Let $F$ be a field, $V$ a f.d. vector space over $F$ and $T \in \mathfrak{gl}(V)$ an $F$-linear transformation from $V$ to $V$. As before let $V_T$ denote $V$ considered as $F[x]$-module where $x$ acts as $T$.

Recall that the existence of the rational canonical form (RCF) of $T$ was derived from the invariant factors decomposition of the $F[x]$-module $V_T$. Today we shall use the decomposition of $V_T$ in the elementary divisors form to establish the existence of the Jordan canonical form (JCF). While RCF exists over any field, to ensure the existence of JCF we need to assume that $F$ is algebraically closed. This is not a very serious restriction since any field can be embedded into an algebraically closed field – we will show this later in the course.

**Definition.** A field $F$ is called algebraically closed if any non-constant polynomial in $F[x]$ has a root in $F$. Equivalently, $F$ is algebraically closed if any irreducible polynomial in $F[x]$ has degree 1.

**12.1. Existence and uniqueness of Jordan canonical form.** So, let $F$ be an algebraically closed field, $V$ a f.d. $F$-vector space and $T \in \mathfrak{gl}(V)$. We apply the classification of modules over PID in ED form to the module $V_T$. Since all irreducible polynomials in $F[x]$ are linear, we get that there exist $\lambda_1, \ldots, \lambda_k \in F$ (not necessarily distinct) and positive integers $d_1, \ldots, d_k$ such that

$$V_T = V_1 \oplus \ldots \oplus V_k \text{ where } V_i \cong F[x]/(x - \lambda_i)^{d_i} \text{ as } F[x]-\text{modules.}$$

As in RCF case, each $V_i$ is $T$-invariant. If we let $T_i = T|_{V_i} \in \mathfrak{gl}(V_i)$, choose a basis $\Omega_i$ of $V_i$ for each $i$ and let $\Omega = \Omega_1 \sqcup \ldots \sqcup \Omega_k$, then $\Omega$ is a basis of $V$ and

$$[T]_{\Omega} = \begin{pmatrix}
[T_1]_{\Omega_1} & 0 & \cdots & 0 \\
0 & [T_2]_{\Omega_2} & \cdots & 0 \\
\vdots & \vdots & \ddots & 0 \\
0 & 0 & \cdots & [T_k]_{\Omega_k}
\end{pmatrix}$$

Thus, as with RCF we are reduced to the case when

$$V_T \cong F[x]/(x - \lambda)^d \text{ as } F[x]-\text{modules.}$$

For each $0 \leq i \leq d - 1$ let $e_i = (x - \lambda)^i$ where $p(x)$ is the image of $p(x)$ in $V_T$. Let $\Omega = \{e_{d-1}, \ldots, e_0\}$ (in this order!) Then $\Omega$ is an $F$-basis of $V$, and
the action of $T$ on $\Omega$ is given by

$$T(e_i) = x(x - \lambda)^i = (x - \lambda)^{i+1} + \lambda(x - \lambda)^i = \lambda e_i + e_{i+1} \text{ if } i < d - 1$$

and

$$T(e_{d-1}) = \lambda e_{d-1}.$$  

So,

$$[T]_\Omega = \begin{pmatrix} \lambda & 1 & \ldots & 0 & 0 \\ 0 & \lambda & 1 & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ 0 & \ldots & \ldots & \lambda & 1 \\ 0 & \ldots & \ldots & 0 & \lambda \end{pmatrix}$$

This $d \times d$ matrix will be denoted by $J(d, \lambda)$. Matrices of this form are called Jordan blocks.

**Definition.** A matrix $A \in \text{Mat}_n(F)$ is said to be in Jordan canonical form if $A$ is block-diagonal where each block is a Jordan block.

**Theorem 12.1** (Existence and uniqueness of JCF). Let $V$ be a f.d. vector space over an algebraically closed field and $T \in \text{gl}(V)$. Then there is a basis $\Omega$ of $V$ and a matrix $J$ in Jordan canonical form s.t. $[T]_\Omega = J$. The matrix $J$ is called the Jordan canonical form of $T$ – it is unique up to permutation of Jordan blocks.

**Proof.** The existence part is proved above. The uniqueness part can be deduced from the uniqueness part in the PID classification theorem (ED form), but we shall also give a different argument later on. □

Here is the matrix form of Theorem 12.1:

**Theorem 12.1’.** Assume that $F$ is algebraically closed. Then any matrix $A \in \text{Mat}_n(F)$ is similar to a matrix in JCF, which is unique up to permutation of Jordan blocks.

