mathbf{P}_1} |I_P| \left|\langle f, \phi_P \rangle\right|^2 \leq \left(\sum_{P \in \mathbf{P}_1} |I_P| \left|\langle f, \phi_P \rangle\right|^2\right)^{1/2} (\text{Cauchy-Schwarz, } |f|_2 = 1) \leq \lambda \left(\sum_{m} |I_{E_m}|\right)^{1/2} \quad \text{(since } |T_1f|_{L^\infty} < 2\lambda\text{).}$$

It follows that $\lambda^2 \mu(\mathbf{P'}) \leq \lambda^2 \sum_m |I_{E_m}| \leq 1$, as desired. □

For any $g$ consider the following tile maximal average

$$(14) \quad M_N(\mathbf{P}, f) := \sup_{\mathbf{P} \subset \mathbf{P}} \frac{1}{|I_P|} \int \sim_{I_P} f .$$

The following Lemmas are reformulation of standard estimates, see e.g. [17].

**Lemma 6.2.** It holds that

$$|T_1f|_{L^\infty(\mathbf{P},\mathcal{S}_{2,\text{lac},\|\cdot\|})} \leq \sup_{E \subset \mathbf{P} \text{ lacunary}} \frac{1}{|I_E|} \left(\sum_{P \in E} |T_1f(P)|^2 |1_{I_P}|\right)^{1/2} |1_{L^1} .$$

By Calderon–Zygmund theory, for every lacunary generating set $E$ it holds that

$$\left|\left(\sum_{P \in E} |T_1f(P)|^2 |1_{I_P}|\right)^{1/2}\right|_{L^1} \leq N \left|\sim_{I_E} f\right|_1$$

Consequently, Lemma 6.3 implies the following standard corollary:

**Lemma 6.3.** It holds that

$$|T_1f|_{L^\infty(\mathbf{P},\mathcal{S}_{2,\text{lac},\|\cdot\|})} \leq M_{N+4}(\mathbf{P}, f) \leq \sup_{\mathbf{P}} \inf_{\mathbf{P}} M_I(f, \sim_{I_E}) .$$

When $\mathbf{P}$ is a generating set, we have the following standard estimates, part (i) is a reformulation of [22, Proposition 3.4]. For convenience, we’ll sketch a proof.

**Lemma 6.4.** (i) Suppose that $E$ is a generating set. Then for every $1 < p \leq \infty$

$$|T_1f|_{L^p(E,\mathcal{S}_{2,\text{lac},\|\cdot\|})} \leq_p N \left|f \sim_{I_E}\right|_p ,$$

and the weak-type endpoint $p = 1$ holds.
(ii) If $E$ is lacunary then for every $1 < p < \infty$
\[ | \sum_{P \in E} |I_P| T_1 f(P) \phi_P|_{L^p(\mathbb{R})} \lesssim_p |T_1 f|_{L^p(E, S_{2, lac} - \mu)} . \]

Proof. (i) For any $F \subset C$ by Lemma 6.3 and Lemma 6.2 we have:
\[ |T_1 f|_{L^p(F, S_{2, lac} - \mu)} \leq C_N \sup_{S \subset C \text{ lac}} \inf_{x \in S} M(f \mathcal{X}_E^N)(x) , \]
thus the endpoint $p = \infty$ follows. By interpolation it suffices to consider the weak-type endpoint at $p = 1$. Fix $\lambda > 0$. We select a sequence of generating subsets $S_1, S_2, \ldots$ of $E$ as follows. If there exists $J \subset I_E$ dyadic such that
\[ \inf_{x \in J} M(f \mathcal{X}_E^N)(x) > \lambda/C_N \]
we choose a maximal $J$, and let $S_1 = \{ P \in E : I_P \subset J \}$ and remove $S_1$ from $E$, then repeat the above selection algorithm. Since $E$ is finite the algorithm will stop and we obtain our sequence $S_1, S_2, \ldots, S_k$. Clearly on $E - S_1 - \cdots - S_k$ we have $S_{2, lac}(T_1 f) \leq \lambda$. Now, $I_{S_j}$’s are pairwise disjoint and $S_j$ are generating sets, therefore
\[ \lambda \mu(\bigcup_j S_j) \leq \lambda \{ M(f \mathcal{X}_E^N) > C_N \lambda \} \leq |f \mathcal{X}_E^N|_1 . \]

(ii) Let $g \in L^{p'}$, by outer Radon-Nikodym/Holder we have
\[ \left\langle \sum_{P \in E} |I_P| T_1 f(P) \phi_P | g \right\rangle \leq |T_1 f|_{L^p(E, S_{2, lac} - \mu)} |T_1 g|_{L^{p'}(E, S_{2, lac} - \mu)} \lesssim_p |T_1 f|_{L^p(E, S_{2, lac} - \mu)} |g|_{L^{p'}(\mathbb{R})} \]
thanks to part (i). The desired estimate follows from duality. □

6.3. Embeddings for $T_2$. The following Theorem generalizes the variation-norm density lemma in [23], which in turn generalizes the density lemma in [12].

Theorem 6.5. Assume that $s \neq 2$, and let $q = \min(2, s')$, $s' = s/(s - 1)$.
For any $p \in (s, \infty]$ it holds that
\[ |T_2 f|_{L^p(P, s_{1, overlap} + S_{d, lac} - \mu)} \lesssim M_N(P, 1_{\text{supp}(f)})^{1/p} \| f \sum_j |d_j|^s \|^1/s \|_p , \]
and the weak-type estimate holds at $p = s$.

By interpolation, it suffices to consider the $L^\infty$ estimate and the weak type estimate at $p = s$. (Note that we could fix $G = \text{supp}(f)$ and invoke interpolation theorems for the linear map $T_2$ from functions on $G$ to outer measure spaces, the factor $M_N(P, 1_G)^{1/p}$ should be thought of as an estimate for the norm of $T_2$.)

The proof strategy will involve three Lemmas: Lemma 6.7 and Lemma 6.8 will be used to handle the $L^\infty$ endpoint, and Lemma 6.6 will be used to handle the weak type estimate at $p = 2$.

For convenience of notation, for each $P$ let $\overline{P}$ denote the following completion
\[ P = \{ R : \exists P_1, P_2 \in P \text{ with } I_{P_1} \subset I_R \subset 30 I_{P_2}, \mathcal{X}_R \cap \mathcal{X}_{\overline{P}_j} \neq \emptyset \} \]  
(15) 

For convenience of notation, for any measurable sequence $(g_j)$ we denote
\[ m_{s, N}(P, (g_j)) := \sup_{R \in P} \frac{1}{|R|} \int_{\mathcal{X}_R^N} \left( \sum_{j : N_j \in R} |g_j|^s \right)^{1/s} , \]
(16)
Clearly, $m_{x,N} \lesssim m_{s,N}$.

Now, the $L^\infty$ endpoint of Theorem 6.5 follows from Using Lemma 6.7 and Lemma 6.8 which gives the estimate

$$|T_2f|_{L^\infty(P, S_{\text{overlap}} + J_{\text{lac}})} \lesssim m_{s,N}(P, (fd_j)).$$

The weaktype estimate at $p = s$ of Theorem 6.5 follows from the following result.

**Lemma 6.6.** Let $p \in [s, \infty]$ and assume $\text{supp}(\sum_j |g_j|^s)^{1/s} \subset G$. For any $\lambda > 0$ there exists $P_1 \subset P$ such that

$$\lambda \mu(P_1 \setminus P)^{1/p} \lesssim_{N,p,s} M_N(P, 1_G)^{1/p'} \left( \sum_j |g_j|^s \right)^{1/s} |L^p(\mathbb{R})|$$

and $m_{s,N}(P, (g_j)) \leq \lambda$.

**Proof.** The endpoint $p = \infty$ is trivial, while the endpoint $p = s$ follows from

$$m_{s,N}(P, (g_j)) \leq M_N(P, 1_G)^{1/s} \sup_{R \in \mathbb{P}} \left( \frac{1}{|R|} \int_{A_R}^{N} \sum_{j \in \omega_R} |g_j|^s \right)^{1/s}$$

and a variation-norm version of the standard density lemma (see e.g., [23]). The general case could be obtained by a simple interpolation argument: let $A = \{ x : (\sum_j |g_j(x)|^s)^{1/s} > \lambda/C \}$ for $C > 0$ large, we use the $p = s$ and $p = \infty$ endpoints to respectively treat $g_j 1_A$ and $g_j 1_{A^c}$. We omit the details. \qed

**Lemma 6.7.** Assume that $s \neq 2$. Let $q = \min(2, s')$. Then

$$|T_2f|_{L^q(P, S_{\text{lac}} \cdot \mu)} \lesssim m_{s,N}(P, (fd_j)).$$

**Proof.** It suffices to show that for every lacunary $E$ and $g : E \rightarrow \mathbb{C}$ it holds that

$$|T_2f|_{L^q(E, S_{\text{lac}} \cdot \mu)} \lesssim |E| \|g\|_{L^q(E, S_{\text{lac}} \cdot \mu)} m_{s,N}(E, (fd_j)).$$

Indeed, taking $g(P) = T_2f(P)|T_2f(P)|^{q-2}$ and applying the above estimate for all lacunary subsets of $P$, we obtain an equivalent form of the desired estimate:

$$\left( S_{\text{lac}}(T_2f)(P) \right)^q \lesssim \left( S_{\text{lac}}(T_2f)(P) \right)^{q-1} m_{s,N}(P, (fd_j)).$$

Below, for brevity we write $|g|_\infty$ for $|g|_{L^\infty(E, S_{\text{lac}} \cdot \mu)}$, and $m_s$ for $m_{s,N}$.

For every $P$ and $x$, it is clear that at most one $j$ will satisfy $N_{j-1} \in \omega_{P, \text{lower}}$ and $N_j \in \omega_{P, \text{upper}}$. Let $d_P(x) = d_j(x) 1_{I_P \in \omega_P}$ if such $j$ exists, otherwise $d_P(x) = 0$. Let $J$ be the collection of all maximal dyadic $J$ such that $3J$ does not contain any $I_P$, $P \in E$. Clearly, $J$ is a partition of $\mathbb{R}$, so the left-hand side of (17) is bounded by

$$\leq \sum_{J \in J} \sum_{P \in E} |I_P| |g(P) \phi_P f|_{L^1(J)} \leq A + B,$$
where $A$ denotes the contribution of $(J, P)$ such that $|I_P| \leq 2|J|$ and the rest is in $B$. Since $|\tilde{\chi}^N_{I_P} f d_P|_1 \leq |I_P|m_{\infty,N}(E,(f d_j))$, we obtain

$$A \leq \sup_P |g(P)|m_{\infty}(E, (f d_j)){\sum_{J \in \mathcal{J}, P, |P| \leq 2|J|} |I_P| \sup_{x \in J} \tilde{\chi}_{I_P}(x)^2 \tilde{\chi}_{I_P}(x)^2}$$

$$\leq \sup_P |g(P)|m_{\infty}(E, (f d_j)){\sum_{J \in \mathcal{J}, P, |I_P| \leq 2|J|} \sup_{x \in J} \tilde{\chi}_{I_P}(x)^2}$$

$$\leq |I_E| \sup_P |g(P)|m_{\infty}(E, (f d_j)) .$$

Note that the above estimate remains true when $E$ is overlapping.

We now estimate $B$ i.e. the contribution of $(J, P)$ with $|I_P| \geq 4|J|$.

First, by geometry, such $J$ must be a subset of $3I_E$. Furthermore, by maximality there is $P_1 \in E$ such that $I_{P_1} \subset 3\pi(J)$ where $\pi(J)$ is the dyadic parent of $J$. Let $R$ be a tile such that $(I_{P_1} \cup \pi(J)) \subset I_R$ and $|I_R| = 4|J|$, and $\xi_E \in \omega_R$. We’ll show that $R \in \mathcal{P}$. To see this, note that for (17) we may assume that $I_{P_1} = I_E$ for some $P_1 \in E$. It follows that $I_{P_1} \subset I_R \subset 10I_3 \subset 3I_P$; it is also clear that $\hat{\omega}_R \cap \hat{\omega}_{P_j} \neq \emptyset$ for each $j = 1, 2$ since they all contain $\xi_E$, thus $R \in \mathcal{P}$ as claimed.

Now, if $P \in E$ such that $|I_P| \geq |I_R| = 4|J|$, by sparseness it follows that $\omega_{P,upper} \subset \hat{\omega}_R$. Now, for $x \in J$ by H"older’s inequality we have

$$\sum_{P \in E: |I_P| \geq 4|J|} |I_P| |g(P)| \phi_P(x)f(x) d_P(x)$$

$$\leq \left( \sum_{j: P \in \omega_R} |f d_j|^s \right)^{1/s} \left( \sum_{j: \alpha_j(x) \geq |I_P| \geq 2|J|} |I_P| |g(P)| \tilde{\phi}_{P,j} |x|^s \right)^{1/s} .$$

Since $T$ is lacunary and $\mathcal{P}$ is sparse, we can find a sequence of integers $O(1) + \log_2(|J|) \leq m_1(x) \leq n_1(x) \leq \cdots \leq m_K(x) \leq n_K(x)$ such that

$$\{\alpha_j \geq |I_P| > 2|J| : N_{j-1} \in \omega_{P,lower}, N_j \in \omega_{P,upper} \} = \{P \in E : 2^{m_j} < |I_P| \leq 2^{n_j} \} .$$

(Note that when $m_j = n_j$ the set is understood to be empty, one example when this may happen is when $\alpha_j$ is too small or $2|J|$ is too large relative to the range of $|I_P|$ imposed by the $N_j, N_{j-1}$ constraints.) Now, let $g_E(x) = \sum_{P \in E} |I_P| |g(P)| \phi_P(x)$. For every $x \in J$ we have

$$(\sum_{1 \leq j \leq K} \sum_{\alpha_j \geq |I_P| > 2|J|} |I_P| |g(P)| \tilde{\phi}_{P,j} |x|^s \right)^{1/s} = (\sum_{1 \leq j \leq K} (|\Pi_{n_j} - \Pi_{m_j}| g_E |x|^s)^{1/s}$$

where $\Pi_{n,j}$ are Fourier projections onto the relevant frequency scales of $E$ (essentially projecting onto $|k - \xi_E| \leq 2^{-n}$, thus larger values of $n$ means narrower bands). Using $m_1 > \log_2{|J|} + O(1)$ and Minkowski’s inequality, the last display is bounded by $M_J(|\Pi_k g_E|_{V^N_k(z)})$. It follows that

$$B \leq \sum_{J \in \mathcal{J}, P \subset 3I_E} M_J(|\Pi_k g_E|_{V^N_k(z)}) (\sum_{j: P \in \omega_R} |f d_j|^s |x|^s)^{1/s} |L^1(J)$$

$$\leq |M(|\Pi_k g_E|_{V^N_k(z)})| L^1(3I_E) m_{+}(E,(f d_j)) .$$
Since $s \neq 2$ we either have $s' > 2$ or $s' < 2$. If $s' > 2$ then by the (continuous) Lépingle inequality we obtain

$$B \lesssim |I_E|^{1/2}|g_E|_2 m_s(E,(fd_j))$$

$$\lesssim |I_E||g|_{L^p(S_{\ell,\text{lac}})} m_s(E,(fd_j)).$$

If $s' < 2$ then by the continuous Pisier–Xu inequality (see [5]) we obtain

$$B \lesssim |I_E|^{1/2} \left( \sum_k \left| \sum_{P \in E: |I_P| = 2^k} |I_P| g(P) \phi_P|^{s'} \right|^{1/s'} m_s(E,(fd_j)) \right)^{1/s'}$$

$$\lesssim |I_E||g|_{L^p(E,S_{\ell,\text{lac}})} m_s(E,(fd_j)). \quad \Box$$

**Lemma 6.8.** If $1 < q \leq \infty$ then

$$|T_2f|_{L^q(P,S_{\ell,\text{overlap}},\mu)} \lesssim m_{x,N}(P,(fd_j)).$$

**Proof.** The proof is entirely similar to the proof of Lemma 6.7, it suffices to show for any overlapping $E$ the following estimates

$$(19) \quad \sum_{P \in E} |I_P| |T_2f(P)g(P)| \lesssim |I_E| \sup_P |g(P)| m_{x,N}(E,(fd_j)).$$

We define $J$ as before and invoke (18) again, and $A$ could be estimated as before. To estimate $B$ we use the following observation: since $E$ is overlapping, for every $x$ there is at most one $j$ and at most one scale of $E$ that contributes to $\sum_{|I_P| = 2^j} |I_P| g(p(x)) dp(x)$. Let $R \in E$ be as before. It follows that

$$B \lesssim \sum \sup_{J \in 3J_E} \sup_{j:N_j \in R \cap [2^j]} \sum_{P \in E: |I_P| = 2^j} |g(P)||\tilde{\chi}_{I_P}^{N+4} d_P f|_{L^1(J)}$$

$$\lesssim \sum_{J \in 3J_E} \sup_{j:N_j \in R} |d_j f|_{L^1(J)} \sup_P |g(P)|$$

$$\lesssim N |I_E| \sup_P |g(P)| m_{x,N}(E,(fd_j)). \quad \Box$$

### 6.4. Outer $L^p$ estimates for discrete variation-norm Carleson operators.

Let $r \neq 2$ and $q = \min(2,r)$, and $\tilde{\phi}_{P,j}$'s are defined relative to $(N_j(x))$ and $K(x)$. We consider the variation-norm operator

$$V_r(g) = \left( \sum_j \sup_{\alpha} \frac{1}{|I_P|} \sum_{P \in P: |I_P| \leq \alpha} |I_P| g(P) \tilde{\phi}_{P,j} |^{r} \right)^{1/r}.$$

**Theorem 6.9.** Let $F$ be a subset of $\mathbb{R}$.

(i) For any $1 < p < r$ it holds that

$$|V_r(g)|_{L^p(F)} \lesssim \sum_{p \in \mathcal{P}} M_N(P, 1_F)^{1/p} |g|_{L^p(P,S_{\ell,\text{lac}},\mu)}.$$

(ii) If $r > 2$ then for all $p > r'$ it holds that

$$|V_r(T_f)|_p \lesssim |f|_{L^p(\mathbb{R})}.$$

**Proof.** (i) Let $s = r'$. We may find $(\alpha_j)$ and $(g_j)$ measurable functions with $\alpha_j \geq 0$ and $\sum_{1 \leq j \leq K} |g_j|^s = O(1)$ such that

$$V_r(g) \leq \sum_j |I_P| g(P) \sum_{P \in P: |I_P| \leq \alpha_j(x)} \tilde{\phi}_{P,j} 1_{|I_P| \leq \alpha_j(x)} g_j$$
Let \( h \in L^p(\mathbb{R}) \) and \( T_2 h(P) = \left\langle \sum_j \bar{\varphi}_{P,j} g_j 1_{[\alpha_j(x), h1_F]} \right\rangle \). It suffices to show that
\[
\sum_P |I_P| g(P) T_2 h(P) \lesssim_{p,N} M_N(P,1_F)^{1/p} |g| L^p(P,1_{\text{lac}}) |h| L^{p'}(\mathbb{R})
\]
Via applications of the classical Holder inequality it follows that
\[
S_1(gT_2 h) \lesssim S_{\text{lac}}(g)(S_{\text{lac}} + S_{\text{overlap}})(T_2 h).
\]
Thus, using outer Radon-Nykodym and outer Holder inequalities, we obtain
\[
\sum_P |I_P| g(P) T_2 h(P) \lesssim |g| L^p(P,1_{\text{lac}}) T_2 h L^{p'}(P,1_{\text{overlap}}) \quad (20)
\]
Now, using Theorem 6.5 and noticing \( s' = r \) it follows that
\[
|T_2 h| L^{p'}(P,1_{\text{overlap}}) \lesssim M_N(P,1_F)^{1/p} |h| \left( \sum_j |a_j|^s \right)^{1/s} L^{p'}(\mathbb{R})
\]
provided that \( p' > s \). This completes the proof of part (i).

