25. Some category theory

25.1. **Categories.** A category $\mathcal{C}$ consists of the following data:

- objects $\text{Ob}(\mathcal{C})$
- for any $X, Y \in \text{Ob}(\mathcal{C})$ a set $\text{Mor}(X, Y) = \text{Mor}_\mathcal{C}(X, Y)$ called morphisms from $X$ to $Y$
- for any triple $X, Y, Z \in \text{Ob}(\mathcal{C})$ a map

$$\text{Mor}(X, Y) \times \text{Mor}(Y, Z) \to \text{Mor}(X, Z)$$

$$(f, g) \mapsto g \circ f$$

called the composition law of morphisms.

The following axioms must be satisfied:

1. The sets $\text{Mor}(X, Y)$ and $\text{Mor}(X', Y')$ are disjoint unless $X = X'$ and $Y = Y'$.
2. Composition of morphisms is associative, that is, for any $f \in \text{Mor}(X, Y)$, $g \in \text{Mor}(Y, Z)$ and $h \in \text{Mor}(Z, W)$ we have

$$h \circ (g \circ f) = (h \circ g) \circ f.$$
3. For any $X \in \text{Ob}(\mathcal{C})$ there is identity morphism $1_X \in \text{Mor}(X, X)$ with the following property: if $Y$ is any object of $\mathcal{C}$, then $f \circ 1_X = f$ for any $f \in \text{Mor}(X, Y)$ and $1_X \circ g = g$ for any $g \in \text{Mor}(Y, X)$.

**Notation:** We will often write $f : X \to Y$ instead of $f \in \text{Mor}(X, Y)$.

Here are some basic examples of categories.

**Examples:**

1. $\mathcal{C} = \text{SET}$, the category of sets. Objects of $\mathcal{C}$ are arbitrary sets and $\text{Mor}(X, Y) = \text{Func}(X, Y)$, all functions from $X$ to $Y$. The composition of morphisms is the usual composition of functions.

2. $\mathcal{C} = \text{GRP}$, the category of groups. Objects are all groups, $\text{Mor}(X, Y)$ is the set of groups homomorphisms from $X$ to $Y$, the composition of morphisms is the usual composition of functions.

3. $\mathcal{C} = \text{TOP}$, the category of topological spaces. Objects are topological spaces, $\text{Mor}(X, Y)$ is the set of continuous functions from $X$ to $Y$, the composition of morphisms is the usual composition of functions.

Here is an example of rather different kind.

4. Let $A$ be a poset with partial order relation $\leq$. Then we can consider the following category $\mathcal{C}$. The objects of $\mathcal{C}$ are simply elements of $A$, and
morphisms are defined by setting
\[ \text{Mor}(x, y) = \begin{cases} \emptyset & \text{if } x \not\leq y \\ \text{The one element set consisting of the pair } (x, y) & \text{if } x \leq y. \end{cases} \]
The composition of morphisms \( \text{Mor}(x, y) \times \text{Mor}(y, z) \to \text{Mor}(x, z) \) is defined as follows:

(i) If \( x \not\leq y \) or \( y \not\leq z \), then \( \text{Mor}(x, y) \times \text{Mor}(y, z) = \emptyset \), so there exists unique map \( \text{Mor}(x, y) \times \text{Mor}(y, z) \to \text{Mor}(x, z) \) (the “do nothing” map)

(ii) If \( x \leq y \) and \( y \leq z \), then \( x \leq z \) by transitivity, so \( |\text{Mor}(x, y)| = |\text{Mor}(y, z)| = |\text{Mor}(x, z)| = 1 \). Again there exists unique map \( \text{Mor}(x, y) \times \text{Mor}(y, z) \to \text{Mor}(x, z) \) given by \((x, y), (y, z)) \mapsto (x, z)\).

Finally, associativity of composition is automatic and identity morphisms \( 1_x \) exist since \( x \leq x \).

25.2. Products and coproducts.

**Definition.** Let \( C \) be a category and \( \{X_\alpha\} \) a collection of objects of \( C \). An object \( X \in \text{Ob}(C) \) is called a product of \( \{X_\alpha\} \) denoted \( \prod_C X_\alpha \) if there exist morphisms \( \pi_\alpha : X \to X_\alpha \) for each \( \alpha \) s.t. for any \( Y \in \text{Ob}(C) \) and any morphisms \( \varphi_\alpha : Y \to X_\alpha \) there is unique morphism \( \varphi : Y \to X \) s.t. for each \( \alpha \) we have \( \varphi_\alpha = \pi_\alpha \varphi \), or equivalently, the following diagram is commutative:

\[
\begin{array}{ccc}
X & \longrightarrow & X \\
\downarrow{\pi_\alpha} & & \downarrow{\pi_\alpha} \\
Y & \longrightarrow & X_\alpha \\
& \varphi \uparrow & \\
& Y & \varphi_\alpha \\
\end{array}
\]

A standard argument shows that if a product \( \prod_C X_\alpha \) exists, it is unique up to \( C \)-isomorphism; however, a product need not exist in general.