From now on we will primarily state our results for matrices (instead of linear transformations).

12.2. **Relation of JCF to minimal and characteristic polynomial.**

**Lemma 12.2.** Let $F$ be an algebraically closed field and $A \in \text{Mat}_n(F)$. Let $\text{Spec}(A)$ be the set of all eigenvalues of $A$.

(a) Let $\lambda \in F$. Then $\lambda \in \text{Spec}(A)$ if and only if $\text{JCF}(A)$ contains at least one Jordan $\lambda$-block (that is, a block of the form $J(d, \lambda)$).

(b) The characteristic polynomial $\chi_A(x)$ is equal to $\prod_{\lambda \in \text{Spec}(A)}(x - \lambda)^{s_\lambda}$ where $s_\lambda$ is the sum of sizes of all Jordan $\lambda$-blocks in $\text{JCF}(A)$.
(c) The minimal polynomial \( \mu_A(x) \) is equal to \( \prod_{\lambda \in \text{Spec}(A)} (x - \lambda)^{m_\lambda} \) where \( m_\lambda \) is the maximum size of a Jordan \( \lambda \)-block in JCF(A).

Proof. Since \( \text{Spec}(A) \), \( \chi_A(x) \) and \( \mu_A(x) \) do not change under conjugation, we can assume that \( A \) is in JCF. In this case (a) and (b) are obvious.

It is also clear that \( \mu_A(x) \) is the least common multiple of the minimal polynomials of the Jordan blocks. Thus, to prove (c) it is enough to show that if \( A = J(d, \lambda) \) for some \( d \in \mathbb{N} \) and \( \lambda \in F \), then \( \mu_A(x) = (x - \lambda)^d \), and the latter can be verified by a simple computation. \( \square \)

12.3. Computing JCF using ranks. Let \( F \) be an algebraically closed field and \( A \in \text{Mat}_n(F) \). We can determine the eigenvalues of \( A \) by computing the characteristic polynomial \( \chi_A(x) \). The sizes of the Jordan blocks are often easy to compute via the following observation.

**Lemma 12.3.** For each \( \lambda \in F \) and \( k \in \mathbb{N} \) let \( n_A(k, \lambda) \) be the number of Jordan \( \lambda \)-blocks of size \( \geq k \) in JCF(A). Then

\[
 n_A(k, \lambda) = \text{rk}((A - \lambda I)^{k-1}) - \text{rk}((A - \lambda I)^k)
\]

(***)

(here \( I \) is the identity matrix of suitable size)

**Remark:** Formula (***), holds for all \( \lambda \in F \), not only for \( \lambda \in \text{Spec}(A) \).

**Proof.** As before, we can assume that \( A \) is in JCF.

**Case 1:** \( A \) has just one Jordan block, that is, \( A = J(d, \mu) \).

If \( \mu \neq \lambda \), then clearly \( n_A(k, \lambda) = 0 \) for each \( k \), and on the other hand the matrix \( (A - \lambda I)^k \) has maximal rank being invertible. Thus, (***), holds.

If \( \lambda = \mu \), then \( n_A(k, \lambda) = 1 \) for \( k \leq d \) and \( n_A(k, \lambda) = 0 \) for \( k > d \). On the other hand, direct computation shows that

\[
 \text{rk}((A - \lambda I)^k) = \begin{cases} 
 d - k & \text{if } k \leq d \\
 0 & \text{if } k > d 
\end{cases}
\]

Again, we see that (***), holds for each \( k \).

**Case 2:** \( A \) has more than one Jordan block.

We can put \( A \) into a block-diagonal form with non-trivial blocks \( B \) and \( C \). Note that

(i) \( n_A(k, \lambda) = n_B(k, \lambda) + n_C(k, \lambda) \)

(ii) \( \text{rk}((A - \lambda I)^k) = \text{rk}((B - \lambda I)^k) + \text{rk}((C - \lambda I)^k) \)

By induction (***), holds for both \( B \) and \( C \). Combining this fact with (i) and (ii), we get that (***), also holds for \( A \). \( \square \)

**Remark:** Lemma 12.3 gives an alternative proof of the uniqueness of JCF (up to permutation of Jordan blocks).
Corollary 12.4. Any matrix \( A \in \text{Mat}_n(F) \) is similar to its transpose \( A^T \).

Proof. Let \( F' \) be an algebraically closed field containing \( F \). By Corollary 10.3 it is enough to prove that \( A \) and \( A^T \) are similar in \( \text{Mat}_n(F') \).