(ii) Since \( r > 2 \), we obtain \( q = 2 \). We say that a subset is major if it has at least half of the total measure. By restricted weak type interpolation [19] (see also Section 7.3.2), it suffices to show that we could find \( \beta \) arbitrarily close to 0 and also arbitrarily close to 1/s such that \( \langle V_\vee(T_1 f), h \rangle \) is of restricted weak type \((\beta,1-\beta)\). That is, there exists \( j_0 \in \{1,2\} \) depending only on \( \beta \) such that given any \( F_1, F_2 \subset \mathbb{R} \) with finite positive Lebesgue measures we could find \( S_1 \subset F_1 \) and \( S_2 \subset F_2 \) both major subsets and furthermore \( S_{j_0} = F_{j_0} \) and
\[
\langle V_\vee(T_1 f), h \rangle \lesssim |F_1|^\beta |F_2|^{1-\beta} \quad \text{if } |f| \leq 1_{S_1} \text{ and } |h| \leq 1_{S_2}.
\]
To get \( \beta \) near 1/s, we let \( S_1 = F_1 \) and \( S_2 = F_2 \setminus E \) where \( E := \{M_1 F_1 > C|F_1|/|F_2|\} \) and \( C \) is large enough, and \( S_2 = F_2 \). Without loss of generality assume that \( 1 + \frac{\text{dist}(I_p,E)}{|I_p|} \sim 2^k \) provided that we have enough decay in the estimate. By convexity, for \( 2 < p < s' \) it follows from part (i) that
\[
\langle V_\vee(g), h \rangle \lesssim |g| L^p(P,1_{\text{lac}}) M_N(P,1_F)^{1/p} \left| h \right|_{L^p}
\]
\[
\lesssim \left| g \right|^{1-\frac{2}{p}} L^p(P,1_{\text{lac}}) \left| g \right|^{\frac{2}{p}} L^{2\times q}(P,1_{\text{lac}}) M_N(P,1_F)^{1/p} |F_2|^{1/p'}
\]
\[
\lesssim 2^{-Nk} (\sup_{I_p} \sup_{x \in I_p} M_1 F_1(x))^{1-\frac{2}{p}} |F_1|^{1/p} |F_2|^{1/p'}
\]
\[
\lesssim 2^{-k} \left( |F_1|/|F_2| \right)^{1-2/p} |F_1|^{1/p} |F_2|^{1/p'} = 2^{-k} |F_1|^\beta |F_2|^{1-\beta},
\]
where \( \beta := 1/p' \), which is arbitrarily close to 1/s if \( p \) is sufficiently close to \( s' \).

To get \( \beta \) near 0, we let \( S_1 = F_1 \setminus E \) and \( S_2 = F_2 \) where \( E := \{M_1 F_2 > C|F_2|/|F_1|\} \) with \( C \) sufficiently large. Similarly we obtain
\[
\langle V_\vee(g), h \rangle \lesssim |g| L^p(P,1_{\text{lac}}) \left| g \right|^{\frac{2}{p}} L^{2\times q}(P,1_{\text{lac}}) M_N(P,1_F)^{1/p} \left| F_2 \right|^{1/p'}
\]
\[
\lesssim 2^{-Nk} |F_1|^{1/p} (2^k |F_2|/|F_1|)^{1/p} |F_2|^{1/p'}
\]
\[
\lesssim 2^{-k} |F_2| \quad \square
\]
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7. Estimates for BC model operators

Theorem 7.1. Let \( T \) be a BC model operator. Then \(|T|_{L^{p_1} \times L^{p_2} \rightarrow L^q} < \infty \) for every \( 1/q = 1/q_1 + 1/q_2 \) such that \( \frac{p_r}{3p - 4} < q_1, q_2 \leq \infty \) and \( q \geq \frac{p_r}{2} \).

For simplicity, we assume that \( \omega_{P,\text{upper}} \) is finite and \( \omega_{P,\text{lower}} \) is a halfline for each \( P \in \mathbb{P} \); other situations are either symmetric or could be reduced to this setting.

The outer measure spaces on \( \mathbb{P} \) are defined as in Section 6.1. Below we discuss the outer measure spaces on \( \mathbb{Q} \), which are similar to the settings on \( \mathbb{P} \), thus we only discuss the needed changes. By further decomposition if necessary, we may assume that \( \mathbb{P} \) and \( \mathbb{Q} \) are very sparse.

For any two tiles \( R, R' \) we say that \( R < R' \) if \( I_R \subseteq I_{R'} \) and \( 5\omega_R \subset 5\omega_{R'} \).

We say that \( R \leq R' \) if \( R < R' \) or \( R = R' \). Clearly, \( \leq \) and \( \leq \) are transitive.

Generating subsets of \( \mathbb{Q} \): A nonempty \( E \subset \mathbb{Q} \) is a generating set if for some \( E \)-tile \( I_E \) (with the same rigidity) the following holds: for every \( Q \in E \) there is \( j = j(Q) \in \{1, 2, 3\} \) such that \( Q_j \subset Q_{E,j} \), where \( Q_j \) and \( Q_{E,j} \) are the \( j \)-tiles of \( Q \) and \( Q_E \). We denote \( I_E = I_{Q_E} \) and \( \xi_E = \epsilon(Q_E) \).

For any fixed \( j \in \{1, 2, 3\} \), we say that \( E \) is \( j \)-overlapping if \( j(Q) = j \) for every \( Q \in E \). We say that \( E \) is \( j \)-lacunary if it is \( j \) overlapping for some \( k \in \{1, 2, 3\} \setminus \{j\} \).

Outer measure on \( \mathbb{Q} \): Let \( \sigma \) be generated from \( \sigma(E) = \inf \sum_j |I_{E,j}| \), infimum taken over all countable coverings of \( E \) by generating sets.

Size: For every \( 0 < t \leq \infty \), we define \( S_t \) just as in the setting for \( \mathbb{P} \). If the defining supremum is taken over all \( j \)-lacunary \( S \), we obtain \( S_t^{[j]} \). Similarly, for \( S_t^{[j,j]} \) the supremum is taken over all \( S \) that is both \( j_1 \) and \( j_2 \) lacunary, or equivalently \( k \) overlapping where \( k \neq j_1, j_2 \).

Strongly disjointness: For any \( j \in \{1, 2, 3\} \), a collection \( (E_m) \) of \( j \)-lacunary sets of \( \mathbb{Q} \) is strongly disjoint if the following holds for any \( E_m, E_n, m \neq n \):

- If \( Q \in E_m \) and \( R \in E_n \) such that \( 3\omega_{Q_j} \cap 3\omega_{R_j} \neq \emptyset \) and \( |I_Q| \leq |I_R| \) then \( I_Q \cap I_{E_n} = \emptyset \). (In particular \( I_Q \cap I_R = \emptyset \).)

An analogue of Lemma 6.1 also holds in the current setting.

In the rest of this section, for every \( E \subset \mathbb{Q} \) let \( \mathbb{P}_E \) denote the set of all \( P \in \mathbb{P} \) such that there exists at least one \( Q \in E \) with \( |I_P| \leq |I_Q| \) and \( \frac{5}{6}\omega_P \cap \frac{5}{6}\omega_{3,Q} \neq \emptyset \). The following observation from [17] will be useful in the proof.

Observation 7.1. Let \( E \) be \( 3 \)-lacunary, then the following holds for every \( P \in \mathbb{P}_E \) and \( Q \in E \): if \( \frac{5}{6}\omega_P \cap \frac{5}{6}\omega_{3,Q} \neq \emptyset \) then \( |I_P| \leq |I_Q| \).

Proof. Assume the contrary, that is for some \( P \in \mathbb{P}_E \) and \( Q, R \in E \) it holds that \( \frac{5}{6}\omega_P \cap \frac{5}{6}\omega_{3,Q} \neq \emptyset \), \( \frac{5}{6}\omega_P \cap \frac{5}{6}\omega_{3,R} \neq \emptyset \), and \( |I_R| \geq 2|I_Q| \). It follows that \( |I_R| \geq 2|I_Q| \), therefore using sparseness of \( Q \) it is clear that \( 5\omega_{R_3} \subset 5\omega_{3,Q} \), which contradicts the fact that \( E \) is \( 3 \)-lacunary and \( Q \) is sparse. \( \Box \)

Now, fix a Schwarz function \( f_3 \) on \( \mathbb{R} \). Let \( a_j(Q) := \langle f_j, \phi_{j,Q} \rangle \) for \( j = 1, 2 \), and \( d(P) = \langle f_3, \tilde{\phi}_P \rangle \) where \( \tilde{\phi}_P := \sum_j \phi_{P,j}d_j \). Let \( K \) and its adjoint be defined by

\[
(Kf)(Q) := \sum_{P \in \mathbb{P}, |I_P| \leq |I_Q|} |I_P|f(P) \langle \phi_P, \phi_{3,Q} \rangle, \quad Q \in \mathbb{Q},
\]

\[
K^*f(P) := \sum_{Q \in \mathbb{Q}, |I_Q| \leq |I_P|} |I_Q|f(Q) \langle \phi_{3,Q}, \phi_P \rangle, \quad P \in \mathbb{P}.
\]
Now \( \langle T_B c(f_1, f_2), f_3 \rangle = \sum_{Q \in Q} |I_Q| a_1(Q) a_2(Q) (Kd)(Q) \), which will be estimated using outer measure techniques.

### 7.1. Outer \( L^p \) estimates for \( K \) and \( K^* \)

Realize that \( Q \) has rank 1. The following two Lemmas are the main estimates of this section.

**Lemma 7.2.** Let \( 1 < q \leq 2 \). For every \( p \in (q', \infty) \) we have

\[
|Kg|_{\ell^q(P, S_c^{[n], \sigma}, \sigma)} \lesssim |g|_{\ell^p(P, S_d, \mu)}.
\]

**Lemma 7.3.** Let \( q \in (1, 2] \). Then for any \( 1 < p < q \) we have

\[
|K^* f|_{\ell^p(P, S_q, \mu)} \lesssim |f|_{\ell^q(P, S_s^{[n], \sigma}, \sigma)}.
\]

We will deduce Lemma 7.2 from Lemma 7.3 using a simple duality argument. By interpolation it suffices to consider weak-type estimates. Fix \( \lambda > 0 \) and \( q' < p < \infty \). Without loss of generality assume \( |Kg|_{\ell^q(Q, S_{2}^{[n], \sigma})} \leq 2\lambda \). Similar to the proof of Theorem 6.1, we may select a strongly disjoint collection of 3-lacunary sets \( (E_m) \) and \( Q' \) containing all \( E_m \) such that \( |Kg|_{\ell^q(Q, Q', S_{2}^{[n], \sigma})} \leq \lambda \) and

\[
\lambda^2 \sigma(Q') \lesssim M := \sum_{m \in \mathbb{N}} \sum_{Q \in \mathcal{E}_m} |I_Q||Kg(Q)|^2.
\]

Now, let \( f(Q) = Kg(Q) \) for \( Q \in \mathbb{N}, E_m \) and zero elsewhere, we obtain via applications of the outer Radon–Nikodym/Holder inequalities

\[
M = \sum_{Q \in Q} |I_Q| f(Q)(Kg)(Q) = \sum_{P \in P} |I_P| (K^* f)(P) g(P)
\]

\[
\lesssim \|K^* f\|_{\ell^p(P, S_q, \mu)} |g|_{\ell^q(P, S_d, \mu)}.
\]

Therefore, by Lemma 7.3 we obtain

\[
M \lesssim \|f\|_{\ell^q(\mathcal{Q}, S_{2}^{[n], \sigma}, \sigma)} |g|_{\ell^p(P, S_d, \mu)}.
\]

Note that \( f \) is supported on \( Q'' := \bigcup E_m \). It is clear that any 3-overlapping subset of \( Q'' \) is essentially an union of spatially disjoint tritiles. Therefore \( S_1^{[n]}(f) \lesssim S_{\times}(f) \lesssim S_{3}^{[n]}(f) \lesssim \lambda \). We obtain

\[
M \lesssim \lambda \sigma(Q') \|f\|_{\ell^p(P, S_d, \mu)}.
\]

Collecting estimates we obtain the desired weak-type estimate

\[
\lambda \sigma(Q') \|f\|_{\ell^p(P, S_d, \mu)} \lesssim |g|_{\ell^q(P, S_d, \mu)}.
\]

In the rest of the section, we prove Lemma 7.3.

**Proof of Lemma 7.3.** By interpolation, it suffices to prove weak-type estimates for any fixed \( p \in [1, q) \). Without loss of generality, assume \( |f|_{\ell^p(Q, S_{2}^{[n], \sigma}, \sigma)} = 1 \).

Fix any \( \lambda > 0 \). We’ll show that there exists \( P' \subset P \) such that \( \mu(P') \lesssim \lambda^p \) and \( |K^* f|_{\ell^q(P, P', S_d, \mu)} \leq \lambda \). Without loss of generality we may assume that

\[
|K^* f|_{\ell^q(P, P', S_d, \mu)} < 2\lambda. \tag{21}
\]

For more details see the proof of the \( L^2 \) case of Lemma 7.4.
The construction of $P'$ is similar to the proof of Theorem 6.1 and we also obtain a strongly disjoint collection of lacunary sets $(E_m)_{m \geq 1}$ contained inside $P'$ with the following property:

$$
\lambda^q \mu(P') \leq \lambda^q \sum_m |I_{E_m}| \leq \sum_{P \in \bigcup E_m} |I_P| |(K^* f)(P)|^q .
$$

Let $h(P) := K^* f(P) |K^* f(P)|^{q-2}$ for $P \in P'$ and zero elsewhere, and let $M$ be the last right hand side, which could be rewritten as

$$
M = \sum_{P \in P} |I_P| (K^* f)(P) h(P) = \sum_{Q \in Q} |I_Q| f(Q) \overline{K(h)(Q)} .
$$

By the classical Holder inequality we have

$$
S_1(fg) \leq S_2^[\alpha] (f) S_2^[\alpha] (g) + S_1^{[1,2]} (f) S_2^{[1,2]} (g) \leq (S_2^[\alpha] + S_1^{[1,2]}) (f) S_2^{[1,2]} (g) .
$$

Using outer Radon–Nikodym, outer Holder, the normalization $|f|_{L^p} = 1$, we have

$$
M \leq |f| K h \|_{L^1(Q,S_{[\alpha]},\sigma)} \leq |K h|_{L^{p'}(Q,S_2^{[\alpha]},\sigma)} .
$$

Using Lemma 7.4 and (21) and the definition for $h$, it follows that

$$
|K h|_{L^{p'}(Q,S_2^{[\alpha]},\sigma)} \leq \lambda^{q/p} \left( \sum_{k} |I_{E_k}| \right)^{1/p} \leq \lambda^{q/p-q/p'} M^{1/p'} .
$$

Collecting estimates we obtain $M^{1/p} \leq \lambda^{q/p-q/p'}$, therefore

$$
\mu(P')^{1/p} \leq \lambda^{-q/p} M^{1/p} \leq \lambda^{-1} . \quad \square
$$

**Lemma 7.4.** Let $q \in (1, 2]$ and $(E_k)$ be strongly disjoint lacunary in $P$. Then for $q_1 \in (q', \infty]$ and every $h : P \to \mathbb{C}$ that vanishes outside $P' := \bigcup E_k$ it holds that

$$
|K h|_{L^{q_1}(Q,S_2^{[\alpha]},\sigma)} \leq |h|_{L^{q_1}(P_1,S_2^{[\alpha],\mu_1})(\sum_{k} |I_{E_k}|)^{1/q_1}}
$$

and weak type estimates hold at $q_1 = q'$.