**Examples:** (1) Let \( C \) be the category of sets (resp. groups, abelian groups, rings). Then \( \prod_C X_\alpha \) always exists and coincides with the usual direct product of sets (resp. groups, abelian groups, rings).

(2) Let \( C \) be the category of fields (with morphisms being field embeddings). Then products in \( C \) do not always exist (in fact, almost never exist).

Coproducts are defined in the same way as products with all arrows reversed:

**Definition.** Let \( C \) be a category and \( \{X_\alpha\} \) a collection of objects of \( C \). An object \( X \in \text{Ob}(C) \) is called a coproduct of \( \{X_\alpha\} \) denoted \( \sqcup_C X_\alpha \) if there exist morphisms \( \iota_\alpha : X_\alpha \to X \) for each \( \alpha \) s.t. for any \( Y \in \text{Ob}(C) \) and any morphisms \( \varphi_\alpha : X_\alpha \to Y \) there is unique morphisms \( \varphi : X \to Y \) s.t. for
each $\alpha$ we have $\varphi_{\iota\alpha} = \varphi_\alpha$, that is, the following diagram is commutative:

\[
\begin{array}{c}
Y \\
\varphi_\alpha \\
X_\alpha \xrightarrow{\iota_\alpha} X
\end{array}
\]

Unlike products, coproducts in familiar categories have rather different descriptions.

Examples: (1) Let $\mathcal{C}$ be the category of sets. Then $\sqcup_{\mathcal{C}} X_\alpha$ is the disjoint union of $\{X_\alpha\}$ (as the notation suggests).

(2) Let $\mathcal{C}$ be the category of groups. Then $\sqcup_{\mathcal{C}} X_\alpha = \star X_\alpha$, the free product of $\{X_\alpha\}$. Informally, this means that given $\alpha' \neq \alpha$, there are no relations between the images of $X_\alpha$ and $X_{\alpha'}$ inside $\star X_\alpha$.

(3) Let $\mathcal{C}$ be the category of abelian groups. Then $\sqcup_{\mathcal{C}} X_\alpha = \oplus X_\alpha$, the direct sum of $\{X_\alpha\}$.

(4) Let $R$ be a commutative ring with 1, and let $\mathcal{C} = R -$ COMMALG be the category of commutative $R$-algebras. Then $\sqcup_{\mathcal{C}} X_\alpha = \otimes X_\alpha$, the tensor product of $\{X_\alpha\}$.

25.3. Motivating direct limits. Let $Y$ be a set and let $\{X_\alpha\}_{\alpha \in A}$ be a collection of subsets of $Y$ which form a chain, that is, for any $\alpha, \beta$ we have $X_\alpha \subseteq X_\beta$ or $X_\beta \subseteq X_\alpha$. Then we can consider $X = \sqcup X_\alpha$, the union of $X_\alpha$ as subsets of $Y$. Our goal is to find a characterization of $X$ similar to that of the disjoint union $\sqcup X_\alpha$.

Let $\leq$ be the order relation on the index set $A$ defined by $\alpha \leq \beta$ if and only if $X_\alpha \subseteq X_\beta$. Note that $\leq$ is a total order on $A$ since $\{X_\alpha\}$ is a chain.

For each $\alpha, \beta \in A$ with $\alpha \leq \beta$ let $\iota_{\alpha, \beta} : X_\alpha \to X_\beta$ be the inclusion map. Note that for any $\alpha \leq \beta \leq \gamma$ the following diagram is commutative:

\[
\begin{array}{c}
X_\alpha \\
\downarrow^{\iota_{\alpha, \beta}} \\
X_\beta \xrightarrow{\iota_{\beta, \gamma}} X_\gamma
\end{array}
\]

Now suppose we are given another set $Y$ and maps $\varphi_\alpha : X_\alpha \to Y$ for each $\alpha \in A$. The natural question is when does there exist a map $\varphi : X = \sqcup X_\alpha \to Y$ s.t. $\varphi|_{X_\alpha} = \varphi_\alpha$ for $\alpha \in A$?
Clearly, such \( \varphi \) exists if and only if \((\varphi_\beta)_{|X_\alpha} = \varphi_\alpha\) for any \( \alpha \leq \beta \). Equivalently, \( \varphi \) exists if and only if for any \( \alpha \leq \beta \) the following diagram is commutative:

\[
\begin{array}{c}
X_\alpha \\
\downarrow_{\iota_{\alpha,\beta}} \\
X_\beta \\
\downarrow_{\varphi_\beta} \\
X
\end{array}
\]

Thus the union \( X = \cup X_\alpha \) satisfies certain universal property similar to the one in the definition of coproduct, except that instead of considering arbitrary collections of morphisms \( \varphi_\alpha : X_\alpha \to Y \) (where \( Y \) is another set), one only considers the collections satisfying the compatibility condition (25.3). This analysis provides a motivation for the concept of direct limit, which will be given in the next lecture.