Homework #7.

Plan for next week: Free groups (continued) and presentations of groups by generators and relations (§ 6.3), start ring theory (§ 7.1).

Problems, to be submitted by Thursday, October, 20th

1. Let $n \geq 3$ and $G = D_{2n}$, the dihedral group of order $2n$.
   (a) Prove that $G$ contains a subgroup isomorphic to $D_{2k}$ for any $k \mid n$.
   (b) Prove that the dihedral group $G$ is nilpotent if and only if $n$ is power of 2.

2. (a) Let $R$ be an associative ring with 1, and let $a, b \in R$ be such that $1 + a$ and $1 + b$ are invertible. Prove the following formula

   $$(1 + a)^{-1}(1 + b)^{-1}(1 + a)(1 + b) = 1 + (1 + a)^{-1}(1 + b)^{-1}(ab - ba).$$

   (b) Let $R$ be an associative ring with 1 and $n \geq 2$ be an integer, and let $U_n(R)$ be the upper unitriangular subgroup of $GL_n(R)$. Prove that $U_n(R)$ is nilpotent of class $n - 1$ (we briefly outlined the proof in class). Note that you will need to apply (a) not to $R$ itself but to the ring of $n \times n$ matrices over $R$.

3. (a) Let $n \in \mathbb{N}$ be an integer, and suppose that for every non-prime divisor $m$ of $n$ there are no simple groups of order $m$. Prove that any group of order $n$ is solvable.
   (b) Prove that any group of order $p^k q$, where $p > q$ are distinct primes, is solvable.

4. Problems 31 and 32 on page 200 of DF. Note: Problem 31 follows very easily from Lemma 11.2 (about the structure of minimal normal subgroups).

5. (a) Let $A$ and $B$ be finitely generated groups. Prove that the wreath product $A \wr B$ is also finitely generated. Hint: Recall that $A \wr B = C \rtimes B$ where $C = \oplus_{b \in B} A_b$ (with each $A_b \cong A$). Let $S$ be a generating set for $A$, $T$ a generating set for $B$, fix $b \in B$, and let $S_b$ be the image of $S$ under an isomorphism $A \to A_b$. Prove that $S_b \cup T$ generates $A \wr B$.
   (b) Use (a) to give a simple example showing that a subgroup of a finitely generated group may not be finitely generated.
6. (a) (practice) In class we outlined the proof of the fact that $\mathbb{Z}_p \text{wr} \mathbb{Z}_p$ is isomorphic to the Sylow $p$-subgroup of $S_{p^2}$. Fill in the details of that proof.

(b) (bonus) Realize the Sylow $p$-subgroup of $S_{p^3}$ in terms of wreath products and prove your answer (you may skip some technical details).