Remark: While Lemma 7.4 is weaker than Lemma 7.2 it will be directly proved as part of the proof of Lemma 7.3.

**Proof.** It suffices to prove the following stronger estimate, which holds for $q_1 > 2$:

$$
(22) \quad |K h|_{L^{q_1}(Q,S_2^{[\alpha]},\sigma)} \leq \left( \sum_{P \in P} |I_P| |h(P)|^{q_1} \right)^{1/q_1} + \sup_{P \in P} |h(P)| \left( \sum_{k} |I_{E_k}| \right)^{1/q_1} .
$$

Clearly, if $q_1 \geq q'$ then the desired conclusion follows from (22).

By interpolation, it suffices to prove weak-type estimates at $q_1 = 2$ and $q_1 = \infty$. Since the right hand side of (22) is not technically an $L^p$ norm, we will detail the interpolation argument. Assume that the weak type estimates hold at $q_1 = 2$ and $q_1 = \infty$. Let $2 < q_1 < \infty$, for each $\lambda > 0$ we decompose $h(P) = h_1(P) + h_2(P)$ where $h_1(P) = h(P) \mathbf{1}_{|h(P)| \leq \lambda C}$ and $C$ is sufficiently large but about the size of the norm for the assumed case $q_1 = \infty$ of (22). Using sublinearity of $K$, it follows that

$$
\sigma(S_2^{[\alpha]}(K h) > \lambda) \leq \sigma(S_2^{[\alpha]}(Th_2) > \lambda) .
$$
By the assumed $L^2$ case of (22), the last display is bounded above by

$$
\lambda^{-2} \left( \sum_{P \in \mathcal{P}} |I_P| \|h(P)\|^2 1_{\lambda < C|I_P|} + \sup_{P} |h(P)|^2 1_{\lambda < C|I_P|} \right) \left( \sum_{k} |E_k| \right)
$$

$$
= \lambda^{-2} \left( \sum_{P \in \mathcal{P}} |I_P| \|h(P)\|^2 1_{\lambda < C|I_P|} + \sup_{P} |h(P)|^2 1_{\lambda < C|I_P|} \right) \left( \sum_{k} |E_k| \right)
$$

here in the second term we are able to move the sum inside because for any $a \geq 0$, $g(x) := x \chi_{a<x}$ is increasing for $x \in [0, \infty)$. Now, multiplying both side with $\lambda^{\xi - 1}$ and integrate over $\lambda \in (0, \infty)$ we will obtain the desired estimate. This completes the interpolation argument.

**Case 1:** $g_1 = \infty$. By a standard characterization for $S_2$ (see e.g. [17, Lemma 6.4]) it suffices to show that if $E \subset Q$ is $3$-lacunary then

$$
\frac{1}{|I|} \left( \sum_{Q \in E} |Kh(Q)|^2 1_{|Q|} \right)^{1/2} \lesssim \sup_{P \in \mathcal{P}} |h(P)|.
$$

Since $Q$ is very sparse, it is clear that for any interval $I$ there is at most one $Q \in E$ such that $I_Q = I$. This remark will be used implicitly below.

Let $P_E$ be defined as in Observation 7.1 and let $P_E' = P' \cap P_E$. It follows from the following pointwise estimate on $E$

$$
\left( \sum_{Q \in E} |Kh(Q)|^2 1_{|Q|} \right)^{1/2} \lesssim \frac{1}{|I|} \int \tilde{h}_I(x)^N |h_E(x)| dx.
$$

Decompose $P_E' = P_{E,1} \cup P_{E,2}$ where $P_{E,1} = \{ P \in P_E : I_P \cap 3I \neq \emptyset \}$. It suffices to show that for every interval $I$ of the same length as $I_E$ it holds for $j = 1, 2$

$$
A_j := \frac{1}{|I|} \int \sum_{P \in P_{E,j}} |I_P| |h(P)\phi_P(x)| dx \lesssim \sup_{P} |h(P)|.
$$

For $A_1$, notice first that for every $P \in P_{E,1}$ we have $I_P \subset 5I$. Since $\phi_P$ is $L^1$-normalized and $\{ I_P, P \in P_{E,1} \}$ are disjoint, we obtain

$$
A_1 \lesssim \frac{1}{|I|} \sum_{P \in P_{E,1}} |I_P| |h(P)| \lesssim \sup_{P \in \mathcal{P}} |h(P)|.
$$

To estimate $A_2$, we decompose the summation over $P_{E,2}$ according to the length of $I_P$. Recall that $\{ I_P : P \in P_{E,2} \}$ are disjoint and disjoint from $3I$. The desired estimate for $A_2$ follows from the following pointwise estimate on $I$:

$$
\sum_{P \in P_{E,2}} |I_P| |\phi_{P_I}(x)| \lesssim \sum_{n} \sum_{P : |I_P| = 2^n} \left( \frac{|I_P|}{|I|} \right)^4 \lesssim 1.
$$

**Case 2:** $q_1 = 2$. Fix $\lambda > 0$. We need to show existence of $Q' \subset Q$ such that

$$
\lambda^2 \sigma(Q') \lesssim N := \sum_{P \in \mathcal{P}} |I_P| |h(P)|^2 + \sup_{P \in \mathcal{P}} |h(P)|^2 \sum_{k} |E_k|
$$

and $|Kh|_{L^\infty(Q,Q')} \lesssim \lambda$. Without loss of generality, assume that $|Kh|_{L^\infty(Q)} \leq 2\lambda.$
Now, similar to Observation 7.1, we may write
the following algorithm:

\[ (26) \]

Now, to show (24) for \( Q \) finite the selection argument will stop. By geometry, \( (G_j) \) is strongly disjoint. Let \( Q'_1 \) be the set of all tritiles removed from \( Q \).

We similarly collect \( S_2 \) a collection of strongly disjoint 2-lacunary sets, the difference is we maximize \( c(\omega_{Q_k}) \) in step (i). Without loss of generality, assume that \( \mu(Q') \leq \mu(Q'_1) \), which we will estimate below.

Let \( k(Q) = Kh(Q) \) for \( Q \in Q'_1 \) and zero elsewhere, clearly

\[
\lambda^2 \mu(Q') \leq \lambda^2 \mu(Q'_1) \leq \lambda^2 \sum_j |I_{G_j}| \leq M,
\]

\[
M := \sum_{Q \in Q'_1} |I_Q||k(Q)|^2 = \sum_{P \in P^*} \sum_{Q \in Q'_1} 1_{|I_P| \leq |I_Q|}|I_P||I_Q||k(Q)|h(P) \langle \phi_P, \phi_{3,Q} \rangle.
\]

Now, to show (24) for \( Q'_1 \) it suffices to show that

\[ (25) \]

We first estimate the analogous double sum \( M_{\text{free}} \) where we don’t include the coupling condition \( |I_P| \leq |I_Q| \). By Cauchy–Schwarz and Lemma 6.1 and the assumptions \( |Kh|_{L^2(Q_2^2, \mu)} \leq \lambda \), it follows that

\[
M_{\text{free}} \leq \left( \sum_{P \in P^*} |I_P||h(P)\phi_P|^2 \right) \sum_{Q \in Q'_1} |I_Q||k(Q)|^2 \|\phi_{3,Q}\|_2 \leq N^{1/2}M^{1/2}.
\]

We now consider the diagonal sum \( M_{\text{diag}} \) where \( |I_P| = |I_Q| \). We may further assume that \( |I_P| + \text{dist}(I_Q, I_P) \sim 2^n |I_Q| \) provided that there is an extra decaying factor in the estimate. Note that in order for \( \langle \phi_P, \phi_{3,Q} \rangle \) to be nonzero the frequency support of \( \phi_P \) and \( \phi_{3,Q} \) must overlap. Thus essentially \( Q \) is determined from \( P \) and vice versa. Using \( |\langle \phi_P, \phi_{3,Q} \rangle| \leq 2^{-n} |I_Q|^{-1/2} |I_Q|^{-1/2} \), and Cauchy–Schwarz,

\[
M_{\text{diag}} \leq 2^{-n} \left( \sum_{P \in P^*} |I_P||h(P)|^2 \right)^{1/2} \left( \sum_{Q \in Q'_1} |I_Q||k(Q)|^2 \right)^{1/2} \leq 2^{-n}N^{1/2}M^{1/2}.
\]

Thus, to prove (25) the roles of \( P^* \) and \( Q'_1 \) are fairly symmetric. We will assume below that \( \sum_m |G_m| \leq \sum_k |G_k| \), the proof for the other case is entirely similar. Using Cauchy–Schwarz, it suffices to show that for every \( G \in \{G_1, G_2, \ldots \} \) we have

\[
M_G := \sum_{Q \in G} \sum_{P \in P^*} 1_{|I_P| \leq |I_Q|}|I_P||I_Q||h(P)||k(Q)||\phi_P, \phi_{3,Q}|| \langle \phi_P, \phi_{3,Q} \rangle \leq \sup_P |h(P)| \left( \sum_{Q \in G} |k(Q)|^2 |I_Q| \right)^{1/2} |I_G|^{1/2} + \sup_Q |k(Q)||I_G| \right) \cdot
\]

Now, similar to Observation 7.1, we may write

\[
M_G = \sum_{Q \in G} |I_Q||k(Q)||h_{G, \phi_{3,Q}}||, \quad h_G := \sum_{P \in P_G} h(P) |I_P| \phi_P.
\]
and $P_G'$ contains all $P \in P'$ such that for some $Q \in G$ we have \( \frac{5}{6} \omega_P \cap \frac{5}{6} \omega_3.Q \neq \emptyset \) and $|I_P| \leq |I_Q|$. Using strong disjointness, it is clear that the intervals of the elements of $P_G'$ are essentially pairwise disjoint.

We now estimate the contribution of those $P \in P_G'$ such that $I_P \cap 4I_G \neq \emptyset$. Clearly we will have $I_P \subseteq 6I_G$. Let $h_{E,0}$ be the corresponding subsum of $h_G$. By Cauchy Schwarz and standard Calderon–Zygmund theory, we have

$$\sum_{Q \in G} |I_Q| k(Q) \langle h_{G,0}, \varphi_{3, Q} \rangle \leq \left( \sum_{Q \in G} |I_Q| k(Q) \right)^{1/2} \left( \sum_{P \in P_G', I_P \subseteq 6I_G} |I_P| |h(P)| \right)^{1/2}$$

$$\leq \left( \sum_{Q \in G} |I_Q| k(Q) \right)^{1/2} |I_G|^{1/2} \sup_P |h(P)| ,$$

in the last estimate we used disjointness of the intervals of elements of $P_G'$.

Consider the contribution of other $P$'s. Let $P_{G,k}$ contains all $P \in P_G'$ such that $I_P \cap 2^{k+2}I_G \neq \emptyset$ but $I_P \cap 2^{k+1}I_G = \emptyset$. Let $\Lambda := \sup_Q |k(Q)| \sup_P |h(P)|$. It suffices to show that for any $Q \in G$ we have

$$\sum_{P \in P_{G,k} : |I_P| \leq |I_Q|} |I_Q| |I_P| k(Q) h(P) \langle \varphi_P, \varphi_{3, Q} \rangle \leq 2^{-k} \Lambda \left( \frac{|I_Q|}{|I_G|} \right)^2 |I_G|$$

We decompose $\varphi_{3, Q} = \varphi_{3, Q}1_{2^{k+2}I_G} + \varphi_{3, Q}(1 - 1_{2^{k+2}I_G})$. Since $k(Q) h(Q) \leq \Lambda$ and since $I_P$'s are essentially pairwise disjoint and contained in $2^{k+3}I_G \setminus 2^k I_G$, the left hand side of the above display is bounded above by

$$\leq \Lambda \sum_{P} |I_P| |I_Q| \left( |\varphi_{1, 2^k I_G}1_{|\varphi_{3, Q}| \infty} + |\varphi_P1_{|\varphi_{3, Q}1_{2^{k+2}I_G} \infty}| \right)$$

$$\leq 2^{-2k} \Lambda \sum_{P : |I_P| \leq |I_Q|} |I_P| |I_Q| \left( \left( \frac{|I_P|}{|I_G|} \right)^2 + \frac{1}{|I_Q|} \left( \frac{|I_P|}{2^k |I_G|} \right)^2 \right)$$

$$\leq 2^{-2k} \Lambda \left( \frac{|I_Q|}{|I_G|} \right)^2 \sum_{P : |I_P| \leq 2^{k+3}I_G} |I_P| \leq 2^{-k} \Lambda \left( \frac{|I_Q|}{|I_G|} \right)^2 |I_G| . \quad \Box$$

7.2. Outer $L^\infty$ estimate for $Kd$. When $d(P) = \langle f_3, \tilde{\varphi}_P \rangle$, we have

Lemma 7.5. For any $\epsilon > 0$ we gave

$$|Kd|_{L^\infty(E;Q,S_{1,2}^{[3, \sigma]})} \leq \epsilon, N \sup_{Q \subseteq Q} \frac{1}{|I_Q|} \int \tilde{\chi}^N_{I_Q} |f_3|^{1+\epsilon})^{1/(1+\epsilon)} .$$

Proof. Let $E \subseteq Q$ be 3-lacunary and define $P_E$ as in Observation 7.1. It follows that $Kd(Q) = \langle T_{P_E}^* f_3, \varphi_{3, Q} \rangle$ where $T_{P_E}^*$ is the adjoint of

$$T_{P_E} f := \sum_j \sum_{P \in P_E} |I_P| \langle f, \varphi_P \rangle \tilde{\varphi}_{P,j} \delta_j .$$

Therefore, similar to the $L^{2, \infty}$ case of Lemma 7.3, it suffices to show that

$$\frac{1}{|I|} \int_I |(T_{P_E}^* f_3)(x)| dx \leq \left( \frac{1}{|I|} \int \tilde{\chi}^N_{I_Q} |f_3|^{1+\epsilon})^{1/(1+\epsilon)} ,$$

for every interval $I$ of the same length as $I_E$. 

Now, it is clear that $P_E$ is an overlapping generating subset of $P$. It follows that for any $x$ only one $j$ and one scale of $P_E$ would contribute to the defining summation of $T_{P_E} f$. Thus, $T_{P_E} f$ is controlled by the maximal function of $\sum_{P \in P_E} |I_P| \langle f, \phi_P \rangle \phi_P$ and so by Calderon–Zygmund theory, $T_{P_E}$ is bounded on $L^p(\mathbb{R})$ for any $1 < p < \infty$. By duality $T_{P_E}^*$ is also bounded on $L^p(\mathbb{R})$. Furthermore, we also have

$$
(27) \quad \langle T_{P_E}^* g, h \rangle \lesssim \sum_{P \in P_E} \frac{1}{|I_P|} \left\langle |g|, \hat{\chi}_P^N \right\rangle \left\langle \hat{\chi}_{I_P}(x)^N, |h| \right\rangle, \quad N > 0,
$$

from there we obtain a pointwise estimate for $T_{P_E}^* g(x)$ which holds for a.e. $x$.

Decompose $f_3 = f_3 1_{7I} + f_3 1_{(7I)^c}$. By boundedness of $T_{P_E}^*$ and Holder inequality, the contribution of $f_3 1_{7I}$ could be easily controlled. For the contribution of $f_3 1_{(7I)^c}$, let $P_{E,1} = \{ P \in P_E : I_P \subset 5I \}$, and $P_{E,2} = P_E \setminus P_{E,1}$. In $P_{E,1}$ clearly $\hat{\chi}_{I_P} \lesssim \hat{\chi}_I$, thus using the resulting pointwise estimate resulting from (27) we obtain

$$
T_{P_{E,1}}(f_3 1_{(7I)^c})(x) \lesssim \sum_{P \in P_{E,1}} |I_P|^{-1} \left\langle |f_3|, \left(\frac{|I_P|}{|I|}\right)^2 \chi_I^N \right\rangle \hat{\chi}_{I_P}(x)^2 \lesssim \frac{1}{|I|} \left\langle |f_3|, \hat{\chi}_I^N \right\rangle.
$$

For $P_{E,2}$ it is clear that $\hat{\chi}_{I_P}(x) \hat{\chi}_{I_P}(y) \lesssim (|x - y|/|I_P|)^{-2} \lesssim \hat{\chi}_I(y)$ for every $x \in I$ and $y \in (7I)^c$. It follows that

$$
|T_{P_{E,2}} f_3|_{L^1(I)} \lesssim \int |f_3(y)| \hat{\chi}_I(y)^N \sum_P \frac{1}{|I_P|} \hat{\chi}_{I_P}(y)^2 \left(\frac{|I_P|}{|I|}\right)^{-2} |\hat{\chi}_{I_P}|_{L^1(I)} \, dy \lesssim \int |f_3(y)| \hat{\chi}_I(y)^N \sum_P \hat{\chi}_{I_P}(y)^2 \left(\frac{|I_P|}{|I|}\right)^2 \, dy \lesssim \int |f_3(y)| \hat{\chi}_I(y)^N \, dy. \quad \square
$$

### 7.3. Proof of Theorem 7.1

#### 7.3.1. The basic range. Let $q = \min(2, r/2) = r/2 \in (1, 2)$. Let $a_3 = \frac{Kd}{q}$. Assume $q_1, q_2, q_3 \geq 1$ such that $\sum 1/q_j = 1$. By classical Holder, we obtain

$$
S_1(a_1 a_2 a_3) \lesssim S_2^{[1]}(a_1) S_2^{[q]}(a_2) S_2^{[q]}(a_3).
$$

Using outer Radon-Nikodym/Holder inequalities, it follows that

$$
(28) \quad \Lambda_{P,Q} \lesssim \prod_{j=1,2,3} |a_j|_{L^q(Q) S_2^{[q]}(a_j)},
$$

provided that $\sum 1/q_j = 1$ and $q_3 > q'$. If $q_1, q_2 > 2$ then via Lemma 7.2 and Carleson embeddings (Theorem 6.1, Theorem 6.3) we obtain $\Lambda_{P,Q} \lesssim \prod_{j} |f_j|_{q_j}$.

