Solutions to selected homework problems.

Problem 2.1: Let \( p \) be any prime and \( V = \mathbb{Z}^2_p \), the standard two-dimensional vector space over \( \mathbb{Z}_p \). How many ordered bases does \( V \) have?

Answer: \((p^2 - 1)(p^2 - p)\).

Solution: First, by Corollary 3.5(c) any basis of \( V \) has two elements.

Lemma: Let \( v, w \in V \). Then \( \{v, w\} \) is a basis of \( V \) \( \iff v \neq 0 \) and \( w \) is not a multiple of \( v \).

Proof of Lemma: First note that by Corollary 3.5(e) \( \{v, w\} \) is a basis \( \iff \{v, w\} \) is linearly independent. Thus, replacing the statement of the lemma by contrapositive, we are reduced to proving the following:

\( \{v, w\} \) is linearly dependent \( \iff v = 0 \) or \( w \) is a multiple of \( v \).

“\( \Rightarrow \)” If \( \{v, w\} \) is linearly dependent, there exist \( \lambda, \mu \in F \), not both 0 s.t. \( \lambda v + \mu w = 0 \). If \( \mu = 0 \), then \( \lambda v = 0 \) and \( \lambda \neq 0 \), so \( v = \lambda^{-1}(\lambda v) = 0 \). And if \( \mu \neq 0 \), then \( w = -\frac{\lambda}{\mu}v \) is a multiple of \( v \). \( \square \)

By Lemma, to find the number of bases we need to count the number of ordered pairs \( (v, w) \) with \( v \neq 0 \) and \( w \) not a multiple of \( v \). The total number of vectors in \( V \) is the number of pairs \( (a, b) \) with \( a, b \in \mathbb{Z}_p \). There are \( p \) choices for \( a \) and \( p \) choices for \( b \) (and there are no dependencies between \( a \) and \( b \)), so \( |V| = p^2 \). Thus, there are \( p^2 - 1 \) nonzero vectors in \( V \), so we have \( p^2 - 1 \) choices for \( v \).

Since \( v \neq 0 \), it has precisely \( p \) multiples (including itself) \( \) – indeed, since \( |\mathbb{Z}_p| = p \), there are at most \( p \) multiples, namely \( 0 \cdot v, 1 \cdot v, \ldots, (p - 1) \cdot v \); on the other hand, all these multiples are distinct: if \( \lambda, \mu \in F \) are such that \( \lambda v = \mu v \), then \( (\lambda - \mu)v = 0 \), and if \( \lambda \neq \mu \), multiplying by \( (\lambda - \mu)^{-1} \), we get \( v = 0 \), which is contradiction.

So, once \( v \) has been chosen there are precisely \( p^2 - p \) choices for \( w \). Therefore, the total number of choices for the ordered pair \( (v, w) \) is \((p^2 - 1)(p^2 - p)\).

Problem 2.2: Prove Lemma 3.2 from class: If \( V \) is a vector space, \( S \) a subset of \( V \) and \( v \in V \), then \( \text{Span}(S \cup \{v\}) = \text{Span}(S) \iff v \in \text{Span}(S) \).

Solution: “\( \Rightarrow \)” Assume that \( \text{Span}(S \cup \{v\}) = \text{Span}(S) \). Since \( v \in S \cup \{v\} \)
and $T \subseteq \text{Span}(T)$ for any set $T$, we get $v \in \text{Span}(S \cup \{v\}) = \text{Span}(S)$.

“$\Leftarrow$” Assume that $v \in \text{Span}(S)$. Since $S \subseteq \text{Span}(S)$, we have $S \cup \{v\} \subseteq \text{Span}(S)$ and therefore $\text{Span}(S \cup \{v\}) \subseteq \text{Span}(\text{Span}(S))$ by Theorem 2.1(e). But $\text{Span}(\text{Span}(S)) = \text{Span}(S)$ by Theorem 2.1(d), so $\text{Span}(S \cup \{v\}) \subseteq \text{Span}(S)$. The opposite inclusion $\text{Span}(S) \subseteq \text{Span}(S \cup \{v\})$ is clear (again by Theorem 2.1(e) since $S \subseteq S \cup \{v\}$). □

**Problem 2.5(b):** Let $V = \mathbb{R}^2$, and let $U$ and $W$ be subspaces of $V$ with $\dim(U) = \dim(W) = 1$. Prove that $W$ is a complement of $U \iff W \neq U$.

