Homework #9. Due on Thursday, April 7th

Reading:
1. For this assignment: Lectures 17-18 (for Lecture 18 look at both class notes and online version), [Pinter, §8,13], [Gilbert, §4.1]
2. For Tuesday’s class: online lecture 19, [Pinter, §13] and [Gilbert, §4.4]
3. For Thursday’s class: online lecture 20, [Pinter, §14] and [Gilbert, §4.5]

Problems:

Problem 1: (a) Let \( f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 1 & 3 & 7 & 2 & 6 & 4 \end{pmatrix} \) in two-line notation. Write \( f \) as in disjoint cycle form.
(b) Write the following element of \( S_9 \) as a product of disjoint cycles:
\[(1, 2, 4, 6, 7)(3, 4, 5, 1, 8)(9, 2, 3, 5)\]

Problem 2: List all elements of \( S_3 \) in disjoint cycle form and compute the multiplication table of \( S_3 \).

Problem 3: Two elements \( f \) and \( g \) of \( S_n \) are said to have the same cycle type if their disjoint cycle forms contain the same number of cycles of each length. For instance, elements \((1, 5, 6)(2, 3)(4, 7)\) and \((1, 7, 8)(4, 5)(3, 6)\) of \( S_8 \) have the same cycle type. Show that elements of \( S_6 \) have 11 distinct cycle types. For each cycle type list one element of that type.

Problem 4: (a) Use the result of Problem 3 to determine possible orders of elements of \( S_6 \). Recall that if \( f \in S_n \) is written as a product of disjoint cycles \( f_1f_2\ldots f_r \) where \( f_1 \) has length \( k_1 \), \ldots, \( f_r \) has length \( k_r \), then the order of \( f \) is the least common multiple of \( k_1, k_2, \ldots, k_r \).
(b) Find the smallest \( n \in \mathbb{N} \) for which \( S_n \) has an element of order 15 and prove your answer.

Problem 5: (a) Let \( f, g \in S_n \) be two transpositions, that is, \( f = (i, j) \) and \( g = (k, l) \) for some \( i, j, k, l \). What are the possible orders of the product \( fg \)? Note: By definition, a transposition is just a cycle of length 2. Hint: Consider three cases depending on the size of the set \( \{i, j\} \cap \{k, l\} \) (note that \( \{i, j\} \cap \{k, l\} \) is empty if and only \( f \) and \( g \) are disjoint cycles).
(b) (optional) Answer the same question when \( f \) is a transposition and \( g \) is a cycle of length 3.

Problem 6: Let \( G \) and \( H \) be finite groups such that \(|G|\) and \(|H|\) are coprime. Prove that any homomorphism \( \varphi : G \to H \) must be trivial, that is, \( \varphi(x) = e_H \) for all \( x \in G \) where \( e_H \) is the identity element of \( H \). Hint: Use the Range-Kernel theorem (see Lecture 16) and Lagrange theorem (applied to a suitable subgroup).
Problem 7: Let $p$ and $q$ be distinct primes, and let $G$ be a group of order $pq$. Prove that one of the following two cases occurs:

(i) $G$ is isomorphic to $\mathbb{Z}_{pq}$.

(ii) for every $x \in G$ either $x^p = e$ or $x^q = e$.

Problem 8: Use Lagrange theorem to prove Fermat’s little theorem: if $p$ is prime, then $n^p \equiv n \mod p$ for any $n \in \mathbb{Z}$. **Hint:** Apply Corollary 18.1(B) to the group $\mathbb{Z}_p^\times = (\mathbb{Z}_p \setminus \{0\}, \cdot)$.

Problem 9: Let $G$ be a finite group of order $n$, and let $\varphi : G \to S_n$ be an injective homomorphism from the proof of Cayley’s theorem:

(a) Describe $\varphi$ explicitly (by computing $\varphi(g)$ for every $g \in G$) for each of the following groups: $G = \mathbb{Z}_4$, $G = S_3$ (in Lecture 18 we did the corresponding computation for $\mathbb{Z}_2 \times \mathbb{Z}_2$).

(b) (bonus) Prove that the following property holds for every group: if $g \in G$ and $m = o(g)$, then $\varphi(g)$ is a product of $\frac{n}{m}$ disjoint cycles of length $m$.

(c) (bonus) There are several different ways to “justify” the terminology “cyclic group”. Use the result of (b) to give one possible explanation of why cyclic groups are called cyclic.