#### 7.3.2. Extending the range. To extend the range, we’ll use restricted weak-type interpolation, following [19]. Let $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ be such that $\sum \alpha_j = 1$ and $\alpha_j \geq 0$. Then we say that a trilinear form $\Lambda(f_1, f_2, f_3)$ satisfies restricted weak-type estimates with exponents $\alpha$ if the following holds: there exists $j_0 \in \{1, 2, 3\}$ such that for every $F_1, F_2, F_3 \subset \mathbb{R}$ finite Lebesgue measures we could find $B \subset F_{j_0}$ with less than half of the measure, so that $|\Lambda_{P,Q}(f_1, f_2, f_3)| \lesssim |F_1|^{\alpha_1} |F_2|^{\alpha_2} |F_3|^{\alpha_3}$ whenever $|f_j| \leq 1_{F_j}$ for every $j$ and furthermore $|f_{j_0}| \leq 1_{F_{j_0} - B}$. When exactly one
of the index is negative, say $\alpha_k < 0$, we say that $\Lambda(f_1, f_2, f_3)$ satisfies restricted weak-type estimates with exponents $\alpha$ if the previous claim holds with $j_0 = k$.

From [19], if $T(f_1, f_2)$ is a bilinear operator such that $\langle T(f_1, f_2), f_3 \rangle$ satisfies restricted weak-type estimates for a finite collection $V$ of triples then

$$|T(f_1, f_2)|_{p_1} \lesssim |f_1|_{L^{p_1}(\mathbb{R})} |f_2|_{L^{p_2}(\mathbb{R})}$$

for any triples of exponents $0 < p_1, p_2, p_3 \leq \infty$ such that $(1/p_1, 1/p_2, 1/p_3)$ is in the interior of the convex hull of $V$.

Thus, to show Theorem 7.1 it suffices to show that $\Lambda_{p, q}(f_1, f_2, f_3)$ satisfies Let $H \subset \{\alpha_1 + \alpha_2 + \alpha_3 = 1\}$ with vertices:

$$H_1 \left(\frac{1}{2}, -\frac{1}{2}, 1\right), \ H_2 \left(-\frac{1}{2}, \frac{1}{2}, 1\right), \ H_3 \left(\frac{1}{2} - \frac{q'}{q}, \frac{1}{2} + \frac{1}{q'}, \frac{1}{q'}\right), \ H_4 \left(\frac{1}{2} + \frac{1}{q'}, \frac{1}{2} - \frac{1}{q'}, \frac{1}{q'}\right).$$

Our proof below will be fairly symmetric across the vertices, so we will show the claim only for $H_2$ and $H_3$. For any $\epsilon > 0$ and $q_1, q_2 > 2$, $q_3 \geq q' + \epsilon$, it follows from [28], Carleson embeddings, and convexity that

$$\Lambda_{p, q} \lesssim |a_1|_{\infty} \sup_{Q \in \mathbb{Q}} \frac{1}{Q^2} \int \hat{\chi}_{L^2}(x)^N |f_2(x)| dx.$$  

By Lemma 6.3 for $j = 1, 2$ we have

$$|a_j|_{L^{\infty}(Q, S_{2j}^{\beta}, \sigma)} \lesssim_N \sup_{Q \in \mathbb{Q}} \frac{1}{Q^2} \int \hat{\chi}_{L^2}(x)^N |f_2(x)|^{1/\sigma} dx.$$  

For $|a_3|_{\infty}$ we’ll use Lemma 7.5 and obtain

$$|a_3|_{L^{\infty}(Q, S_{2}^{\eta}, \sigma)} \lesssim_N \sup_{Q \in \mathbb{Q}} \frac{1}{Q^2} \int \hat{\chi}_{L^2}(x)^N |f_3(x)|^{1/\sigma} dx.$$  

We now consider neighborhood of $H_1$ and $H_2$. Near these vertices we have $\alpha_1 < 0$, so we may choose $G_1 = F_1 \setminus B$ where $B = \bigcup_{j=2}^3 M(1, F_j) > C|F_2|/|F_1|$ and $C$ is large enough to ensure $|B| < |F_1|/2$.

Without loss of generality assume that $2^m \leq 1 + \text{dist}(I_Q, B) + |I_Q| < 2^{m+1}$ for some $m \geq 0$ integer, provided that we could obtain extra decaying factors.

From (30) we obtain $|a_2|_{L^{\infty}} \leq 2^m \sup_{Q \in \mathbb{Q}} |M_1 F_2(x)| \leq 2^m |F_2|/|F_1|$ and similarly using (31) we have $|a_3|_{L^{\infty}} \lesssim (2^m |F_2|/|F_1|)^{1/(1+\epsilon)}$. It follows from (29) that

$$\Lambda_{p, q} \lesssim 2^{-m} |F_1|^{\alpha_1} |F_2|^{\alpha_2} |F_3|^{\alpha_3}$$

$$\alpha_1 = \frac{1}{q_1} + \frac{2}{q_2} + \frac{q' + \epsilon}{q_3(1 + \epsilon)} - 1 - \frac{1}{1 + \epsilon}, \ \alpha_2 = 1 - \frac{1}{q_2}, \ \alpha_3 = 1 - \frac{1}{1 + \epsilon} - \frac{q' + \epsilon}{q_3(1 + \epsilon)} + \frac{1}{q_3}.$$

Thus, letting $(\epsilon, q_1, q_3)$ close to $(0, 2, q')$, we can make $\alpha$ arbitrarily close to $H_3$. Similarly, letting $(\epsilon, q_1, q_3)$ close to $(0, 2, \infty)$ we can make $\alpha$ arbitrarily close to $H_2$. This completes the proof of Theorem 7.1.
8. Estimates for CC model operators

**Theorem 8.1.** Let $T$ be a CC model operator. Then $|T|_{L^{p_1}_1 \times L^{p_2}_2 \to L^{p_3}_3} < \infty$ for every $1/q = 1/q_1 + 1/q_2$ such that $\frac{2r}{3r-4} < q_1, q_2 < \infty$ and $q_3 > \frac{r}{2}$.

For simplicity, we’ll assume that $\omega_{j,P,\text{upper}}$ are finite intervals and $\omega_{j,P,\text{lower}}$ are halflines for $j = 1, 2$; the other cases are either symmetric or could be reduced to this situation. We will use the same set up for outer measures space in Section 6.1 for both $P_1$ and $P_2$. For convenience of notation, let $\mu_1$ and $\mu_2$ be the corresponding outer measures.

For every $x$, $P \in P_1$, and $P' \in P_2$, there is at most one $1 \leq j \leq K(x)$ such that

$$N_{j-1} \in \omega_{1,P,\text{lower}} \cap \omega_{2,P',\text{lower}}, \quad N_j \in \omega_{1,P,\text{upper}} \cap \omega_{2,P',\text{upper}}.$$

Let $d_{P,P'}(x) = d_j(x)$ if such $j$ exists, and zero otherwise; define $d_P$ and $d_{P'}$ similarly.

Fix $f_3$ Schwartz on $\mathbb{R}$ and let $a_1(P) = \langle f_1, \phi_1, P \rangle$ and $a_2(P') = \langle f_2, \phi_2, P' \rangle$, and

$$Kf(P) := \sum_{P' \in P_2: |I_P| \geq 16 |I_{P'}|} |I_P|f(P') \langle \phi_{1,P,\phi_{2,P',d_{P',P}}, f_3} \rangle$$

$$K^*g(P') := \sum_{P \in P_1: |I_P| \geq 16 |I_{P'}|} |I_P|g(P) \langle \phi_{1,P,\phi_{2,P',d_{P',P}}, f_3} \rangle.$$

Clearly, $\langle T(f_1, f_2), f_3 \rangle = \sum_{P \in P_1} |I_P|a_1(P)(Ka_2)(P)$. To prove Theorem 8.1, we first establish outer $L^p$ estimates for $K$.

For convenience of notation, assume that $f_3$ is supported on a fixed set $F_3$. Let $b_P(x) := f_3 d_{j,P,\text{overlap}} f$ if there exists (a unique) $j$ such that $N_{j-1} \in \omega_{1,P,\text{lower}}$ and $N_j \in \omega_{1,P,\text{upper}}$, and let $b_P(x) = 0$ otherwise. Also, define

$$M_N(F_3) := M_N(P_1, 1 F_3)^{1/p_1} M_N(P_2, 1 F_3)^{1/p_2}.$$(Recall the definition of $M_N$ from (14).)

8.1. **Outer $L^p$ estimates for $K$.** The following Lemma is the main estimate of the current section, and we’ll always assume that $1/p_1 + 1/p_2 + 1/p_3 = 1$ and $2 < p_1, p_2 < 2r/(4 - r)$, and $\infty > p_3 > r/(r - 2)$. All implicit constants may depend on these exponents.

**Lemma 8.2.** It holds that

$$|Kf|_{L^{p_3}_3(P_1, S_{1,\text{overlap}} + S_{2,\text{lac}, \mu_1})} \lesssim_M M_N(F_3) |f|_{L^{p_2}_2(P_2, S_{2,\text{lac}, \mu_2})} f_3 |_{p_3}.$$

The proof of Lemma 8.2 consists of two parts. The factor $M_N(F_3)$ should be thought of an estimate for the norm of $K : L^{p_2}_2(P_2) \times L^{p_3}_3(F_3) \rightarrow L^{p_1}_1(P_1)$, capturing the interaction of $\text{supp}(f)$ (i.e. $P_2$), $\text{supp}(Kf)$ (i.e. $P_1$) and $\text{supp}(f_3)$. Thus, by interpolation (Lemma 5.2) it suffices to show the weak-type estimates (note that the constant $M_N(F_3)$ could be naturally absorbed inside the (outer)measures on the right hand side), and we will treat the contribution of $S_{1,\text{overlap}}$ in Section 8.1.1 and the contribution of $S_{2,\text{lac}}$ in Section 8.1.2.

We first fix some notations. For each $j$ and any $f : P_2 \rightarrow \mathbb{C}$ and $\alpha > 0$ let

$$T_{j,\alpha}(f) = \sum_{P \in P_2: |I_P| \leq \alpha} |I_P|f(P') \phi_{2,P',j}, \quad T_j^*f = \sup_{\alpha} |T_{j,\alpha}f|.$$

For every $E \subset P_1$ let $P_3(E)$ contains all $P' \in P_2$ such that for some $P \in E$ we have $|I_P| \leq 1/16$ and $\frac{2}{3} \omega_{1,P,\text{upper}} \cap \frac{2}{3} \omega_{2,P',\text{upper}} \neq \emptyset$.
8.1.1. The overlapping setting. In this section we prove that
\[ |Kf|_{L^{p_1}(\mathbb{P}_1, S_{1, \text{overlap}, \mu_1})} \leq N \quad M_N(\mathbb{P}_1, 1, F_3)^{1/p_1} \left| f \right|_{L^{p_2}(\mathbb{P}_2, S_{2, \text{lac}, \mu_2})} \left| f \right|_{L^{p_3}(\mathbb{R})} \cdot \]
Using Hölder inequality, it follows from Lemma 6.3 that
\[ |Kf|_{L^p(\mathbb{P}_1, S_{1, \text{overlap}, \mu_1})} \leq N \quad M_N(\mathbb{P}_1, 1, F_3)^{1/p_1} \sup_{\mathbb{P} \in \mathbb{P}_1} \left| \int \chi_{N}^{\mathbb{P}} \sum_{j: \mathbb{N} \in \mathbb{P}} |f d j T_j^s(f)|^{p_1} \right|^{1/p_1} \]
Therefore, it follows from Lemma 6.6 that
\[ |Kf|_{L^{p_1, x}(\mathbb{P}_1, S_{1, \text{overlap}, \mu_1})} \leq N \quad M_N(\mathbb{P}_1, 1, F_3)^{1/p_1} \left| f \right|_{p_3} \left( \sum_{1 \leq j \leq K} |T_j^s(f)|^s \right)_{L^{p_2}(F_3)}, \]
so using \( \sum j |d_j|^r = O(1) \) it follows that
\[ |Kf|_{L^{p_1, x}(\mathbb{P}_1, S_{1, \text{overlap}, \mu_1})} \leq N \quad M_N(\mathbb{P}_1, 1, F_3)^{1/p_1} \left| f \right|_{p_3} \left( \sum_{1 \leq j \leq K} |T_j^s(f)|^s \right)_{L^{p_2}(F_3)}, \]
where \( s > 0 \) such that \( 1/p_1^r \leq 1/s + (r - 2)/r \). Since \( 1/p_1^r = 1/p_2 + 1/p_3 < 1/p_2 + (r - 2)/r \), we could choose \( s \) such that \( s > p_2 \) (which is larger than 2). The desired estimate now follows from Theorem 6.9.

**Lemma 8.3.** Uniform over \( \mathbb{P} \subset \mathbb{P}_1 \) it holds that
\[ |K(f)|_{L^p(\mathbb{P}, S_{1, \text{overlap}, \mu_1})} \leq N \quad m_{\mathbb{P}, N}(\mathbb{P}, (f d j T_j^s(f))) \cdot \]
**Proof.** This follows from a simple adaptation of the proof of Lemma 6.3 and the fact that for every \( j \) and every \( P \) we have \( \left| T_j^s(f) \right|_{L^{p_2}(\mathbb{R})} \leq T_j^s(f) \). \( \square \)

8.1.2. The lacunary setting. In this section we prove that
\[ |Kf|_{L^{p_1, x}(\mathbb{P}_1, S_{2, \text{lac}, \mu_1})} \leq N \quad M_N(\mathbb{P}_1, 1, F_3)^{1/p_1} \left| f \right|_{L^{p_2}(\mathbb{P}_2, S_{2, \text{lac}, \mu_2})} \left| f \right|_{L^{p_3}(\mathbb{R})} \cdot \]
Fix \( \lambda > 0 \). Without loss of generality assume that \( |Kf|_{L^p(\mathbb{P}, S_{2, \text{lac}, \mu_1})} \leq 2\lambda \).

Apply a variant of the selection argument in the proof of Theorem 6.1 we may find \( A \subset \mathbb{P}_1 \) such that \( |Kf|_{L^p(\mathbb{P}, S_{2, \text{lac}, \mu_1})} \leq \lambda \) and a strongly disjoint collection of generating subsets \( (E_m) \) covering \( A \) such that
\[ \lambda^2 \mu_1(A) \leq \lambda^2 \sum m |E_m| \leq M := \sum_{P \in E_m} |P| |Kf(P)|^2. \]
Let \( g(P) = \overline{Kf(P)} \) for \( P \in \mathbb{Q}_1 := \bigcup E_m \) and \( g(P) = 0 \) otherwise. We obtain
\[ M = \sum_{P \in \mathbb{Q}_1} |P| |Kf(P)g(P)| = \sum_{P \in \mathbb{P}_2} |P| |f(P')(K^s g)(P')|, \]
Using Lemma 8.4 and the assumption that \( |Kf|_{L^p(\mathbb{Q}_1, S_{2, \text{lac}, \mu_1})} \leq 2\lambda \) we obtain
\[ |K^s g|_{L^{p_1, x}(\mathbb{P}_2, S_{2, \text{lac}, \mu_2})} \leq N \quad M_N(F_3) \lambda \left( \sum m |E_m| \right)^{1/p_1} |f_3|_{p_3} \cdot \]
Using outer Radon-Nikodym/Holder, it follows that
\[ M \leq |f|_{L^{p_2}(\mathbb{P}_2, S_{2, \text{lac}, \mu_2})} |K^s g|_{L^{p_1}(\mathbb{P}_2, S_{2, \text{lac}, \mu_2})} \leq M_N(F_3) \lambda |f|_{L^{p_2}(\mathbb{P}_2, S_{2, \text{lac}, \mu_2})} |f_3|_{p_3} \lambda^{-2} M^{1/p_1}, \]
from this the desired estimates for \( M \) (and hence for \( \mu_1(A) \)) easily follow.
Lemma 8.4. It holds that (with $|g| \infty = |g| \mathcal{L}^\infty(Q, S_{2, lac})$)

$$
|K^*g|_{\mathcal{L}^{p_2}(P, S_{1, overlap} + S_{2, lac})} \lesssim N \left( M_N(F_3) |f_3|_{p_3} |g| \left( \sum_n |I_{E_n}| \right)^{1/p_1} \right).
$$

Proof. By simple modifications of the argument in Section 8.1.1 together with Theorem 6.9 part (i), we obtain

$$
|K^*g|_{\mathcal{L}^{p_2}(P, S_{1, overlap})} \lesssim M_N(P_2, 1_{F_3})^{1/p_2} |f_3|_{p_3} V_{p_1}(g) |1_{F_3}|_{p_1} \lesssim M_N(P_1, 1_{F_3})^{1/p_1} M_N(P_2, 1_{F_3})^{1/p_2} |f_3|_{p_3} |g| \mathcal{L}^{p_1}(S_{2, lac}),
$$

so it remains to consider the contribution of $S_{2, lac}$, and by interpolation it suffices to consider weak-type estimates.

Fix $\alpha > 0$. Without loss of generality we may assume that $|K^*g| \mathcal{L}^\infty(S_{2, lac}) \leq 2\alpha$.

By a standard argument, we may find $B \subset P_2$ with $|K^*g| \mathcal{L}^\infty(B, S_{2, lac}) \leq \lambda$ and a collection of strongly disjoint lacunary generating sets $(G_n)$ covering $B$ such that

$$
\alpha^2 \mu_2(B) \lesssim \alpha^2 \sum_n |I_{G_n}| \lesssim N := \sum_n \sum_{P \in G_n} |I_P| |K^*g| \mathcal{L}^{p_1}(P')^2.
$$

Let $h(P') = \overline{K^*g}(P')$ for $P' \in Q_2 = \bigcup G_n$, and let $h(P') = 0$ otherwise.