**Solution:** “$\Rightarrow$” By contradiction. Suppose that $W = U$. Since we assume that $W$ is a complement of $U$, we have $U \cap W = \{0\}$ which together with $W = U$ implies that $U = U \cap U = \{0\}$ and similarly $W = \{0\}$. But then $U + W = \{0\} \neq V$, contrary to the assumption that $W$ is a complement of $U$.

“$\Leftarrow$” Suppose that $W \neq U$. Then at least one of the following holds: $W$ is not contained in $U$ or $U$ is not contained in $W$. WLOG assume that $W$ is not contained in $U$. Then $U \cap W$ is a proper subspace of $W$, so by Theorem 1.11 (book) $\dim(U \cap W) < \dim(W)$. Since $\dim(W) = 1$, the only possibility is that $\dim(U \cap W) = 0$ which means that $U \cap W = \{0\}$. Also note that $\dim(U) + \dim(W) = 1 + 1 = 2 = \dim(V)$. Hence, by Problem 2.4(c) we conclude that $V = U \oplus W$, so $W$ is a complement of $U$.

**Problem 3.1** For each of the following maps $T$ do the following: Prove that $T$ is linear and find a basis for $\text{Ker}(T)$ and $\text{Im}(T)$.

(a) $T : P_6(\mathbb{R}) \rightarrow P_6(\mathbb{R})$ given by $T(f(x)) = f'(x)$

(b) $T : P_6(\mathbb{Z}_3) \rightarrow P_6(\mathbb{Z}_3)$ given by $T(f(x)) = f'(x)$ (where as before $\mathbb{Z}_3$ is the field of congruence classes mod 3).

(c) $T : P_3(\mathbb{R}) \rightarrow \mathbb{R}$ given by $T(f(x)) = f(2)$, that is, $T$ is the evaluation map at $x = 2$.

(d) $T : P_3(\mathbb{R}) \rightarrow P_4(\mathbb{R})$ given by $T(f(x)) = (x + 1)p(x)$, that is, $T$ is the multiplication by $x + 1$.

**Answer:**

(a) $\text{Ker}(T)$ has basis $\{1\}$ and $\text{Im}(T)$ has basis $\{1, x, x^2, x^3, x^4, x^5\}$.

(b) $\text{Ker}(T)$ has basis $\{1, x^3, x^6\}$ and $\text{Im}(T)$ has basis $\{1, x, x^3, x^4\}$.

(c) $\text{Ker}(T)$ has basis $\{x - 2, (x - 2)^2, (x - 2)^3\}$ and $\text{Im}(T)$ has basis $\{1\}$. 

2
(d) \( \text{Ker}(T) = \{0\} \), so has the empty set \( \emptyset \) as its only basis and \( \text{Im}(T) \) has basis \( \{(x+1), x(x+1), x^2(x+1), x^3(x+1)\} \).

Of course, the choice of basis is not unique, and in the case of \( \text{Ker}(T) \) in (c) and \( \text{Im}(T) \) in (d) there is no particularly natural choice of for a basis.

**Justification for (c)** First note that \( \text{Im}(T) = \mathbb{R} \) since for any \( \alpha \in \mathbb{R} \) there exists \( f(x) \in P_3(\mathbb{R}) \) s.t. \( f(2) = \alpha \) (e.g. the constant polynomial \( f(x) = \alpha \)). So, \( \dim(\text{Im}(T)) = 1 \), and by the rank-nullity theorem \( \dim(\text{Ker}(T)) = \dim(P_3(\mathbb{R})) - \dim(\text{Im}(T)) = 4 - 1 = 3 \).

The polynomials \( x-2, (x-2)^2, (x-2)^3 \) vanish at 2, so they lie in \( \text{Ker}(T) \). They are also linearly independent (e.g. by HW#1.7 since they have distinct degrees), and since there are 3 = \( \dim(\text{Ker}(T)) \) of them, they must form a basis.

**Problem 3.6:** Let \( V \) be a vector space and \( T : V \to V \) a linear map. A subspace \( W \) of \( V \) is called \( T \)-invariant if \( T(W) \subseteq W \) where \( T(W) = \{T(w) : w \in W\} \).

(a) Prove that \( \text{Ker}(T) \) and \( \text{Im}(T) \) are \( T \)-invariant subspaces

(b) Assume that \( \dim(V) < \infty \) and \( W \) is a \( T \)-invariant subspace of \( V \) s.t. \( V = W \oplus \text{Ker}(T) \). Prove that \( W = \text{Im}(T) \). **Hint:** First show that \( \text{Im}(T) \subseteq W \).

