Homework Assignment # 3.

Plan for next week: Tensor, symmetric an exterior algebras (11.5), modules over PID (12.1).

Problems, to be submitted by Thu, February 11th.

Problem 1. Solve Problem 8 on pp.375-376 of [DF]. Note that there is a small mistake (or rather a misprint) in the setup that you need to find and fix.

Problem 2. (a) Let $V$ be a finite-dimensional vector space over $\mathbb{C}$ (complex numbers). Note that $V$ can also be considered as a vector space over $\mathbb{R}$, but $\dim_{\mathbb{R}}(V) = 2 \dim_{\mathbb{C}}(V)$. Prove that $V \otimes_{\mathbb{C}} V$ is not isomorphic to $V \otimes_{\mathbb{R}} V$ as vector spaces over $\mathbb{R}$ and compute their dimensions over $\mathbb{R}$.
(b) Let $R$ be an integral domain and $F$ its field of fractions. Prove that $F \otimes_{R} F \sim F \otimes_{F} F \sim F$ as $F$-modules. Note that the $F$-module structure on $F \otimes_{R} F$ is given by the extension of scalars construction (type I tensor product).

Problem 3. Let $R$ be a commutative domain, and let $M$ be a free $R$-module with basis $e_1, \ldots, e_k$. Prove that the element $e_1 \otimes e_2 + e_2 \otimes e_1 \in M \otimes M$ is not representable as a simple tensor $m \otimes n$ for some $m, n \in M$.

Problem 4 (practice). Let $I$ and $J$ be ideals of a (commutative) ring $R$, and let $\pi_I : R \rightarrow R/I$ and $\pi_J : R \rightarrow R/J$ be canonical projections.
(a) Prove that every element of $R/I \otimes_R R/J$ can be written as a simple tensor $\pi_I(1) \otimes \pi_J(r)$ for some $r \in R$.
(b) Prove that $R/I \otimes_R R/J \cong R/(I + J)$ (as $R$-modules).
(c) Show that there is a surjective $R$-module homomorphism $I \otimes_R J \rightarrow IJ$ such that $i \otimes j \mapsto ij$.
(d) Give an example where $\varphi$ in (c) is not an isomorphism.

Problem 5. Let $R$ be a commutative ring (with 1) and $n, m \in \mathbb{N}$. Prove that $R^n \otimes_R R^m \cong R^{nm}$ as $R$-algebras. As usual $R^k = R \oplus \ldots \oplus R$, $k$ times.

Problem 6. The purpose of this problem is to classify 2-dimensional $\mathbb{R}$-algebras (\$=\text{reals})$, that is, $\mathbb{R}$-algebras which are 2-dimensional as vector spaces over $\mathbb{R}$.
Let $A$ be a 2-dimensional $\mathbb{R}$-algebra (as always, with 1).
(a) Let $u \in A$ be any element which is not an $\mathbb{R}$-multiple of 1. Prove that
(i) $u$ generates $A$ as an $\mathbb{R}$-algebra, that is, the minimal $\mathbb{R}$-subalgebra of $A$ containing $u$ and 1 is $A$ itself.

(ii) $u$ satisfies a quadratic equation $au^2 + bu + c = 0$ for some $a, b, c \in \mathbb{R}$ with $a \neq 0$.

(b) Show that there exists $v \in A$ such that $v^2 = -1$, $v^2 = 1$ or $v^2 = 0$. **Hint:** take any $u$ as in (a), and look for $v$ of the form $\alpha u + \beta$ with $\alpha, \beta \in \mathbb{R}$.

(c) Deduce from (b) that $A$ is isomorphic as an $\mathbb{R}$-algebra to $\mathbb{R}[x]/(x^2 + 1)$, $\mathbb{R}[x]/(x^2 - 1)$ or $\mathbb{R}[x]/x^2$.

(d) Prove that the algebras $\mathbb{R}[x]/(x^2 + 1)$, $\mathbb{R}[x]/(x^2 - 1)$ and $\mathbb{R}[x]/x^2$ are pairwise non-isomorphic. **Hint:** this can be done with virtually no computations involved.