We obtain

$$
N = \sum_{\substack{\mu Q_1, \mu Q_2 \mid I_P \mid I_P'} |I_P| |I_P'| h(P') g(P) \langle \phi_{1, p}, \phi_{2, p'}, d_{P, P'}, f_3 \rangle,
$$

thus it suffices to show that

$$
N \lesssim M_N(F_3) |f_3|_{p_3} |h| \mathcal{L}^\infty(S_{2, lac}) |g| \mathcal{L}^\infty(S_{2, lac}) \left( \sum_n |I_{E_n}| \right)^{1/p_1} \left( \sum_n |I_{G_n}| \right)^{1/p_2}. \tag{1}
$$

Note that since $Q_1$ and $Q_2$ are unions of strongly disjoint lacunary sets, all overlapping sets are essentially a collection of spatially disjoint tiles, therefore $S_{1, overlap} \lesssim S_{2, lac}$ in each of them. This observation will be used implicitly below.

We first show that the unconstrained double sum $N_{free}$ over $P, P'$ (where there is no constraint between $P$ and $P'$) satisfies the desired estimate. Indeed, since $2 < p_1, p_2 < 2/(3r - 4)$ and $1/p_1 + 1/p_2 = 1 - 1/p_3 > 2/r$ we may find $s, t > 2$ such that $2/r = 1/s + 1/t$ and $t > p_1$ and $s > p_2$. Using Theorem 6.9 we have

$$
N_{free} \lesssim M_N(F_3) |f_3|_{p_3} |h| \mathcal{L}^\infty(S_{2, lac}) |g| \mathcal{L}^\infty(S_{2, lac}) \left( \sum_n |I_{E_n}| \right)^{1/p_1} \left( \sum_n |I_{G_n}| \right)^{1/p_2}.
$$

We now consider diagonal sums when $|I_{P'}| = C |I_P|$ for some fixed $C \in [2^{-4}, 2^4]$. The proof below is symmetric in $P, P'$, so we’ll assume $C \leq 1$. We say that $P$ is linked to $P'$ if the corresponding summand is nonzero, clearly that all linked pairs satisfy $\omega_{1, P, upper} \cap \omega_{2, P', upper} \neq \emptyset$. Since these are dyadic intervals and $|\omega_{1, P, upper}| \leq |\omega_{2, P', upper}|$, we obtain $\omega_{1, P, upper} \subset \omega_{2, P', upper}$. Without loss of generality, assume that for some $k$ it holds that $2^k |I_P| \leq |I_P| + \text{dist}(I_P, I_P') < 2^{k+1} |I_P|$ for all linked pairs, provided that we have sufficient decay in the estimates. It follows that for each $P'$ there is at most $O(1)$ linked $P$ and vice versa, and by further dividing if necessary we may assume that exactly one $P$ is linked to exactly one $P'$, and let $P' = F(P)$ and $P = F^{-1}(P')$. It follows that $h(P')$
and \(c(P, P') := |I_P| \langle \phi_1, p \phi_2, p d_P, P, f_3 \rangle\) are functions on \(P\). Using outer Radon–Nikodym/Holder and the triangle inequalities, it follows that the corresponding sum is bounded by

\[
N_{\text{diag}, k} \lesssim \|g\|_{L^p(Q_1, S_{2, \mu_1})} |h \circ F|_{L^p(Q_1, S_{2, \mu_1})} |c|_{L^p(Q_1, S_{x, \mu_1})}.
\]

Now, we note that since \(Q_1\) is an union of strongly disjoint lacunary sets, all the overlapping generating subsets of \(Q_1\) has \(O(1)\) elements, therefore \(S_2 \lesssim S_{2, \text{lac}}\) on \(Q_1\). For any generating set \(E_2 \subset Q_2\) it is clear that \(F^{-1}(E_2)\) could be covered by \(O(1)\) generating sets of \(Q_1\), whose top intervals are contained in some bounded enlargement of \(2^k I_{E_2}\). Therefore \(\mu_1(F^{-1}(E_2)) \lesssim 2^k \mu_2(E_2)\). Conversely, if \(E_1\) is a generating set in \(Q_1\) then \(F(E_1)\) could be generously covered by \(O(2^k)\) generating sets in \(Q_2\), and the length of the top intervals of these covering sets are comparable to \(|I_{E_1}|\). Therefore \(S_{2, \text{lac}}(h \circ F)(E_1) \lesssim 2^k S_{2, \text{lac}}(h)(F(E_1))\). Therefore by pull-back (essentially the same proof as [8, Proposition 3.2]) we obtain

\[
|h \circ F|_{L^p(Q_1, S_{2, \mu_1})} \lesssim 2^{2k} |h|_{L^p(Q_2, S_{2, \text{lac}, \mu_2})}.
\]

On the other hand, notice that for any \(P \in Q_1\) we have

\[
c(P, P') \lesssim N 2^{-Nk} \frac{1}{|I_P|} \int (\chi_{I_P} \tilde{\chi}_{I_P})^N \sup_{j=N, \eta \in \omega_1, P, \text{upper}} |d_j| f_3.
\]

Therefore using (perhaps a version of) Lemma 6.6 it follows that

\[
|c|_{L^p(Q_1, S_{x, \mu_1})} \lesssim N 2^{-Nk} \sup_{P \in Q_1} \left( \frac{1}{|I_P|} \int (\chi_{I_P} \tilde{\chi}_{I_P}) \right)^{p_3} |f_3|_{p_3} ^{p_3} \lesssim N 2^{-Nk} M_N(|F_3| f_3 |p_3).
\]

Here we used \(p_3 > r/(r-2)\). This leads to the desired estimate for \(N_{\text{diag}, k}\).

Consequently, for the purpose of proving the desired estimates for \(N\) the roles of \(Q_1\) and \(Q_2\) are symmetric, and we may assume that

\[
M_N(Q_1, F_3) \mu_1(Q_1) \subseteq M_N(Q_2, F_3) \mu_2(Q_2).
\]

It follows from Lemma 8.5 that

\[
|K h|_{L^p(Q_1, S_{2, \text{lac}, \mu_1})} \lesssim (M_N(Q_1, F_3) \mu_1(Q_1) \mu_2(Q_2)) |h|_{L^p(S_{2, \text{lac})}} |f_3|_{p_3}.
\]

Recall that on \(Q_1\) and \(Q_2\) we have \(S_2 \lesssim S_{2, \text{lac}}\), so using a combination of outer Radon-Nikodym and outer Holder inequalities, we obtain

\[
N_{\text{aff, diag}} = \sum_{P \in Q_1} |I_P| g(P)(K h)(P)
\]

\[
\lesssim |g|_{L^p(Q_1, S_{2, \text{lac}, \mu_1})} |K h|_{L^p(Q_1, S_{2, \text{lac}, \mu_1})}
\]

\[
\lesssim M_N(F_3) \mu_1(Q_1) \mu_2(Q_2) |g|_{L^p(S_{2, \text{lac})}} |h|_{L^p(S_{2, \text{lac})}} f_3 |p_3
\]

\[
\lesssim M_N(F_3) \left( \sum_m |I_{E_m}| |g|_{L^p(S_{2, \text{lac})}} |h|_{L^p(S_{2, \text{lac})}} f_3 |p_3
\]

as desired. \(\square\)
Lemma 8.5. Let \( f \) be supported on \( Q_2 \). Then
\[
|Kf|_{L^p(x)}(Q_1,s_{2,\text{lac},\mu_1}) \\
\leq (MN(Q_1,F_3)^{1/p_1} + MN(F_3)^{1/p_2}) |f|_{L^p(Q_2)} f_{3,p_3}.
\]
To prove Lemma 8.5, we first show the following estimate for \( |Kf|_{\infty} \).

Lemma 8.6. Uniform over \( P \subset P_1 \), it holds for any \( s < 2 \) and \( N > 0 \) that
\[
|Kf|_{L^p(x)}(P,s_{2,\text{lac},\mu_1}) \leq m_{s,N}(P,(f_{3,d_j^p}f)) |f|_{L^p(P)} + m_{s,N}(P,(f_{3,d_j^p}f)).
\]

Below we deduce Lemma 8.5 from Lemma 8.6. By interpolation, it suffices to prove weak-type estimates. We may assume \( |f|_{L^p(E)} = 1 \) by scaling. Using Lemma 8.6 for \( s = p_1 \), for any \( \lambda > 0 \) we may find \( A \subset Q_1 \) such that
\[
m_{s,N}(Q_1,A,(f_{3,d_j^p}f)) \leq \lambda \quad \text{and} \quad \lambda_{p_1}(A) \leq MN(Q_1,F_3)^{1/p_1} \left( \sum_j |f_{3,d_j^p}f|^{1/p_1} \right)^{1/p_1}.
\]
Thus, arguing as in Section 8.1.1 we obtain
\[
\lambda_{p_1}(A) \leq MN(Q_1,F_3)^{1/p_1} MN(P_2,F_3)^{1/p_2} |f|_{L^p(Q_2)} f_{3,p_3} \leq MN(F_3)^{1/p_2} |f|_{L^p(P)} f_{3,p_3}.
\]
Since \( p_3 > r/(r-2) \), it follows from Lemma 8.6 that we could find \( B \subset Q_1 \) with
\[
m_{s,N}(Q_1,B,(f_{3,d_j^p}f)) \leq \lambda \quad \text{and} \quad \lambda_{p_1}(B) \leq \mu_1(Q_1).
\]
Clearly we have \( \mu_1(B) \leq \mu_1(Q_1) \), therefore
\[
\lambda_{p_1}(B) \leq MN(Q_1,F_3)^{1/p_3} |f|_{L^p(P)} f_{3,p_3}.
\]
and this completes our proof of Lemma 8.5.

Proof of Lemma 8.6. Let \( E \subset P \) be lacunary. Using sparseness of \( P_2 \), it is clear that \( P_2(E) \) is a lacunary subset of \( P_2 \). Let \( u_j := f_{3,d_j^p}f \) for convenience. Similar to the proof of Lemma 8.7, it suffices to show that if \( |g|_{L^p(E)} = 1 \) then
\[
1 |I_E| \sum_{P \in E} |I_P|Kf(P)g(P) \leq m_{s,N}(P,(f_{3,d_j^p}f)) |f|_{L^p(P)} f_{3,p_3} + m_{s,N}(P,(u_j)) .
\]
Let \( J \) and \( b_P \) be defined as usual, we also start with
\[
\sum_{P \in E} |I_P|Kf(P)g(P) \leq \sum_J \sum_{P \in E} |I_P|g(P) \phi_j, P b_{P,J} |_{L^1(J)} \leq A + B .
\]
where \( A \) denote the contribution of \( |I_P| \leq 2 |J| \) and the rest is in \( B \). Using \( |b_P(x)| \leq \sup_{J : N \in \Omega_1,P} u_j \), it is not hard to see that
\[
A \leq |I_E|m_{s,N}(E,(u_j)) .
\]
We now estimate $B$. For convenience, let $G_E(x) = \sum_{P \in E} |I_P|g(P)\phi_{1,P}(x)$ and
\[
V^s(g_E)(x) := \sup_{K,n_0 < \ldots < n_K} \left( \sum_{j=1}^K \left| (\Pi_{n_j} - \Pi_{n_{j-1}})g_E(x) \right|^s \right)^{1/s}
\]
for $0 < s < \infty$, where $\Pi_j$ denotes a suitable Fourier projection on to the relevant frequency scale of $E$ (see also proof of Lemma 3.7).

For each $J$, let $\omega_J = \bigcup_{P \in E, |I_P| > 4|J|} \omega_{1,P,upper}$. As in the proof of Lemma 6.7 there exists $R \in \mathcal{P}$ with $|I_R| \approx |J|$, $\text{dist}(I_R, J) \leq |J|$, and $\omega_J \subseteq \tilde{\omega}_R$. By decomposing $T_{j|I_P|16} = T_{j|I_P|4} + (T_{j|I_P|16} - T_{j|I_P|4})$ we obtain the decomposition $b_P = \ell_{P,J} + s_{P,J}$, and we will estimate the contribution of each term.

Contribution of $\ell_{P,J}$: We decompose further $\ell_{P,J} = \ell_{P,J,\text{core}} + \ell_{P,J,\text{tail}}$ by decomposing $f = f_{E,\text{core}} + f_{E,\text{tail}}$ where $f_{E,\text{core}}$ is the restriction of $f$ to the subset of $P_2(E)$ containing all $P^r$ with $|I_P| < 3|E|$. By the Holder inequality, we have
\[
\sum_J \sum_{|I_P| > 4|J|} |I_P|g(P)\phi_{1,P,\ell_{P,J,\text{core}}} |L^1(J) | \lesssim \sum_J |B_J(g, f_{E,\text{core}})(\sum_j |f_{3j}d_j| \lesssim \omega_{1,2,J}^{1/2} |L^1(J)|,
\]
\[
B_J(g, f) := \sum_J \sum_{P,P^r, } |I_P||I_{P^r}|g(P)f(P^r)\phi_{1,P,J} \phi_{2,P^r,J} \lesssim \omega_{1,2,J}^{1/2},
\]
the constraints in the sum are $|I_P| \leq |I_P|/16$, $|I_P| > 4|J|$, $|I_P^r| \geq |J|/4$. Let $F_{E,\text{core}} = \sum_{P^r} |I_P^r|f_{E,\text{core}}(P^r)\phi_{2,P^r}$ and define $F_{E,\text{tail}}$ similarly. Clearly,
\[
B_J(g, f_{E,\text{core}})(x) \lesssim M_J(V^{r/2}(G_E,F_{E,\text{core}})), \quad (r > 2 \text{ is needed for convexity}),
\]
where as in [3] we define the bilinear variation-norm $V^{r/2}(h_1, h_2)$ to be
\[
\sup_{K: N_0 < \ldots < N_K} \left( \sum_{k=1}^K \sum_{N_{k-1} < j < N_k} \left( \Delta_j h_1 \Delta_j h_2 \right)^{r/2} \right)^{2/r},
\]
and $(\Delta_j)_{j > 0}$ and $(\Delta_j)_{j \geq 0}$ are two suitable families of Littlewood-Paley projections relative to $\xi_E$. By variation-norm estimates for paraproducts [3], it follows that
\[
\sum_J \sum_{|I_P| > 4|J|} |I_P|\phi_{1,P,\ell_{P,J,\text{core}}} |L^1(J)| \lesssim m_{r/2} N(E, \{f_3d_j\}) |M(V^{r/2}(G_E, F_{E,\text{core}})| \omega_{1,2} \lesssim M_{r/2} N(E, \{f_3d_j\}) |G_E|_{L^p(\mathbb{R})} |F_{E,\text{core}}|_{L^p(\mathbb{R})}.
\]
(33)

By Lemma 6.4 we have $|F_{E,\text{core}}|_{L^p(\mathbb{R})} \leq |I_E|^{1/p} |f|_{L^p(P_2, S_2, lac, P_2)}$ and a similar estimate for $|G_E|_{L^p(\mathbb{R})}$, giving the desired estimate for the contribution of $\ell_{P,J,\text{core}}$.

For contribution of $\ell_{P,J,\text{tail}}$, notice that for every $x \in J_3 |E|$ it holds that
\[
B_J(g, f_{E,\text{tail}})(x) \lesssim (\sum_J \sum_P |I_P|g(P)\phi_{1,P,J}|^{r/2}(\sum_P |I_P^r|f_{E,\text{tail}}(P^r)\phi_{2,P^r}|) \lesssim_N M_J(V^r(G_E)) \sup_{P: |I_P| > 4E = 2|I_E|/16} \left| f(P^r) \widetilde{\chi}_{I_p}(\phi(I_E)) \right|^{N+5}.
\]
It follows that
\[\sum_J \sum_{P \in E; |P| \geq 4|J|} \left| I_P \right| g(P) \phi_{1,P} \ell_{P,J,\text{tail}} |_{L^1(J)} \lesssim \]
\[ \lesssim m_{\frac{r}{r-2}, \mathcal{N}(\mathbb{E}, (f_3 d_j))} | M(V^r(G_E)) |_{L^{r, x}(3 \ell_E)} \sup_P | f(P) | \chi_{\ell_E} (\epsilon(I_E))^{N+5} \]
which implies the desired estimate for the contribution of \( \ell_{P,J,\text{tail}} \) via the continuous Lépingle inequality (see e.g. [1, 10]).

The desired claim follows from generalized Carleson embeddings and duality.

8.2.1. The basic range. We first show the range \( 2 < q_1, q_2 < 2r/(4 - r) \) and \( 1 < q_2 < r/2 \) of Theorem 8.1. Note that now \( 2r/(3r - 4) < q'_3 < 2 \) and \( q'_3 > r/(r - 2) \).

Using outer Radon Nikodym/Holder inequalities and Lemma 8.2, it follows that
\[ \langle T_{CC}(f_1, f_2), f_3 \rangle = \Lambda_{P_1, P_2} := \sum_{P \in E} |I_P| a_1(P) K(a_2)(P) \]
\[ \lesssim M_N(F_3) \left| a_1 \right|_{L^{q_1}} \left| a_2 \right|_{L^{q_2}} \left| f_3 \right|_{L^{q'_3}(\mathbb{R})} \]
where in the above display (and in subsequent displays) the outer norm for \( a_j \) is over \( \mathcal{P}_j, S_{2,\text{lac}}, \mu_j \) and the outer norm for \( K a_2 \) is over \( \mathcal{P}_1, S_{1,\text{overlap}} + S_{2,\text{lac}}, \mu_1 \).

Thus the desired claim follows from generalized Carleson embeddings and duality, and the trivial bound \( M_N(F_3) \leq 1 \).