(c) Give an example with \( \dim(V) < \infty \) where the sum \( \text{Im}(T) + \text{Ker}(T) \) is NOT direct.

(d) Use (b) and (c) to conclude that a \( T \)-invariant subspace may NOT have a \( T \)-invariant complement.

**Solution:** (a) We know that \( 0 \in \text{Ker}(T) \). Hence for any \( v \in \text{Ker}(T) \) we have \( T(v) = 0 \in \text{Ker}(T) \), so \( \text{Ker}(T) \) is \( T \)-invariant. Now take any \( w \in \text{Im}(T) \). Since \( \text{Im}(T) \subseteq V \), we have \( T(w) \in T(\text{Im}(T)) \subseteq T(V) = \text{Im}(T) \), so \( \text{Im}(T) \) is \( T \)-invariant.

(b) done in class on September 29th

(c) Take any \( n \geq 1 \) and consider \( T : P_n(\mathbb{R}) \to P_n(\mathbb{R}) \) given by \( T(f(x)) = f'(x) \). Then any nonzero constant polynomial lies in both \( \text{Ker}(T) \) and \( \text{Im}(T) \), so \( \text{Ker}(T) \cap \text{Im}(T) \neq \{0\} \) and thus the sum \( \text{Ker}(T) + \text{Im}(T) \) cannot be direct.

(d) Let us take any map \( T : V \to W \) where the sum \( \text{Ker}(T) + \text{Im}(T) \) is not direct (e.g. take the above map from (c)). We know that \( \text{Ker}(T) \) is \( T \)-invariant. Suppose that \( \text{Ker}(T) \) has a \( T \)-invariant complement, that is, there is a \( T \)-invariant subspace \( W \) s.t. \( V = \text{Ker}(T) \oplus W \). Then by (b) \( W = \text{Im}(T) \). This contradicts the assumption that the sum \( \text{Ker}(T) + \text{Im}(T) \) is not direct.
Problem 3.7: Recall that $\mathfrak{sl}_n(F)$ denotes the space of all $n \times n$ matrices over $F$ with trace 0. In Problem 2 of HW#6 it was proved that $\dim(\mathfrak{sl}_n(F)) = n^2 - 1$ after a considerable amount of work. Now give a short proof of this fact by applying the rank-nullity theorem to a suitable linear map.

Solution: Let $Mat_n(F)$ be the vector space of all $n \times n$ matrices over $F$, and consider the map $T : Mat_n(F) \to F$ given by $T(A) = \text{tr}(A)$. Then $\text{Ker}(T) = \mathfrak{sl}_n(F)$ (by definition) and $\text{Im}(T) = F$ since any $\alpha \in F$ is the trace of some $A \in Mat_n(F)$ (e.g. $\alpha = \text{tr}(\alpha e_{11})$). So, $\dim(\text{Im}(T)) = 1$ and by the rank-nullity theorem $\dim(\text{Ker}(T)) = \dim(Mat_n(F)) - \dim(\text{Im}(T)) = n^2 - 1$.

Problem 4.3: Let $V$ be a finite-dimensional vector space and $k \leq \dim(V)$ a positive integer. Let $T : V \to V$ be a linear transformation. Prove that the following are equivalent:

(a) There exists a $T$-invariant subspace $W$ of $V$ with $\dim(W) = k$ (recall that the notion of a $T$-invariant subspace is defined in Problem#6 of Homework#3).

(b) There exists a basis $\mathcal{B}$ of $V$ s.t. the matrix $[T]_\mathcal{B}$ has the block-diagonal form

$$\begin{pmatrix}
A_{k \times k} & B_{k \times (n-k)} \\
0_{(n-k) \times k} & C_{(n-k) \times (n-k)}
\end{pmatrix}$$

where subscripts indicate matrix sizes and $0_{(n-k) \times k}$ is the $(n-k) \times k$ zero matrix.

Solution: “(b)⇒(a)” Suppose that $\mathcal{B} = \{v_1, \ldots, v_n\}$ and $[T]_\mathcal{B} = (a_{ij})_{1 \leq i,j \leq n}$.

