Today: $R$ denotes a commutative ring.

4. Tensor products and bilinear maps

Definition. Let $M$ and $N$ be $R$-modules and $L$ an abelian group.

(a) A map $\varphi : M \times N \to L$ is called $R$-balanced if

(i) $\varphi(m_1 + m_2, n) = \varphi(m_1, n) + \varphi(m_2, n)$

(ii) $\varphi(m, n_1 + n_2) = \varphi(m, n_1) + \varphi(m, n_2)$

(iii) $\varphi(m, rn) = \varphi(rm, n)$ for any $r \in R$, $m \in M$, $n \in N$

(b) Now suppose that $L$ is also an $R$-module. Then a map $\varphi : M \times N \to L$ is called $R$-bilinear if $\varphi$ is $R$-balanced and $\varphi(m, rn) = r\varphi(m, n)$.

Example 4.1: The map $\iota : M \times N \to M \otimes_R N$ given by $\iota(m, n) = m \otimes n$ is $R$-bilinear. This follows from defining relations of tensor products.

Example 4.2: Suppose $M$ and $N$ are finitely generated free $R$-modules. Let $\{x_1, \ldots, x_k\}$ be a basis of $M$ and $\{y_1, \ldots, y_t\}$ be a basis of $N$.

Let $L$ be another $R$-module, and choose arbitrary elements $l_{ij} \in L$, with $1 \leq i \leq k$ and $1 \leq j \leq t$. Then there exists unique $R$-bilinear map $\varphi : M \times N \to L$ such that $\varphi(x_i, y_j) = l_{ij}$. In fact, $\varphi$ is given by the formula

$$\varphi(\sum r_i x_i, \sum s_j y_j) = \sum_{i,j} r_i s_j l_{ij}.$$ 

Theorem 4.1. Let $M$ and $N$ be $R$-modules and $L$ an abelian group. Then there is a bijection

$$\Omega = \left\{ \begin{array}{l} R\text{-balanced maps} \\ \varphi : M \times N \to L \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{group homomorphisms} \\ \Phi : M \otimes_R N \to L \end{array} \right\} = \Delta$$

which maps an $R$-balanced map $\varphi \in \Omega$ to a group homomorphism $\Phi \in \Delta$ such that $\Phi(m \otimes n) = \varphi(m, n)$ for any $m \in M$, $n \in N$

Proof. “$\longrightarrow$” (a map $f : \Omega \to \Delta$). Recall that $M \otimes_R N = F/I$ where $F$ is the free abelian group on $M \times N$ and $I$ is the subgroup generated by $\{(m, n_1 + n_2) - (m, n_1) - (m, n_2)\}$ etc. 

Now let $\varphi : M \times N \to L$ be $R$-balanced. Since $F$ is a free $\mathbb{Z}$-module on $M \times N$, there is a group homomorphism $\Phi : F \to L$ such that $\Phi((m, n)) = \varphi(m, n)$. Then $I \subset \ker \Phi$ precisely because $\varphi$ is $R$-balanced. For instance,

$$\Phi((m, n_1 + n_2) - (m, n_1) - (m, n_2)) = \Phi((m, n_1 + n_2)) - \Phi((m, n_1)) - \Phi((m, n_2))$$

$$= \varphi(m, n_1 + n_2) - \varphi(m, n_1) - \varphi(m, n_2) = 0,$$
where the first equality holds since $\Phi$ is a group homomorphism and the last equality holds since $\varphi$ is $R$-balanced.
Thus, $\Phi$ induces a group homomorphism $\Phi: M \otimes_R N = F/I \to L$ such that $\Phi(m \otimes n) = \varphi(m, n)$. We set $f(\varphi) = \Phi$.

$\Phi$ (a map from $g: \Delta \to \Omega$). This is easy – just set $(g(\Phi))(m, n) = \Phi(m \otimes n)$.

Then $g(\Phi)$ is $R$-balanced by defining relations in $M \otimes_R N$.

Thus, we defined two maps $f: \Omega \to \Delta$ and $g: \Delta \to \Omega$, and it remains to check that $f$ and $g$ are mutually inverse.

By construction for any $\varphi \in \Omega$ we have $g(f(\varphi))(m, n) = f(\varphi)(m \otimes n) = \varphi(m, n)$.

Thus $g \circ f = id_{\Omega}$. Similarly, $f(g(\Phi))(m \otimes n) = \Phi(m \otimes n)$. Since a group homomorphism $M \otimes_R N \to L$ is uniquely determined by its values on simple tensors, we must have $f \circ g = id_{\Delta}$.

Here is the variation of Theorem 4.1 dealing with bilinear maps.

**Theorem 4.2.** Let $M$ and $N$ be $R$-modules and $L$ another $R$-module. Then there is a bijection

$$\Omega = \left\{ \begin{array}{c} \text{R-bilinear maps} \\ \varphi: M \times N \to L \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{R-module homomorphisms} \\ \Phi: M \otimes_R N \to L \end{array} \right\} = \Delta$$

which maps a R-bilinear map $\varphi \in \Omega$ to an R-module homomorphism $\Phi \in \Delta$ such that $\Phi(m \otimes n) = \varphi(m, n)$ for any $m \in M$, $n \in N$.