8.2.2. Extending the range. Let \( \mathcal{M} \subset \{ \alpha_1 + \alpha_2 + \alpha_3 = 1 \} \) have the following vertices
\[ M_1(\frac{3r - 4}{2r}, \frac{1}{2}, -\frac{r - 2}{r}), \quad M_2(\frac{1}{2}, \frac{3r - 4}{2r}, -\frac{r - 2}{r}) \]
\[ M_3(0, \frac{3r - 4}{2r}, -\frac{r - 2}{r}), \quad M_4(0, 0, 1), \quad M_5(\frac{3r - 4}{2r}, 0, -\frac{r - 2}{r}) \]

Similar to Section 7.3.2, to show Theorem 8.1 it suffices to prove restricted-weak type estimates for one \( \alpha = (\alpha_1, \alpha_2, \alpha_3) \) in any neighborhood of any given vertex of \( \mathcal{M} \). Our starting point will be 35, where \( \mathcal{P}_1 \) and \( \mathcal{P}_2 \) are symmetric, thus it suffices to consider \( M_1, M_3, M_4 \). Below let \( F_1, F_2, F_3 \) have finite positive Lebesgue measures. Let \( C_0 > 0 \) be a finite large absolute constant such that \( |M(f)|_{1, \infty} < \frac{C_0}{4} |f|_1 \) in the maximal inequality.

In the following, let \( 2 < p_1, p_2 < 2r/(4 - r) \) and \( p_3 > \frac{r}{r-2} \) with \( \sum 1/p_j = 1 \).
Near $M_1$. Let $B = \bigcup_{j=1,2} \{ M(1_{F_j}) > C_0 |F_j|/|F_3| \}$, clearly $|B| < |F_3|/2$. Let $G_3 = F_3 \setminus B$. We may assume that for some $k_1, k_2 \geq 0$ it holds that

$$2^{k_1} \leq 1 + \frac{\operatorname{dist}(I_P, B^c)}{|I_P|} < 2^{k_1+1} |I_P|, \quad \forall P \in \mathcal{P}_1$$

and similarly $2^{k_2} \leq 1 + \frac{\operatorname{dist}(I_{P'}, B^c)}{|I_{P'}|} < 2^{k_2+1}$ for every $P' \in \mathcal{P}_2$, provided that we have sufficient decay in the estimate. Let $|f_1| \leq 1_{F_1}$, $|f_2| \leq 1_{F_2}$, and $|f_3| \leq 1_{F_3-B}$.

By convexity, it follows from (35) and Carleson embeddings that

$$\Lambda_{\mathcal{P}_1, \mathcal{P}_2} \lesssim \left| F_3 \right|^\frac{1}{r_1} \prod_{j=1,2} M_N(P_j, 1_{F_3 \setminus B})^\frac{1}{r_j} |a_j|_{\frac{1}{r_j}} |a_j|_{\frac{2}{r_2}}$$

$$\lesssim 2^{-N(k_1+k_2)} |F_1|^{-\frac{1}{p_1}} |F_2|^{-\frac{1}{p_2}} |F_3|^{-\frac{1}{r}} \left( \frac{F_3}{F_1} \right)^{\frac{1}{r}} |F_1|^{\frac{1}{r} \frac{N}{p_1}} M_N(P_2, F_3)^{\frac{1}{p_2}} |a_2|_{p_2}$$

thus $\Lambda_{\mathcal{P}_1, \mathcal{P}_2}$ satisfies restricted weak-type estimate for $\alpha = (1/p_1', 1/p_2', -1/p_3)$, which could be made arbitrarily close to $M_1$.

Near $M_3$. Let $B = \{ M(1_{F_3}) > C_0 |F_3|/|F_1| \}$, clearly $|B| < |F_1|/2$. We may assume that (36) holds for some $k_1 \geq 0$, provided that we have sufficient decay in the estimate. Let $|f_1| \leq 1_{F_1-B}$ and $|f_2| \leq 1_{F_2}$ and $|f_3| \leq 1_{F_3}$, we’ll show that

$$\Lambda_{\mathcal{P}_1, \mathcal{P}_2}(f_1, f_2, f_3) \lesssim \left| F_2 \right|^\frac{1}{r_2} \left| F_3 \right|^\frac{1}{r_2}$$

which implies desired estimates by letting $p_2$ close to $2r/(4-r)$.

To show (37), we first use convexity and (35) and Carleson embeddings to obtain

$$\Lambda_{\mathcal{P}_1, \mathcal{P}_2}(f_1, f_2, f_3) \lesssim \left| F_3 \right|^\frac{1}{r_3} M_N(P_1, 1_{F_3})^\frac{1}{r_1} |a_1|_{\frac{1}{r_1}} |a_1|_{\frac{2}{r_2}} M_N(P_2, F_3)^{\frac{1}{p_2}} |a_2|_{p_2}$$

$$\lesssim 2^{-Nk_1} |F_3|^\frac{1}{r_3} \left( \frac{F_3}{F_1} \right)^{\frac{1}{r} \frac{N}{p_1}} M_N(P_2, F_3)^{\frac{1}{p_2}} |a_2|_{p_2}$$

(38)

It follows in particular that $\Lambda(f_1, f_2, f_3) \lesssim 2^{-Nk_1} |F_2|^{1/p_2} |F_3|^{1/p_3}$ for any $2 < p_2 < \frac{2r}{4-r}$, which would imply the desired estimate (37) if $|F_3| \leq |F_2|$. When $|F_3| > |F_2|$ we will carry out essentially another layer of restricted weak-type interpolation, which we details below. Let $\tilde{B}_1 = F_3 \cap \{ M1_{F_2} > C_0 |F_2|/|F_3| \}$, clearly $|\tilde{B}_1| < |F_3|/2$. We will show that

$$\Lambda_{\mathcal{P}_1, \mathcal{P}_2}(f_1, f_2, f_3 \mathbb{1}_{\tilde{B}_1}) \lesssim 2^{-Nk_1} |F_2|^{1/p_2} |F_3|^{1/p_3}$$

(39)

To show (39), we may assume that for some $k_2 \geq 0$ it holds for every $P' \in \mathcal{P}_2$ that

$$2^{k_2} \leq 1 + \frac{\operatorname{dist}(I_{P'}, B_1^c)}{|I_{P'}|} < 2^{k_2+1},$$

provided that we have enough decay in the estimates.

It follows from (38) that

$$\Lambda_{\mathcal{P}_1, \mathcal{P}_2}(f_1, f_2, f_3 \mathbb{1}_{\tilde{B}_1}) \lesssim 2^{-Nk_1} M_N(P_2, F_3 - \tilde{B}_1)^{\frac{1}{r}} |a_2|_{\frac{1}{r}} |a_2|_{\frac{2}{r_2}} |F_3|^{\frac{1}{r_2}}$$

$$\lesssim 2^{-Nk_1} 2^{-Nk_2} \left( \frac{|F_2|}{|F_3|} \right)^{\frac{1}{r} \frac{N}{p_1}} |F_2|^{\frac{1}{r_2}} |F_3|^{\frac{1}{r_2}}$$

$$\lesssim 2^{-N(k_1+k_2)} |F_2|^{\frac{1}{r_2}} |F_3|^{\frac{1}{r_2}}$$

proving (39). Now, if $|\tilde{B}_1| > |F_2|$ then we continue to let $\tilde{B}_2 = \tilde{B}_1 \cap \{ M1_{F_2} > C_0 |F_2|/|\tilde{B}_1| \}$ and similarly $\tilde{B}_3, \ldots, \tilde{B}_m$ such that $|F_3| > 2|\tilde{B}_1| > \cdots > 2^m |\tilde{B}_m|$ until
$|\tilde{B}_m| \leq |F_2|$. (This process has to stop since $|F_3|/|F_2|$ is finite.) We obtain (here for convenience of notation let $\tilde{B}_0 = F_3$):

$$
\Lambda_{P_1, P_2}(f_1, f_2, f_3) \lesssim |\Lambda_{P_1, P_2}(f_1, f_2, f_31_{\tilde{B}_m})| + \sum_{j=0}^{m-1} |\Lambda_{P_1, P_2}(f_1, f_2, f_31_{\tilde{B}_j - \tilde{B}_{j+1}})| \\
\lesssim 2^{-Nk_1} |F_2|^{\frac12} |\tilde{B}_m|^{1-\frac{1}{p_2}} + \sum_{j=0}^{m-1} 2^{-Nk_1} |F_2|^{1-\frac{1}{p_2}} (2^{-j} |F_3|)^{\frac12} \\
\lesssim 2^{-Nk_1} |F_2|^{1-\frac{1}{p_2}} |F_3|^{\frac12},
$$

as desired.

Near $M_4$. Let $B = \{M_1 F_3 > C_0 |F_3|/|F_1|\}$ which satisfies $|B| < |F_1|/3$. Let $|f_1| \leq 1_{F_1-B}$, $|f_2| \leq 1_{F_2}$, $|f_3| \leq 1_{F_3}$; it suffices to show that

$$
(40) \quad \Lambda_{P_1, P_2}(f_1, f_2, f_3) \lesssim \epsilon |F_3|^{1-\frac{1}{r'}} ,
$$

for any $0 < \epsilon \leq 1/p_2$. As before, we’ll assume (36). For any $2 < p_2 < 2r/(4-r)$ we also have (38), which would imply the desired estimate if $|F_3| \geq |F_2|$. When $|F_3| < |F_2|$ we will use interpolation. Similar to the analysis near $M_3$, it suffices to show that if $\tilde{B}_1 = F_2 \cap \{M_1 F_3 > C_0 |F_3|/|F_2|\}$ (which satisfies $|\tilde{B}_1| < |F_2|/2$) then

$$
\Lambda_{P_1, P_2}(f_1, f_21_{\tilde{B}_1}, f_3) \lesssim 2^{-Nk_1} |F_2|^{1-\frac{1}{r'}} |F_3|^{1-\epsilon}
$$

and we will again assume that for some $k_2 \geq 0$ it holds for every $P' \in P_2$ that $2^{k_2} \leq 1 + \frac{\text{dist}(I_{P'}, B_1)}{|P'|^\frac12} < 2^{k_2}$. Now it follows from (38) that

$$
\Lambda_{P_1, P_2}(f_1, f_21_{\tilde{B}_1}, f_3) \lesssim 2^{-Nk_1} M_N(P_2, F_3)^{\frac{1}{p_2}} |a_2|^{1-\frac{2}{p_2}} |a_2|^{\frac{2}{p_2}} |F_3|^{\frac{1}{p_2}}
\lesssim 2^{-Nk_1} 2^{-Nk_2} \min(1, |F_3|^{\frac{1}{p_2}} |F_2|^{\frac{1}{p_2}} |F_3|^{\frac{1}{p_2}})
\lesssim 2^{-N(k_1+k_2)} |F_2|^{1-\frac{1}{r'}} |F_3|^{1-\epsilon}.
$$

This completes the proof of Theorem 8.1. □

9. ESTIMATES FOR LM MODEL OPERATORS

**Theorem 9.1.** Let $T(f_1, f_2)$ be an LM model operator. Then $|T|_{L^{q_1}\times L^{q_2} \rightarrow L^{q_3}} < \infty$ for all $1/q_3 = 1/q_1 + 1/q_2$ with $q_1, q_2 > 2r/(3r-4)$ and $q_3 > r'/2$.

**Proof.** We sketch the proof of this theorem, which is a simple bilinear extension of the proof of Theorem 6.1. For a tritile $Q \in \mathcal{Q}$ in the definition of $T$ we let $\omega_Q$ be the convex hull of $\omega_{Q_1}, \omega_{Q_2}, \omega_{Q_3}$. Let $D^{-1} x = x/2$ the dilation by $1/2$ with respect to the origin, and define $\omega_{Q, lower} = \omega_{1, Q, lower} \cap \omega_{2, Q, lower} \cap D^{-1} (\omega_{3, Q, lower})$ and define $\omega_{Q, upper}$ similarly. Without loss of generality we may assume that $\omega_{Q, lower}$ is a half-line and $\omega_{Q, upper}$ is a finite interval for every $Q \in \mathcal{Q}$. We now could define $\omega_Q$ to be the convex hull of $20 \omega_Q$ and $20 \omega_{Q, upper}$, and from here we may define generating subsets of $\mathcal{Q}$, and construct outer measure spaces on $\mathcal{Q}$ using the usual outer measure and sizes as in Section 6.1.

Let $A$ denote the collection of intervals $J$ such that for some $P_1, P_2 \in \mathcal{Q}$ we have $I_{P_1} \subset J \subset 30 I_{P_2}$. Let $p_1, p_2 > 2$ and $p_3 > r/(r-2)$ such that $\sum 1/p_j = 1$; note
that this implies \( p_1, p_2 < 2r/(4 - r) \). By routine applications of outer measure techniques and embedding theorems in Section 3, we obtain

\[
\langle T(f_1, f_2), f_3 \rangle \lesssim_N \left| F_3 \right| \frac{1}{|J|} \sup_{j \in A} M_J(1_{\text{supp}(f_3)})^{\frac{1}{2^N}} \prod_{j=1,2}^{|F_j|} \frac{1}{|J|} \sup_{j \in A} M_J(f_j)^{\frac{1}{2^N}}. 
\]

Using similar arguments as in previous sections, it follows that restricted weak-type estimates holds for at least one \((\alpha_1, \alpha_2, \alpha_3)\) in any neighborhood of any of the following points, which then implies the theorem (below \( \beta_r := (3r - 4)/(2r) \)):\
\[ H_1(\beta_r, -\beta_r, 1), \quad H_2(-\beta_r, \beta_r, 1), \quad H_3(1/2, \beta_r, 1/2-\beta_r), \quad H_4(\beta_r, 1/2, 1/2-\beta_r). \]

Appendix A. Reduction to Discrete Operators

We’ll show below that the (variation-norm) bilinear Fourier operators with symbols \( m_{CC}, m_{CC \times C}, m_{BC}, m_{CC} \) are controlled by the respective discrete operators.

Recall that \( A \) is the set of all admissible triple \((side, m, n)\) in \( H \). It is clear that

- If \( \alpha \in A \) then \( L_1 \leq \min(m_\alpha, n_\alpha) \leq 2L_1 \).
- If \( k > 2L_1 \) then \((left, 2L_1, k)\) and \((right, k, 2L_1)\) are not in \( A \).

Let \( D_{left}, D_{right} \) be the sets of left and and right-sided dyadic intervals. By definition, if \( \alpha = (side, m, n) \) then \( I_{M,N}(\alpha) \) consists of \( I \in D_{side} \) with \( M \in I_{lower, \alpha} := I - (m + 1)|I| \) and \( N \in I_{upper, \alpha} := I + (n + 1)|I| \).

A.1. Discretization of \( m_{CC} \). We will discuss the discretization for \( m_{CC,1} \), the discretization for other \( m_{CC,j} \) are similar. We first make several observations regarding the decomposition of \( 1_{M<\xi<N} \) in Section 3.1.1.

Lemma A.1 (Observation 1). If \( I \in I_{side, m, n} \) and \( \xi \in \frac{5}{4}I \) then

\[
m/(4n) < |\xi - M|/|\xi - N| < 4m/n.
\]

Proof. This follows from \((m - 1/8)|I| \leq |\xi - M|, |\xi - N| \leq (m + 2 + 1/8)|I| \). □

Lemma A.2 (Observation 2). Let \( I \in I_{side, m, n} \) with \( n > 4m \). Assume that \( J \in I_{side', m', n'} \) and \( sup J < inf I \). Then \( n'/m' > n/m \).

Similarly, if \( m > 4n \) and \( J \) is on the right of \( I \) then \( m'/n' > m/n \).

Proof. Assume \( n > 4m \), the other case is symmetric. Without loss of generality we may assume that \( J \) is adjacent to \( I \). Since \( n > m \), it follows that \( |J| = |I| \) or \( |J| = |I|/2 \). In the first case the desired estimate is trivial, and in the second case we have \( m' = 2m \) and \( n' > 2n + 2 \), which also implies the desired estimate. □

As a corollary of Lemma A.2, it follows that the intervals in \( A_2 \) are strictly in between the elements of \( A_1 \) and the elements of \( A_3 \).

The next observation concerns cancellation when summing bump functions \( \phi_\alpha \) over the set of \( \alpha \in A \) such that \( I \in I_{M,N}(\alpha) \), where \( M, N, I \) are fixed.

Lemma A.3 (Observation 3). \((i)\) Given any \( C \geq 4 \) we can find \( O(L_1) \) many tuples \((a_j, b_j, u_j, v_j) \in \mathbb{Z}^2 \times \{1/2, 1, 2\}^2 \), with \( L_1 + 1 \leq a_j, b_j \leq L_1 \), such that the following holds for every left dyadic interval \( I \) and every \( M < N \):

\[
\sum_{\alpha \in A_{left, m} \geq Cn_\alpha} \phi_\alpha(\xi)1_{I \in I_{M,N}(\alpha)} = \sum_{J} \phi_{a_j, b_j}(\xi)1_{M < \xi < N} - (a_j - 1/2)|I| 1_{N \in I + b_j|I|}. 
\]

Furthermore, an analogous statement also holds for right dyadic intervals.

\((ii)\) A similar statement also holds for \( \sum_{m \leq n \leq C} \).

Remark: the key idea here is that the left hand sides in the above equalities involve infinitely many terms, while the right hand sides contain only $O(L_1)$ terms.

Proof. Let $L_-$ and $L_+$ be the left and right neighbors of $I$ in $\mathcal{H}$. Below we only consider the $\sum_{m_\alpha \geq Cn_\alpha}$, the other sum could be handled similarly.