By definition of $[T]_\mathcal{B}$ we have $T(v_j) = \sum_{i=1}^n a_{ij}v_i$ for all $1 \leq j \leq n$. On the other hand, the assumption about the block-diagonal form of $[T]_\mathcal{B}$ from (b) implies that $a_{ij} = 0$ for $k+1 \leq i \leq n$ and $1 \leq j \leq k$. This means that

$$T(v_j) = \sum_{i=1}^k a_{ij}v_i \text{ for all } 1 \leq j \leq k. \quad (\ast \ast \ast)$$

Let $W = \text{Span}(v_1, v_2, \ldots, v_k)$; note that $\dim(W) = k$ since $\{v_1, \ldots, v_k\}$ is linearly independent, being a subset of a basis of $V$. By (\ast\ast\ast) $T(v_j) \in W$ for all $1 \leq j \leq k$, and since $T$ is linear (and $W$ is a subspace), we conclude that $T(w) \in W$ for all $w \in \text{Span}(v_1, v_2, \ldots, v_k) = W$. So, $T(W) \subseteq W$, and thus $W$ is a $T$-invariant subspace with $\dim(W) = k$.

“(a)⇒(b)” Choose an ordered basis $\{v_1, \ldots, v_k\}$ of $W$ and extend it to an ordered basis $\{v_1, \ldots, v_n\}$ of $V$; call the latter basis $\mathcal{B}$. Since $W$ is $T$-invariant, for each $1 \leq j \leq k$ we have $T(v_j) \in W$, so $T(v_j) = \sum_{i=1}^k a_{ij}v_i = \sum_{i=1}^k a_{ij}v_i + \sum_{i=k+1}^n 0 \cdot v_i$. This implies that the $(i,j)$ entry of $[T]_\mathcal{B}$ is equal
to 0 whenever $1 \leq j \leq k$ and $k + 1 \leq i \leq n$, so $[T]_B$ has the required block-diagonal form.

**Problem 4.5:** Let $V$ be a finite-dimensional vector space and $T : V \to V$ a linear map. Prove that the following are equivalent:

(i) $T$ is a projection, that is, $T = p_{U,W}$ for some $U$ and $W$ with $U \oplus W = V$ (note that $p_{U,W}$ is defined in Problem 4.4).

(ii) $T^2 = T$ (where $T^2 = T \cdot T$ is the composition of $T$ with itself)

(iii) There is an ordered basis $\beta$ of $V$ s.t. $[T]_\beta = e_{11} + \ldots + e_{kk}$ for some $k \leq \dim V$, that is, $[T]_\beta$ is the diagonal matrix whose first $k$ diagonal entries are equal to 1 and the remaining diagonal entries are equal to 0.

**Solution:** We’ll prove that (i)$\Rightarrow$(iii)$\Rightarrow$(ii)$\Rightarrow$(i).

(i)$\Rightarrow$(iii): choose some ordered bases $\{u_1, \ldots, u_k\}$ of $U$ and $\{w_1, \ldots, w_l\}$ of $W$. We claim that their ordered union $\beta = \{u_1, \ldots, u_k, w_1, \ldots, w_l\}$ (with elements of $W$ listed first) is a basis of $V$ – this is not hard to prove directly, but we can also deduce it from previous homework problems. Indeed, by Problem 2.1(d) $\text{Span}(\beta) = \text{Span}(\{u_1, \ldots, u_k\}) + \text{Span}(\{w_1, \ldots, w_l\}) = U + W$, so $\beta$ spans $U + W = V$. Since $V = U \oplus W$ (the sum is direct), by Problem 2.4, $\dim(V) = \dim(U) + \dim(W) = k + l = |\beta|$, so $\beta$ is a basis of $V$ by Corollary 3.5(d).

Since $T = p_{U,W}$, we have $T(u_j) = u_j$ for $1 \leq j \leq k$ and $T(w_j) = 0$ for $1 \leq j \leq l$. We conclude that $[T]_\beta$ has 1 as its $(j,j)$-entry for all $1 \leq j \leq k$ and all other entries are 0. Therefore, $[T]_\beta = e_{11} + \ldots + e_{kk}$.

(iii)$\Rightarrow$(ii): Since $[T]_\beta = e_{11} + \ldots + e_{kk}$, by direct computation we have $([T]_\beta)^2 = [T]_\beta$. On the other hand, by Theorem 2.14(book) $([T]_\beta)^2 = [T^2]_\beta$. So, $[T^2]_\beta = [T]_\beta$, and since a linear map is uniquely determined by its matrix with respect to a given a basis, we conclude that $T^2 = T$.