Note that for each \( \lambda \in F' \) and \( k \in \mathbb{N} \) we have \(((A - \lambda I)^k)^T = (A^T - \lambda I)^k\) (since \((BC) = C^T B^T\)), and therefore

\[
\text{rk}((A^T - \lambda I)^k) = \text{rk}((A - \lambda I)^k).
\]

Lemma 12.3 now implies that \( A \) and \( A^T \) have the same JCF (over \( F' \)) and hence must be similar in \( \text{Mat}_n(F') \). \( \square \)

12.4. Root subspaces. In the last two subsections we fix an algebraically closed field \( F \), a f.d. vector space \( V \) over \( F \) and \( T \in \mathfrak{gl}(V) \). For each \( \lambda \in F \) let

\[
V_\lambda = \{ v \in V : (T - \lambda I)^k v = 0 \text{ for some } k \in \mathbb{N} \}.
\]

The subspaces \( V_\lambda \) are called root subspaces (or generalized eigenspaces) of \( T \).

Note that \( V_\lambda \neq \{0\} \) if and only if \( \lambda \) is an eigenvalue of \( T \) and that \( V_\lambda \) always contains the eigenspace \( E_\lambda = \{ v \in V : T v = \lambda v \} \).

Observation 12.5. The following are equivalent:

(i) \( T \) is diagonalizable, that is, \( T \) is represented by a diagonal matrix with respect to some basis

(ii) All Jordan blocks in \( \text{JCF}(T) \) have size 1

(iii) \( V_\lambda = E_\lambda \) for each eigenvalue \( \lambda \) of \( T \)

(iv) The minimal polynomial \( \mu_T(x) \) has no multiple roots.

Proof. Follows immediately from what we have already proved. \( \square \)

Lemma 12.6. The space \( V \) is a direct sum of the root subspaces \( V_\lambda \):

\[
V = \oplus_{\lambda \in \text{Spec}(T)} V_\lambda.
\]

Proof. By Theorem 12.1 we have a decomposition

\[
V = V_1 \oplus \ldots \oplus V_k
\]

such that each \( V_i \) is \( T \)-invariant, and if \( T_i = T|_{V_i} \), then there is a basis \( \Omega_i \) of \( V_i \) such that \([T_i]|_{\Omega_i} \) is some Jordan block \( J(d_i, \lambda_i) \).

It is clear that for each \( \lambda \in \text{Spec}(T) \) the root subspace \( V_\lambda \) is the (direct) sum of all \( V_i \) for which \( \lambda_i = \lambda \). This combined with (**) yields the lemma. \( \square \)
12.5. **A few words on computing a Jordan basis.**

**Definition.** A basis $\Omega$ of $V$ is called a **Jordan basis** for $T$ if $[T]_{\Omega}$ is in JCF.

Below we discuss how to compute Jordan basis in two simple cases. A few more complicated cases will be discussed in the homework, and a general algorithm is given in Kevin McCrimmon’s ‘General exam guide’.

**Case 1: $T$ is diagonalizable.** In this case we simply compute eigenvalues of $T$, then for each eigenvalue $\lambda$ compute the eigenspace $E_\lambda$ (by solving the equation $Tv = \lambda v$) and pick a basis for each $E_\lambda$. By Observation 12.5 and Lemma 12.6 the union of these bases is a Jordan basis for $T$.

**Case 2: JCF$(T)$ has just one block.** Let $\lambda$ be the unique eigenvalue of $T$. Clearly, we may replace $T$ by $T - \lambda I$ (since any Jordan basis for $T - \lambda I$ is also a Jordan basis for $T$), and thus we may assume that $\lambda = 0$.

Let $n = \dim V$. We know that if $\{e_0, \ldots, e_{n-1}\}$ is a Jordan basis for $T$, then $Te_0 = 0$ and $Te_i = e_{i-1}$ for $i > 0$. The following lemma is a partial converse of this statement which also provides an algorithm for finding a Jordan basis:

**Lemma 12.7.** The following hold:

(i) For any $0 \leq k \leq n$ we have $\text{Im} T^k = \ker T^{n-k}$

(ii) Let $v_{n-1}$ be any vector which does not lie in $\text{Im} T = \ker T^{n-1}$, and set $v_i = Tv_{i+1}$ for $n - 2 \geq i \geq 0$. Then $\{v_0, \ldots, v_{n-1}\}$ is a Jordan basis for $T$.

**Proof.** This is part of Homework #6. \(\square\)