Since $C \geq 4$ we have $m_\alpha \geq 4L_1$, while $2L_1 \geq n_\alpha \geq L_1$. Our key observation is the fact that: in the sum, the ratios $u(I) = \frac{|I_-|}{|I|}$ and $v(I) = \frac{|I_+|}{|I|}$ depends only on side$_\alpha$ and $n_\alpha$. In other words, knowing the side of $I$ and knowing $n_\alpha$ (only a finite number of possible values) we could determine $\phi_\alpha(\xi) = \phi_{\alpha[I_-,x]}(\xi)$ completely and thus we have the freedom to sum the indicator constraints on $M$, $1_{M \in I - (m+1)I}$, over $m \geq Cn_\alpha$. The following table details this observation

<table>
<thead>
<tr>
<th>side of $I$</th>
<th>values of $n_\alpha$</th>
<th>Corresponding values of $(u(I), v(I))$</th>
</tr>
</thead>
<tbody>
<tr>
<td>left</td>
<td>$L_1$</td>
<td>$(2, \frac{1}{2})$</td>
</tr>
<tr>
<td>left</td>
<td>$L_1 &lt; n_\alpha \leq 2L_1$</td>
<td>$(2, 1)$</td>
</tr>
<tr>
<td>right</td>
<td>$L_1$</td>
<td>$(1, \frac{1}{2})$</td>
</tr>
<tr>
<td>right</td>
<td>$L_1 &lt; n &lt; 2L_1$</td>
<td>$(2, 1)$</td>
</tr>
</tbody>
</table>

(note that if $I$ is right-sided then $n_\alpha < 2L_1$ by definition of $A$). □

A.1.1. Discretization for $m_{CC,1}$: Using Lemma [A.3] we may decompose $m_{CC,1}$ into

$$\sum_{|I| \leq |J|/16} \phi_I(\xi_1) \varphi_J(\xi_2) 1_{M \in c(I)-a_1|I|,N \in I+b_1|I|} 1_{M \in c(J)-a_2|J|,N \in J+b_2|J|}$$

in the sum $I$ and $J$ belong to fixed collection of dyadic intervals, $\phi_I$, $\varphi_J$ are given bump functions supported in $(5/4)I$ and $(5/4)J$, and $L_1 \leq a_1, b_1, a_2, b_2 \leq L_1$.

It follows from a standard Fourier sampling argument that we can decompose

$$\int \int e^{i\omega(\xi_1+\xi_2)} \phi_I(\xi_1) \varphi_J(\xi_2) \hat{f}_1(\xi_1) \hat{f}_2(\xi_2) d\xi_1 d\xi_2$$

into finitely many wavelet sums

$$\sum_{P, P' \text{ tiles: } \omega_P = I, \omega_{P'} = J} \frac{|I_P||I_{P'}|}{|I|} \langle f_1, \phi_{1,P} \rangle \langle f_2, \phi_{2,P'} \rangle \phi_{1,P}(x) \phi_{2,P'}(x),$$

where $\phi_{1,P}$, $\phi_{2,P'}$ are $L^1$-normalized wave functions adapted to the tiles $P$, $P'$. Thus the resulting bilinear operator for $m_{CC,1}$ is controlled by a finite sum over $CC$ discrete operators.

A.2. Discretization of $m_{BC}$. For convenience let $\xi_3 = -\xi_1 - \xi_2$ below. Thanks to fast decay of $a_k$, it suffices to consider the contribution of one fixed $k$. In other words, it suffices to consider symbols

$$\hat{m}_{BC} = \sum_{\alpha \in A} \sum_{|S| \leq |I|} \phi_{1,S}(\xi_1) \phi_{2,S}(\xi_2) \varphi_{3,S}(-\xi_3) \phi_{\alpha,1}(-\xi_3) 1_{\{M \in I_{lower,\alpha}\}} 1_{\{N \in I_{upper,\alpha}\}}$$

in the sum $S$ is in a fixed collection of shifted dyadic cubes with the property [10], and $I$ is in a fixed collection of dyadic intervals, and $\phi_{j,S}$ are uniformly $C^m$ bump functions supported in $\frac{5}{6}S_1$, where $n$ could be chosen arbitrarily large.
For any $I$, $S$, $\alpha \in A$, and any Schwarz $f_1, f_2, f_3$, we have
\[
\int \int e^{ix(\xi_1 + \xi_2)} \phi_{3,S}(\xi_1 + \xi_2) \phi_{\alpha,I}(\xi_1 + \xi_2) \left( \prod_{j=1}^{2} \phi_j,S(\xi_j) \right) \hat{f}_3(x) \, dx
= \int \int e^{ix(\xi_1 + \xi_2)} \phi_{\alpha,I}(\xi_1 + \xi_2) \phi_{3,S}(\xi_1 + \xi_2) \left( \prod_{j=1}^{2} \hat{f}_j(\xi_j) \right) \, dx
= C \int \left( f_3 * \phi_{3,S} \ast \phi_{\alpha,I} \right) (2y) \left( \prod_{j=1}^{2} f_j \ast \phi_{j,S}(y) \right) \, dy
= C \sum_{m \in \mathbb{Z}} \int_{0}^{1} \frac{1}{\ell(S)} \left( f_3 * \phi_{3,S} * \phi_{\alpha,I} \right) \left( \prod_{j=1}^{2} f_j \ast \phi_{j,S} \left( \frac{t-m}{\ell(S)} \right) \right) \, dt
= C \int_{0}^{1} \sum_{Q \in Q_{t,S}} |Q| \left( f_3 * \phi_{\alpha,I} \ast \phi_{3,Q} \right) \left( \prod_{j=1}^{2} \langle f_j, \phi_{j,Q} \rangle \right) \, dt,
\]
where
- for each $t \in [0,1)$, $Q_{t,S}$ is the collection of all tritiles $Q = (Q_1, Q_2, Q_3)$ with $Q_1 = I_Q \times S_1$, $Q_2 = I_Q \times S_2$, $Q_3 = I_Q \times S_3$, and $I_Q$ is a shifted dyadic interval of length $\ell(S)^{-1}$ (with $t$-dependent shift);
- for any $Q \in Q_{t,S}$ and for each $j = 1, 2, 3$, define $\phi_{j,Q}(x) := \phi_{j,S}(x - c(I_Q))$ which are $L^1$-normalized wave packets adapted to $Q_j = I_Q \times S_j$ (with frequency support in $\frac{\alpha}{3} S_j$).

Recall that for every $S \in \mathcal{S}$ the distance between $S_1$ and $S_2$ is comparable to $L_2\|\ell(S)\| \sim L_1\|\ell(S)\|$ (due to the Whitney condition \eqref{Whitney_condition}). Clearly, the collection $Q_{t}$ will be of rank 1 (with uniform constants over $0 \leq t \leq 1$) if $L_1$ is sufficiently large.

Now, for any $b$-shifted dyadic interval $I$, by a Fourier sampling argument $(f_3 * \phi_{3,S} \ast \phi_{\alpha,I})(x)$ equals a sum over finitely many terms of the form $\sum_{P \in \mathcal{P}_I} \langle f_3, \phi_P \rangle \hat{\phi}_P(x)$, where $\mathcal{P}_I$ is the collection of all rectangles $P = J \times I$ formed using standard dyadic intervals $J$ of length $|J|^{-1}$, and $\{ \phi_P, P \in \mathcal{P}_I \}$ is a collection of Fourier wave packets adapted to $P \in \mathcal{P}_I$ with $\text{supp}(\hat{\phi}_{J \times I}) \subset \frac{\alpha}{3} I$.

It follows that, modulo a multiple by some absolute constant,
\[
\int \left( \int e^{ix(\xi_1 + \xi_2)} \hat{m}_{BC} \hat{f}_1(\xi_1) \hat{f}_2(\xi_2) \, d\xi_1 d\xi_2 \right) \, f_3(x) \, dx
\]
can be decomposed into finitely many terms of the following form:
\[
\int_{0}^{1} \sum_{Q \in Q_{t,S}} |Q| \left( \langle f_1, \phi_{1,Q} \rangle \langle f_2, \phi_{2,Q} \rangle \langle C(f_3), \phi_{3,Q} \rangle \right) \, dt,
\]
where
\[
C(f_3)(x) := \sum_{\alpha, I} \sum_{P \in \mathcal{P}_I} \sum_{n \in \mathbb{N}} 1_{\{2M \ell(I_{lower,\alpha})\}} 1_{\{2N \ell(I_{upper,\alpha})\}} \left( \langle f_3, \phi_P \rangle \hat{\phi}_P(x) \right),
\]
and $Q_{t,S} := \bigcup_{S \in \mathcal{S}} Q_{t,S}$.

Let $\mathcal{P}$ be the union of $\mathcal{P}_I$ over $I \in \mathcal{D}$.

Using Lemma \ref{Lemma_A.3}, we can write $C(f_3)(x)$ as a sum over finitely many terms of the form
\[
\pm \sum_{P \in \mathcal{P}} |I_P| \left( \langle f_3, \phi_P \rangle \hat{\phi}_P(x) \right) 1_{\{2M \ell(\omega_P)\}} 1_{\{2N \ell(\omega_P)\}},
\]
where \( \{ (\omega_{P,\text{lower}}, \omega_{P}, \omega_{P,\text{upper}}), P \in P \} \) is rigid. Thus to bound the resulting bilinear operator for \( m_{BC} \), it suffices to estimate discrete model \( BC \) operators.

A.3. Discretization of \( m_{LM} \). In Lemma A.4 and Lemma A.5 we'll show that the supports of \( m_{CC} \) and \( m_{BC} \) are inside \( \{ M \leq \xi_1 \leq \xi_2 \leq N \} \) and they do not intersect except at possibly \((\xi_1, \xi_2) = (M, M) \) and \((N, N) \). It will follow that

\[
m_{LM} = (\chi_{M < \xi_1 < \xi_2 < N} - m_{CC})(\chi_{M < \xi_1 < \xi_2 < N} - m_{BC}) \tag{42}
\]

which will be used in subsection A.3.3 to reduce the bilinear multiplier operator with symbol \( m_{LM} \) to LM model operators.

A.3.1. Support of \( m_{CC} \). It will follow from Lemma A.4 below that \( m_{CC} = 1 \) on \( R_1 \) (defined in (7)), and \( m_{CC} \) is supported inside the following enlargement of \( R_1 \):

\[
R'_1 := \left\{ (\xi_1, \xi_2) \in [M, N]^2 : \min(\xi_1 - M, \xi_2 - N) \leq \frac{1}{3}(\xi_1 - \xi_2) \right\}.
\]

Note that \( \overline{R'_1} \subset R'_1 \), and \( R'_1 \subset \{ M \leq \xi_1 < \xi_2 \leq N \} \).

**Lemma A.4.** (i) If a summand in (8) is non-zero for some \((\xi_1, \xi_2) \in R_1\), then this summand must appear in one of \( m_{CC,k} \), \( 1 \leq k \leq 5 \).

(ii) All summands of \( m_{CC,k} \) are supported in \( R'_1 \).

**Proof.** (i) Suppose that for some \( \alpha, \beta \) and \( I \in I(\alpha) \) and \( J \in I(\beta) \) and \((\xi_1, \xi_2) \in R_1 \) we have \( \phi_{\alpha,j}(\xi_1)\phi_{\beta,j}(\xi_2) \neq 0 \).

Without loss of generality we may assume that \((\alpha, \beta) \notin A_1 \times A_3 \), so \( \alpha \notin A_1 \) or \( \beta \notin A_3 \). By symmetry, we may assume that \( \beta \notin A_3 \).

We first show that \( \alpha \in A_1 \). Since \( \beta \notin A_3 \), it follows from Lemma A.1 that

\[
|\xi_2 - M| \leq 16|\xi_2 - N|, \text{ therefore } |\xi_2 - N| \geq |M - N|/15 \geq |\xi_1 - \xi_2|/15.
\]

Since \((\xi_1, \xi_2) \in R_1\), it follows from the definition of \( R_1 \) that

\[
|\xi_1 - M| \leq |\xi_1 - \xi_2|/200 \leq |M - N|/200.
\]

It follows that \( |\xi_1 - M| \leq |\xi_1 - N|/199 \). Using Lemma A.1 again, it follows that

\[
m_{\alpha/n_{\alpha}} \leq 4|\xi_1 - M|/|\xi_1 - N| < 1/4
\]

thus \( \alpha \in A_1 \) as claimed above.

It remains to show that \( |I| < |J|/16 \). As a corollary of the condition \( \alpha \in A_1 \) and \( \beta \notin A_3 \), we have \( m_{\beta} \leq 4n_{\beta} \leq 8L_1 \), while clearly \( m_{\alpha} \geq L_1 \). Using \( \xi_1 \in \frac{5}{4}I \) and \( \xi_2 \in \frac{5}{4}I \), we have

\[
|I|/|J| \leq \frac{|\xi_1 - M|/(m_{\alpha} - 1/8)}{|\xi_2 - M|/(m_{\beta} + 9/8)} \leq \frac{11|\xi_1 - M|}{|\xi_2 - M|} < \frac{1}{16}.
\]

This completes the proof of claim (i).

(ii) Without loss of generality we may assume that \( \alpha \in A_1 \).

By the definition of \( m_{CC,j} \)'s we have \( |I| \leq |J|/16 \). It suffices to show that

\[
(5/4)I \times (5/4)J \subset R'_1.
\]

To see this, take any \((\xi_1, \xi_2) \in (5/4)I \times (5/4)J \).
Since \( \alpha \in A_1 \) implies that \( m_\alpha = \min(m_\alpha, n_\alpha) \leq 2L_1, \) while clearly \( m_\beta \geq L_1 \geq 1. \) It follows that

\[
\frac{|\xi_1 - M|}{|\xi_2 - M|} \leq \frac{|I(m_\alpha + 9/8)|}{|J(m_\beta - 1/8)|} \leq \frac{1}{4}.
\]

Thus \( |\xi_1 - M| \leq |\xi_1 - \xi_2|/3, \) as desired. \( \square \)

A.3.2. Support of \( m_{BC} \). Consider the following enlargement of \( R_2 \) (defined in (9)):

\[
R_2' := \left\{(\xi_1, \xi_2) : |\xi_1 - \xi_2| \leq \frac{1}{2} \min\left(|\frac{1}{2}(\xi_1 + \xi_2) - M|, \left|\frac{1}{2}(\xi_1 + \xi_2) - N\right|\right)\right\}
\]

Lemma A.5. Let \( m_{BC} \) be defined by Definition 3.2. Then:

(i) \( m_{BC} \) is supported inside \( R_2' \).

(ii) Any summand of \( \alpha, \beta, \gamma \) whose support intersects \( R_2 \) must appear in \( m_{BC} \).

Proof. (i) Take any \( (\xi_1, \xi_2) \) in the support of \( m_{BC} \), then for some \( S \in S \) and \( I \in \mathbf{I}_{2M, 2N}(\alpha), \alpha \in A \), such that \( \ell(S) \leq |I| \) we have

\[
\phi_{1,S,k}(\xi_1)\phi_{2,S,k}(\xi_2)\phi_{3,S,k}(\xi_1 + \xi_2)\phi_{\alpha,I}(\xi_1 + \xi_2) \neq 0.
\]

It follows that \( (\xi_1, \xi_2) \in \frac{5}{6}S_1 \times \frac{5}{6}S_2 \), therefore using (10) we obtain

\[
|\xi_1 - \xi_2| = \sqrt{2}\text{dist}((\xi_1, \xi_2), \{\xi_1 = \xi_2\}) < 5L_2\ell(S).
\]

On the other hand, since \( \xi_1 + \xi_2 \in \frac{5}{4}I \) and \( m_\alpha, n_\alpha \geq L_1 \), we have

\[
\min\left(|\frac{1}{2}(\xi_1 + \xi_2) - M|, \left|\frac{1}{2}(\xi_1 + \xi_2) - N\right|\right) \geq \frac{1}{2}(L_1 - 1/8)|I|.
\]

Since \( \ell(S) \leq |I| \) and since \( L_2 = L_1/40 \), it is not hard to check that \( (\xi_1, \xi_2) \) satisfies the defining property of \( R_2' \).

(ii) Suppose that \( (\xi_1, \xi_2) \in R_2 \) such that

\[
\phi_{1,S,k}(\xi_1)\phi_{2,S,k}(\xi_2)\phi_{3,Q,k}(\xi_1 + \xi_2)\phi_{\alpha,I}(\xi_1 + \xi_2) \neq 0.
\]

We will show that \( \ell(S) \leq |I| \).

First, using \( (\xi_1, \xi_2) \in R_2 \) and using \( \xi_1 + \xi_2 \in (5/4)I \), it follows that

\[
|\xi_1 - \xi_2| \leq \frac{1}{100} \min\left(|\frac{1}{2}(\xi_1 + \xi_2) - M|, \left|\frac{1}{2}(\xi_1 + \xi_2) - N\right|\right)
\]

\[
\leq \frac{1}{100}(2L_1 + 1/8)|I|.
\]

Since \( (\xi_1, \xi_2) \in (4/5)S_1 \times (4/5)S_2 \), it follows that

\[
|\xi_1 - \xi_2| = \sqrt{2}\text{dist}((\xi_1, \xi_2), \{\xi_1 = \xi_2\}) \geq \sqrt{2}L_2\ell(S).
\]

Collecting estimates and using \( L_2 = L_1/40 \), it is clear that \( \ell(S) \leq |I| \), thus the corresponding summand is part of \( m_{BC} \). \( \square \)

A.3.3. Reduction to discrete \( T_{LM} \), part I: decomposition into simpler trilinear symbols. Recall that \( D_t \) is the dilation \( D_tA := \{2^t x : x \in A\} \).