(ii)$\Rightarrow$(i): Assume that $T^2 = T$. We claim that

$$V = \text{Ker}(T) \oplus \text{Im}(T).$$

Take any $v \in \text{Ker}(T) \cap \text{Im}(T)$. Then $T(v) = 0$ and $v = T(u)$ for some $u$, so $v = T(u) = T^2(u) = T(T(u)) = T(v) = 0$. Hence $\text{Ker}(T) \cap \text{Im}(T) = \{0\}$. On the other hand, by the rank-nullity theorem $\dim(\text{Ker}(T)) + \dim(\text{Im}(T)) = \dim(V)$. Combining the two results, we conclude that $V = \text{Ker}(T) \oplus \text{Im}(T)$ by Problem 2.4(c). (It is also not hard to show directly that $V = \text{Ker}(T) + \text{Im}(T)$: indeed, any $v \in V$ can be written as $v = (v - T(v)) + T(v)$ and $v - T(v) \in \text{Ker}(T)$ for $T(v - T(v)) = T(v) - T^2(v) = 0$).
Now let $U = \text{Im}(T)$ and $W = \text{Ker}(T)$. Then $T(w) = 0$ for all $w \in W$ and $T(u) = u$ for all $u \in U$ (for any $u \in U$ can be written as $u = T(z)$ for some $z$ and $T(u) = T(T(z)) = T(z) = u$). Therefore, $T = p_UW$ by definition.

**Problem 5.6:** Prove Proposition 10.4: Let $V$ be a finite-dimensional vector space and $W$ a subspace of $V$. Then

$$\dim(W) + \dim(\text{Ann}(W)) = \dim(V).$$

See Problem 14 in § 2.6 for a hint.

**Solution:** Let $n = \dim V$ and $m = \dim W$. Following the hint in the book, choose a basis $\{v_1, \ldots, v_m\}$ of $W$ and extend it to a basis $\{v_1, \ldots, v_m, v_{m+1}, \ldots, v_n\}$ of $V$. Let $\{v_1^*, \ldots, v_n^*\}$ be the dual basis of $V^*$, and let

$$B = \{v_{m+1}^*, \ldots, v_n^*\}$$

Let us show that $B$ is a basis of $\text{Ann}(W)$ (this would imply that $\dim(\text{Ann}(W)) = n - m = \dim V - \dim W$, as desired).

Note that $B$ is linearly independent (being a subset of a basis of $V^*$), so we only need to check that $\text{Ann}(W) = \text{Span}(B)$.

**Part 1:** $\text{Span}(B) \subseteq \text{Ann}(W)$. First take any element of $B$, that is, $v_i^*$ with $m + 1 \leq i \leq n$. Then $v_i^*(v_j) = 0$ for all $1 \leq j \leq m$ (by definition of dual basis), and by linearity $v_i^*(\lambda_1 v_1 + \ldots + \lambda_m v_m) = 0$ for all $\lambda_1, \ldots, \lambda_m \in F$. So, $v_i^*(w) = 0$ for all $w \in \text{Span}(v_1, \ldots, v_m) = W$, so $v_i^* \in \text{Ann}(W)$.

Thus, we proved that $B \subseteq \text{Ann}(W)$, and since $\text{Ann}(W)$ is a subspace (by Problem 5.5), it follows that $\text{Span}(B) \subseteq \text{Ann}(W)$.

**Part 2:** $\text{Ann}(W) \subseteq \text{Span}(B)$. Take any $f \in \text{Ann}(W)$. Since $\{v_1^*, \ldots, v_n^*\}$ is a basis of $V^*$, we can write

$$f = \lambda_1 v_1^* + \ldots + \lambda_n v_n^*$$

for some $\lambda_1, \ldots, \lambda_n \in F$. Since $f \in \text{Ann}(W)$, we must have $f(v_i) = 0$ for $1 \leq i \leq m$. Fix such $i$ and evaluate both sides of (***), at $v_i$. Since $v_k^*(v_i) = 0$ for $k \neq i$ and 1 for $k = i$, we get that $f(v_i) = \lambda_i$ (and recall that $f(v_i) = 0$). So, $\lambda_i = 0$ for $1 \leq i \leq m$, and therefore, $f = \sum_{i=m+1}^n \lambda_i v_i^* \in \text{Span}(B)$. □