**Proof.** Very similar to that of Theorem 4.1.

**4.1. Applications of Theorem 4.2.**

**Example 4.3:** Let $M, N$ be finitely generated free $R$-modules, $X = \{x_1, \ldots, x_k\}$ a basis of $M$ and $Y = \{y_1, \ldots, y_t\}$ a basis of $N$. Then $\{x_i \otimes y_j\}$ is a basis of $M \otimes_R N$.

**Remark:** Finite generation assumption is not essential.

**Proof.** We know from Example 3.2 that $\{x_i \otimes y_j\}$ generates $M \otimes_R N$, so we only need to check linear independence. Suppose that $\sum_{i,j} r_{ij} x_i \otimes y_j = 0$ where $r_{ij} \in R$ and not all $r_{ij}$ are zero.

WOLOG $r_{11} \neq 0$. By Example 4.2 there exists an $R$-bilinear map $\varphi: M \times N \to R$ such that $\varphi((x_1, y_1)) = 1$ and $\varphi((x_i, y_j)) = 0$ if $(i, j) \neq (1, 1)$.
By Theorem 4.2 there is an $R$-module homomorphism $\Phi: M \otimes N \to R$ such that $\Phi(x_i \otimes y_j) = \varphi((x_i, y_j))$. Then

$$\Phi(\sum_{i,j} r_{ij} x_i \otimes y_j) = \sum_{i,j} r_{ij} \varphi(x_i, y_j) = r_{11} \neq 0,$$
which is a contradiction. □

Example 4.4: Prove that for any $R$-module $M$ we have $R \otimes_R M \cong M$ (as $R$-modules).

Claim. Any element of $R \otimes_R M$ is equal to $1 \otimes m$ for some $m \in M$.

Proof of the claim. Any element of $R \otimes_R M$ can be written as

$$\sum r_i \otimes m_i = \sum (1 \cdot r_i) \otimes m_i = \sum 1 \otimes r_i m_i = 1 \otimes \sum r_i m_i.$$  

□

Proof of Example 4.4: Define the map $\Phi : M \rightarrow R \otimes_R M$ by $\Phi(m) = 1 \otimes m$. Clearly, $\Phi$ is an $R$-module homomorphism, and $\Phi$ is surjective by the Claim. To prove that $\Phi$ is injective it is enough to show that $1 \otimes m \neq 0$ for $m \neq 0$. By Theorem 4.2 it is enough to find an $R$-bilinear map $\varphi : R \times M \rightarrow M$ such that $\varphi(1, m) \neq 0$. The map $\varphi(r, m) = rm$ has such property. □

Proposition 4.3. Let $M, N, P$ be $R$-modules. Then there exist natural $R$-module isomorphisms

(i) $M \otimes N \cong N \otimes M$;
(ii) $(M \otimes N) \otimes P \cong M \otimes (N \otimes P)$
(iii) $(M \otimes N) \otimes P \cong (M \otimes P) \oplus (N \otimes P)$

Proof. We shall prove (i); see [DF] for (ii) and (iii). Consider the map $\varphi : M \times N \rightarrow N \otimes M$ given by $\varphi(m, n) = n \otimes m$. The map $\varphi$ is clearly $R$-bilinear, and thus there exists an $R$-module homomorphism $f : M \otimes N \rightarrow N \otimes M$ such that $f(m \otimes n) = n \otimes m$ for each $m \in M, n \in N$. Similarly, there is an $R$-module homomorphism $g : N \otimes M \rightarrow M \otimes N$ such that $g(n \otimes m) = m \otimes n$. The composition $gf : M \otimes N \rightarrow M \otimes N$ is an $R$-module homomorphism, which fixes all simple tensors and hence fixes everything. So $gf = id_M$, and similarly $fg = id_N$ □

4.2. Generalizations of Theorems 4.1 and 4.2. Theorems 4.1 and 4.2 have natural generalizations dealing with multilinear maps. For instance, here is the generalization of Theorem 4.2.

Definition. Let $k \geq 2$ and let $M_1, \ldots, M_k$ and $L$ be $R$-modules. A map $\varphi : M_1 \times \ldots \times M_k \rightarrow L$ is called $R$-multilinear if for any $1 \leq i \leq k$ we have

$$\varphi(m_1, \ldots, m_{i-1}, m_i + rm_i', m_{i+1}, \ldots, m_k) =$$

$$\varphi(m_1, \ldots, m_{i-1}, m_i, m_{i+1}, \ldots, m_k) + r\varphi(m_1, \ldots, m_{i-1}, m_i', m_{i+1}, \ldots, m_k)$$

for all $m_j \in M_j, 1 \leq j \leq k, m_i' \in M_i$ and $r \in R$. 

Theorem 4.2'. Then there is a bijection $\varphi \leftrightarrow \Phi$ between

\[
\left\{ \text{R-multilinear maps} \quad \varphi : M_1 \times M_2 \times \ldots \times M_k \to L \right\} \quad \text{and} \quad \left\{ \text{R-module homomorphisms} \quad \Phi : M_1 \otimes M_2 \otimes \ldots \otimes M_k \to L \right\}
\]

s.t. $\Phi(m_1 \otimes m_2 \otimes \ldots \otimes m_k) = \varphi(m_1, m_2, \ldots, m_k)$ for all $m_i \in M_i, 1 \leq i \leq k$. 