Lemma A.6. \( m_{LM} \) can be decomposed into finitely many symbols of the form

\[
\sum_{\alpha, \beta, \gamma} \sum_{I \in D_{1,t}} \sum_{J \in D_{2,t}} \sum_{K \in D_{3,t}} 1_{\text{constraints on } I, J, L} \phi_{\alpha,I}(\xi_1)\phi_{\beta,J}(\xi_2)\phi_{\gamma,L}(\xi_1 - \xi_2) \times
\]

\[
1_{\{M \in lower, \alpha, N \in upper, \alpha\}} \times 1_{\{M \in lower, \alpha, N \in lower, \alpha\}} \times 1_{\{2M \in lower, \alpha, 2N \in upper, \alpha\}} ,
\]

- \( D_1, D_2, D_3 \) are three fixed collections of dyadic intervals;
• \( \phi_{1,1}, \phi_{2,1}, \phi_{3,1} \) are \( C^n \)-bump functions adapted to these intervals, with support in \( \frac{5}{4} I, \frac{5}{4} J, \frac{5}{4} L \); and \( n \) could be chosen arbitrarily large
• the constraints on \( I, J, L \) read as follows: the length \( I, J, L \) are comparable, and the distance between \( I, J, D_{-1} L \) are \( O(|I|) \).

Note that the dependence on \( M \) and \( N \) are only in the last three factors of the summands. To prove Lemma A.6, we analyze the factors in the factorization \( [42]. \)

**Part I: First factor.** Using Lemma A.4, it follows that

\[
\chi_{M \leq \xi_1 < \xi_2 < N - mCC} = \chi_{\xi_1 < \xi_2} (\chi_{M \leq \xi_1, \xi_2 < N - mCC}) \\
(44) = \chi_{\xi_1 < \xi_2} \sum_{\alpha, \beta \in A} \sum_{(I, J) \in \mathcal{I}(\alpha) \times \mathcal{I}(\beta)} \phi_{\alpha, I}(\xi_1) \phi_{\beta, J}(\xi_2),
\]

the constraints on \( (I, J) \in \mathcal{I}(\alpha) \times \mathcal{I}(\beta) \) read as follows:

• If \( \alpha \notin A_1 \) and \( \beta \notin A_3 \) then there are no constraints.
• If \( \alpha \in A_1 \) and \( \beta \in A_3 \) then no \( I, J \) are allowed.
• If \( \alpha \in A_1 \) and \( \beta \notin A_3 \) then one requires \( |I| > |J|/16 \).
• If \( \alpha \notin A_1 \) and \( \beta \in A_3 \) then one requires \( |J| > |I|/16 \).

Recall that \( \mathcal{H} \) is the union of \( \mathcal{I}(\alpha) \) over \( \alpha \in A \). We will show that

**Lemma A.7.** Assume that \((I, J) \in \mathcal{I}(\alpha) \times \mathcal{I}(\beta) \) contributes to the right hand side of \([44]\). Then (i) \( |I| \sim |J| \) and (ii) if \( \xi_1 < \xi_2 \) and \( \phi_{\alpha, I}(\xi_1) \phi_{\beta, J}(\xi_2) \neq 0 \) then

\[
\text{dist}(\{\xi_1, \xi_2\}, \{(M, M), (N, N)\}) \sim L_1|I|.
\]

**Proof.** (i) Without loss of generality we may assume that the ratio of the lengths of \( I \) and \( J \) is not in \( \{1, \frac{1}{2}, 2\} \). As the lengths of adjacent intervals in \( \mathcal{I} \) differ by a ratio of 1 or 2 or 1/2, it follows that \( I \neq J \) and they are not adjacent.

Now, we show that \( \sup I \leq \inf J \). Assume towards a contradiction that \( \sup J \leq \inf I \). Since there is at least one interval in \( \mathcal{H} \) between \( I \) and \( J \) and since the lengths of adjacent intervals in \( \mathcal{H} \) differ by a factor in \( \{1/2, 1, 2\} \), it follows that \( \inf \frac{5}{4} I > \sup \frac{5}{4} J \). Consequently, \((5/4)I \times (5/4)J \cap \{(\xi_1, \xi_2) : \xi_1 < \xi_2\} = \emptyset\), contradicting the fact that \((I, J) \) contributes to \([44]\).

Now, if \( \alpha, \beta \in A_2 \) then clearly \( |I| \) and \( |J| \) are comparable to \( |M - N|/L_1 \), so they are comparable. Therefore we may assume that either \( \alpha \notin A_2 \) or \( \beta \notin A_2 \).

We will show that

either \( \alpha \in A_1 \) or \( \beta \in A_3 \).

Note that from the constraints we must have \((\alpha, \beta) \notin A_1 \times A_3 \) (otherwise there won’t be any \((I, J)\)), thus the above two properties can not hold simultaneously. Assume towards a contradiction that \( \alpha \notin A_1 \) and \( \beta \notin A_3 \). Since \( \alpha \notin A_2 \) or \( \beta \notin A_2 \), it follows that \( \alpha \in A_3 \) or \( \beta \in A_1 \).

• If \( \alpha \in A_3 \) then using Lemma A.2 and \( \sup I \leq \inf J \) it follows that

\[ m_\beta/n_\beta > m_\alpha/n_\alpha \geq 4, \]

thus \( \beta \in A_3 \), contradict to the above assumption.

• If \( \beta \in A_1 \) then \( n_\alpha/n_\alpha \geq n_\beta/m_\beta \geq 4 \) and thus \( \alpha \in A_1 \), contradiction again.

Below, without loss of generality assume that \( \alpha \in A_1 \). Thus \( \beta \in A_1 \) or \( A_2 \).

If \( \beta \in A_1 \) then since \( J \) is on the right of \( I \) we easily have \( |I| \leq |J| \), while \( |J| < |I|16 \) by the above constraints. Thus \( |I| \sim |J| \).
If $\beta \in A_2$, then since $\bigcup_{\gamma \in A_1} I(\gamma)$ has $O(1)$ elements of comparable lengths, $|J|$ is comparable to $|I_0|$ the length of the right most interval inside $\bigcup_{\gamma \in A_1} I(\gamma)$. Note that either $I = I_0$ or $I$ is on the left of $I_0$, thus $|I| \preccurlyeq |I_0|$. Combining with the constraint $|I| < 16|I|$, we obtain $|J| \sim |I|$.

(ii) Let $\alpha, \beta \in A$ such that $I \in I(\alpha)$ and $J \in I(\beta)$. It is clear that

$$\text{dist}((\xi_1, \xi_2), \{(M, M), (N, N)\}) \sim \min(\xi_1 - M + \xi_2 - M, |\xi_1 - N| + |\xi_2 - N|).$$

If $(\alpha, \beta) \in (A_1 \cup A_2)^2$ then using $|I| \sim |J|$ we obtain

$$\min(\xi_1 - M + \xi_2 - M, |\xi_1 - N| + |\xi_2 - N|) \sim |\xi_1 - M| + |\xi_2 - M| \sim L_1|I|. $$

Similarly, if $(\alpha, \beta) \in (A_2 \cup A_3)^2$ then the desired claim follows. Since $(\alpha, \beta) \notin A_1 \times A_3$ (by the constraints), the remaining case is $(\alpha, \beta) \in A_3 \times A_1$, however this can’t happen either since $I$ is on the left of $J$. \hfill \square

Part II: The second factor. For convenience of notation, let $\xi_3 := -\xi_1 - \xi_2$. Thanks to Lemma A.5 we may write

$$\chi_{M < \xi_1, \xi_2 < N - m_{BC}} = \chi_{M < \xi_1, \xi_2 < N} \chi_{\xi_1, \xi_2 < 2M} \chi_{\xi_1 + \xi_2 < 2N} - m_{BC})$$

(45) = $\chi_{M < \xi_1, \xi_2 < N} \sum_{k, \alpha} a_k \sum_{S \in S} \sum_{L \in 2 \xi_1, 2 \xi_2 < N, 2} \phi_{j, S, k}(\xi_3) \prod_{j=1}^3 \phi_{j, S, k}(\xi_j)$.

Lemma A.8. Suppose that $(S, L) \in (S, L_{2M, 2N}(\alpha))$ and contributes to the right hand side of (45). Then (i) $\ell(S) \sim |L|$ and (ii) if $(\xi_1, \xi_2) \in (M, N)^2$ is in the support of the corresponding summand then

$$\text{dist}((\xi_1, \xi_2), \{(M, M), (N, N)\}) \sim L_1|L|.$$

Proof. (i) Note that $|L| \leq \ell(S)/2$ by definition, thus it remains to show $\ell(S) = O(|L|)$. Without loss of generality, assume that $|L - 2N| < |L - 2M|$. Then

$$\text{dist}(L, 2N) \sim L_1|L|.$$

Since $\phi_{j, S, k}$ is supported inside $(5/6)S_j$, it follows that $(5/6)S_1 \times (5/6)S_2 \cap (M, N)^2 \neq \emptyset$. Take any $(\xi_1, \xi_2)$ in this intersection. Then $\xi_1 < \xi_2 < N$, therefore using $(\xi_1 + \xi_2) \in \text{supp}(\phi_{\alpha, L}) = (5/6)L$, it follows that

$$|\xi_1 - \xi_2| \leq |\xi_1 - N| + |\xi_2 - N| = |\xi_1 + \xi_2 - 2N| \leq 3|L| + 2\text{dist}(L, 2N) \leq L_1|L|.$$

Now, let $\ell$ be the line $\{\xi_1 = \xi_2\}$. Since $(\xi_1, \xi_2) \in S_1 \times S_2$, it follows from (10) that

$$\ell(S) \leq L_2^{-1}\text{dist}((\xi_1, \xi_2), \ell) \leq L_2^{-1}|\xi_1 - \xi_2| \leq (L_1/L_2)|L|.$$

(Recall that $L_1 = O(L_2)$.) This completes the proof of (i).

(ii) Let $A$ be the distance from $(\xi_1, \xi_2)$ to the set of two corners $(M, M)$ and $(N, N)$. Then the desired lower bound for $A$ follows from

$$A \geq \text{dist}((\xi_1, \xi_2), \ell) \geq L_1\ell(S).$$

Using the triangle inequality and the Whitney property (10), we also have

$$A \leq \text{dist}((\xi_1, \xi_2), \ell) + \text{dist}((\xi_1 + \xi_2)/2, (M, N)) \leq L_2\ell(S) + L_1|L|.$$
Thanks to (i) we obtain the upper bound \( A \lesssim L_1 \ell(S) \), as desired. \( \square \)

Part III (final step). We now examine \( m_{LM} \). Using (44) and Lemma A.7 and Lemma A.8 it follows that \( m_{LM}(M, N, \xi_1, \xi_2) \) could be written as

\[
\begin{align*}
&= \chi_{\xi_1 < \xi_2} \chi_{M < \xi_1, \xi_2 < N} \left( \sum_{\alpha, \beta} \sum_{I, J} \phi_{\alpha, I}(\xi_1) \phi_{\beta, J}(\xi_2) 1_{\text{constraints on } I, J} \right) \times \\
&\quad \times \left( \sum_{k, \gamma} a_k \sum_{S, L} \phi_{1, S, k}(\xi_1) \phi_{2, S, k}(\xi_2) \phi_{3, S, k}(\xi_1 + \xi_2) \phi_{\gamma, L}(\xi_1 + \xi_2) 1_{\text{constraints on } S, L} \right)
\end{align*}
\]

here there are constraints on \( I, J, S, L \) relative to \( \alpha, \beta, \gamma \). Observe that the summation over \( (k, \gamma, S, L) \) is zero outside \( \{\xi_1 < \xi_2\} \), while the summation over \( (\alpha, \beta, I, J) \) is zero outside \( \{M < \xi_1, \xi_2 < N\} \). Therefore we could drop the factor \( \chi_{\xi_1 < \xi_2} \chi_{M < \xi_1, \xi_2 < N} \) in the right hand side and obtain

\[
m_{LM} = \sum_{\alpha, \beta, \gamma, k} a_k \sum_{I \in I, N(a)} \sum_{J \in I, N(\beta)} \sum_{L \in I, 2N(\gamma)} \sum_{S \in S} 1_{\text{constraints on } I, J, S, L} \times \\
\quad \times \left[ \phi_{\alpha, I}(\xi_1) \phi_{1, S, k}(\xi_1) \right] \times \left[ \phi_{\beta, J}(\xi_2) \phi_{2, S, k}(\xi_2) \right] \times \left[ \phi_{\gamma, L}(\xi_1 + \xi_2) \phi_{3, S, k}(\xi_1 + \xi_2) \right].
\]

Since \( a_k \) decays rapidly, for the purpose of proving Lemma A.6 we may drop the summation over \( k \) and consider only the contribution of one \( k \).

Recall that \( D_y \) denotes the dilation by \( 2^y \) with respect to 0, i.e. \( D_y \xi = 2^y \xi \). In particular \( D_{-1} L = \{ x : x \in L \} \). We claim that in any non-zero summand in \( m_{LM} \), it holds that

(i) \(|I| \sim |J| \sim \ell(S) \sim |L| \).

(ii) The spatial distances between any two of \( I, J, S_1, S_2, D_{-1}(S_3), D_{-1}(L) \) are bounded above by \( O(L_1 |I|) \).

Indeed, (i) follows from Lemma A.7 and Lemma A.8

\[
\ell(S) \sim |L| \sim \frac{1}{L_1} \text{dist}((\xi_1, \xi_2), (M, M), (N, N)) \\
|I| \sim |J| \sim \frac{1}{L_1} \text{dist}((\xi_1, \xi_2), (M, M), (N, N))
\]

For (ii), by the Whitney property (10) \( \text{dist}(S_1, S_2) = O(L_2 \ell(S)) \), thus the distances between \( D_{-1}(S_3) \) and \( S_1, S_2 \) are \( O(L_1 \ell(S)) \). The desired claim now follows from examining the factors in the summation.

It follows that by decomposing \( m_{LM} \) into \( O(1) \) sums we may assume that \( J, L, S_1, S_2, S_3 \) are completely determined from \( I \): comparable length and nearby location.

Since \( a_k \) decays rapidly we can ignore the summation over \( k \), and below we will even drop the dependence on \( k \) of the inner sums for brevity of notations. We end up with a symbol of the form

\[
\sum_{I \in D_1} \sum_{J \in D_2} \sum_{L \in D_3} \psi_{1, I}(\xi_1) \psi_{2, J}(\xi_2) \psi_{3, L}(\xi_1 + \xi_2) 1_{\text{constraints}}
\]

here \( D_1, D_2, D_3 \) could be \( D_{left} \) or \( D_{right} \), are three fixed collections of (standard) dyadic intervals, \( \psi_{1, I} \) is a \( C^n \) bump function adapted to \( I \) and is supported on \( \frac{2}{3} I \), and \( \psi_{2, J} \) and \( \psi_{3, L} \) satisfy similar properties. The contraints read as follows: for fixed bounded integers \( m_1, n_1, m_2, n_2 \) it holds that
This completes the proof of Lemma A.6. □

A.3.4. Reduction to $T_{LM}$, part II: completion of the proof. Using Lemma A.6, we may decompose $m_{LM}$ into boundedly many $m_{m_1,n_1,m_2,n_2}$, defined by

$$\sum_{\alpha,\beta,\gamma} \sum_{I,J,L} \psi_{1,I}(\xi_1) \psi_{2,J}(\xi_2) \psi_{3,L}(\xi_1 + \xi_2) 1_{\text{constraints on } I,J,L} \times$$

$$\times 1_{\{M \in \text{lower,}\alpha, N \in \text{upper,}\alpha\}} 1_{\{M \in \text{lower,}\beta, N \in \text{upper,}\beta\}} 1_{\{2M \in \text{lower,}\gamma, 2N \in \text{upper,}\gamma\}}$$

and the ‘constraints’ on $I, J, L$ specify the location and the length of $J$ and $L$ relative to $I$ using $m_1, n_1, m_2, n_2$ as discussed in the last section. Note that the decomposition of $m_{LM}$ is independent of $M$ and $N$.

For each fixed $I, J, L$, we will sum the summands over $\alpha, \beta, \gamma$. Using Lemma A.3, we will divide $m_{m_1,n_1,m_2,n_2}$ into 8 symbols, such that in the summation each of $I, J, L$ are required to be in one of $D_{left}, D_{right}$, and each of these 8 symbols could be further decomposed into $O(L_1)$ symbols having the following structure:

$$\sum_{I \in D_1} \sum_{J \in D_2} \sum_{L \in D_3} \psi_{1,I}(\xi_1) \psi_{2,J}(\xi_2) \psi_{3,L}(-\xi_1 - \xi_2) \times$$

$$\times 1_{\{M \in \text{lower,}\alpha, N \in \text{lower,}\alpha\}} 1_{\{M \in \text{lower,}\beta, N \in \text{lower,}\beta\}} 1_{\{2M \in \text{lower,}\gamma, 2N \in \text{lower,}\gamma\}}$$

where $\psi_j$ are bump functions supported in $[-c, c]$, $(1/2 < c < 5/8)$ and the collection of interval triples $\{(I_{lower}, I, I_{upper}), I \in D_1\}$ is rigid, and the same holds for the other two collections of interval triples. Note that the rigidity type of these collections are not opposite, i.e. we won’t have both (infinite, finite) and (finite, infinite). Due to the flexibility of (finite, finite) rigidity (which could be converted into two terms of any of the other types), we could assume that the rigidity type of the three collections are the same.

We now split the intervals $I, J, L$ if necessary to ensure $|I| = |J| = |L|$: to do this we will split $\psi_{1,I}, \psi_{2,J}, \psi_{3,L}$ and we will correspondingly split all bounded intervals in $I_{lower}, I_{upper}, J_{lower}, J_{upper}, L_{lower}, L_{upper}$ (but we won’t split the half lines). Note that if we split $\psi_{1,I}$ into $2^k$ bump function adapted to the corresponding subintervals of $I$ ($k \geq 0$), then each new bump function in theory could be supported in an interval as large as the $(1 + 2^k(c - 1/2))$ enlargement of the adapting subinterval. Since $k$ is bounded, by choosing $c$ sufficiently close to 1/2 when partitioning $1_{M < \xi < N}$ we could ensure that $1 + 2^k(c - 1/2) < 5/4$.

After the splitting, the rest of the discretization is similar to the discretization of $m_{BC}$. We omit the details.

References

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