Homework #2. Due on Thursday, February 4th, in class

Reading:
1. For this assignment: Online lectures 2 (ordered rings part) and 3 and [Gilbert, §2.1.2.2].
2. For next week’s classes: Online lectures 4 and 5, [Gilbert, §2.3,2.4] and [Pinter, §22]

Problems:

Problem 1: Prove by induction that the following equalities hold for any $n \in \mathbb{N}$:

(a) $1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$

(b) $a + ar + ar^2 + \ldots + ar^{n-1} = a \frac{1-r^n}{1-r}$ where $a, r \in \mathbb{R}$ and $r \neq 1$

Problem 2: Consider the following “proof” by induction: For each $n \in \mathbb{N}$ let $P(n)$ be the statement

$$\sum_{i=0}^{n} 2^i = 2^{n+1}. \quad (\ast \ast \ast)$$

Claim: $P(n)$ is true for all $n \in \mathbb{N}$.

Proof: “$P(n-1) \Rightarrow P(n)$.” Assume that $P(n-1)$ is true for some $n \in \mathbb{N}$. Then $\sum_{i=0}^{n-1} 2^i = 2^{n-1}$. Adding $2^n$ to both sides, we get $\sum_{i=0}^{n} 2^i = 2^n + 2^n$, whence $\sum_{i=0}^{n} 2^i = 2^{n+1}$, which is precisely $P(n)$. Thus, $P(n)$ is true.

By the principle of mathematical induction, $P(n)$ is true for all $n$. □

(a) Show that the statement $P(n)$ is false (it is actually false for any $n$).

(b) Explain why the above “proof” does not contradict the principle of mathematical induction, that is, find a mistake in the above “proof” (Hint: the mistake is in the general logic).

Problem 3: In online lecture 3 it is proved that for every $n \in \mathbb{N}$ there exist $a_n, b_n \in \mathbb{Z}$ such that $(1 + \sqrt{2})^n = a_n + b_n \sqrt{2}$. Moreover, it is shown that such $a_n$ and $b_n$ satisfy the following recursive relations: $a_1 = b_1 = 1$ and $a_{n+1} = a_n + 2b_n$, $b_{n+1} = a_n + b_n$ for all $n \in \mathbb{N}$.

(a) Use the above recursive formulas and mathematical induction to prove that $a_n^2 - 2b_n^2 = (-1)^n$ for all $n \in \mathbb{N}$.

(b) Prove that for all $n \in \mathbb{N}$ there exist $c_n, d_n \in \mathbb{Z}$ such that $(1 + \sqrt{3})^n = c_n + d_n \sqrt{3}$.

(c) (bonus) Find a simple formula relating $c_n$ and $d_n$ (similar to the one in (a)) and prove it.
Problem 4: Given \( n, k \in \mathbb{Z} \) with \( 0 \leq k \leq n \), define the binomial coefficient \( \binom{n}{k} \) by

\[
\binom{n}{k} = \frac{n!}{k!(n-k)!}
\]

(recall that \( 0! = 1 \)).

(a) Prove that \( \binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1} \) for any \( 1 \leq k < n \) (direct computation).

(b) Now prove the binomial theorem: for every \( a, b \in \mathbb{R} \) and \( n \in \mathbb{N} \),

\[
(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k = \binom{n}{0} a^n + \binom{n}{1} a^{n-1} b + \ldots + \binom{n}{n-1} a b^{n-1} + \binom{n}{n} b^n.
\]

**Hint:** Use induction on \( n \). For the induction step write

\[(a+b)^{n+1} = (a+b)^n \cdot (a+b) \]

and use part (a).

Note: In Problems 5(a) and 6(a) below you are allowed to make an extra assumption that \( R \) is a commutative ring with 1 (this does not make the proof considerably easier, but makes it possible to quote several previously established results).

Problem 5:

(a) Let \( R \) be an ordered ring. Prove that \( x^2 > 0 \) for every nonzero \( x \in R \). **Hint:** Consider two cases.

(b) Use (a) to prove that \( \mathbb{C} \) (complex numbers) is not an ordered ring (no matter how we try to define the set of positive elements).

Problem 6:

(a) Let \( R \) be an ordered ring. Prove that if \( xy = 0 \) for some \( x, y \in R \), then \( x = 0 \) or \( y = 0 \). **Hint:** prove this by contrapositive.

(b) Let \( R \) be any ring which satisfies the conclusion of (a) \( (xy = 0 \Rightarrow x = 0 \) or \( y = 0 \) for all \( x, y \in R \)\). Prove that multiplicative cancellation law holds in \( R \), that is, if \( xz = yz \) for some \( x, y, z \in R \), then \( x = y \) or \( z = 0 \).

Problem 7: Let \( R \) be a finite ring (that is, a ring, with finitely many elements), and suppose that \( |R| > 1 \). Prove that \( R \) cannot be an ordered ring. There is a hint on the next page, but first try to solve it without looking at the hint.
**Hint for 7:** Assume by way of contradiction that $R$ is ordered; since $|R| > 1$, $R$ must have at least one nonzero element, hence by axiom (O1) there must be at least one $x \in R$ such that $x > 0$. Now use Problem 3(a) from HW #1 repeatedly and transitivity of $>$ to reach a contradiction.