20. Normal subgroups

20.1. Definition and basic examples. Recall from last time that if $G$ is a group, $H$ a subgroup of $G$ and $g \in G$ some fixed element the set $gH = \{gh : h \in H\}$ is called a left coset of $H$.

Similarly, the set $Hg = \{hg : h \in H\}$ is called a right coset of $H$.

**Definition.** A subgroup $H$ of a group $G$ is called normal if $gH = Hg$ for all $g \in G$.

The main motivation for this definition comes from quotient groups which will be discussed in a couple of weeks.

Let us now see some examples of normal and non-normal subgroups.

**Example 1.** Let $G$ be an abelian group. Then any subgroup of $G$ is normal.

**Example 2.** Let $G$ be any group. Recall that the center of $G$ is the set $Z(G) = \{x \in G : gx = xg \text{ for all } g \in G\}$.

By Homework#6.3, $Z(G)$ is a subgroup of $G$. Clearly, $Z(G)$ is always a normal subgroup of $G$; moreover, any subgroup of $Z(G)$ is normal in $G$.

**Example 3.** $G = S_3$, $H = \langle (1,2,3) \rangle = \{e, (1,2,3), (1,3,2)\}$.

Let $g = (1,2)$. Then

\[
gH = \{(1,2), (1,2)(1,2,3), (1,2)(1,3,2)\} = \{(1,2), (1,3,2), (1,3)\}
\]

\[
Hg = \{(1,2), (1,2,3)(1,2), (1,3,2)(1,2)\} = \{(1,2), (1,3), (2,3)\}.
\]

Note that while there exists $h \in H$ s.t. $gh \neq hg$, we still have $gH = Hg$ as sets.

The above computation does not yet prove that $H$ is normal in $G$ since we only verified $gH = Hg$ for a single $g$. To prove normality we would need to do the same for all $g \in G$. However, there is an elegant way to prove normality in this example, given by the following proposition.

**Proposition 20.1.** Let $G$ be a group and $H$ a subgroup of index 2 in $G$. Then $H$ is normal in $G$.

*Proof. This will be one of the problems in Homework#10.*

Recall from Lecture 19 that the index of $H$ in $G$, denoted by $[G : H]$, is the number of left cosets of $H$ in $G$ and that if $G$ is finite, then $[G : H] = \frac{|G|}{|H|}$. In
Example 3 we have $|G| = 6$ and $|H| = 3$, so $[G : H] = 2$ and Proposition 20.1 can be applied.

Finally, we give an example of a non-normal subgroup:

**Example 4.** $G = S_3$, $H = \langle (1, 2) \rangle = \{e, (1, 2)\}$.

To prove this subgroup is not normal it suffices to find a single $g \in G$ such that $gH \neq Hg$. We will show that $g = (1, 3)$ has this property.

We have $gH = \{(1, 3), (1, 3)(1, 2)\} = \{(1, 3), (1, 2, 3)\}$ and $Hg = \{(1, 3), (1, 2)(1, 3)\} = \{(1, 3), (1, 3, 2)\}$. Since $\{(1, 3), (1, 2, 3)\} \neq \{(1, 3), (1, 3, 2)\}$ (as sets), $H$ is not normal.

20.2. Conjugation criterion of normality.

**Definition.** Let $G$ be a group and fix $g, x \in G$. The element $gxg^{-1}$ is called the conjugate of $x$ by $g$.

**Theorem 20.2** (Conjugation criterion). Let $G$ be a group and $H$ a subgroup of $G$. Then $H$ is normal in $G \iff$ for all $h \in H$ and $g \in G$ we have $ghg^{-1} \in H$. In other words, $H$ is normal in $G \iff$ for every element of $H$, all conjugates of that element also lie in $H$.

**Proof.** “⇒” Suppose that $H$ is normal in $G$, so for every element $g \in G$ we have $gH = Hg$. Hence for every $h \in H$ we have $gh \in gH = Hg$, so $gh = h'g$ for some $h' \in H$. Multiplying both sides on the right by $g^{-1}$, we get $ghg^{-1} \in H$. Thus, we showed that $ghg^{-1} \in H$ for all $g \in G, h \in H$, as desired.

“⇐” Suppose now for all $g \in G, h \in H$ we have $ghg^{-1} \in H$. This means that $ghg^{-1} = h'$ for some $h' \in H$ (depending on $g$ and $h$). The equality $ghg^{-1} = h'$ can be rewritten as $gh = h'g$. Since $h'g \in Hg$ by definition, we get that $gh \in Hg$ for all $h \in H, g \in G$, so $gH \subseteq Hg$ for all $g \in G$.

Since the last inclusion holds for all $g \in G$, it will remain true if we replace $g$ by $g^{-1}$. Thus, $g^{-1}H \subseteq Hg^{-1}$ for all $g \in G$. Using Lemma 19.1 (associativity of multiplication of subsets in a group), multiplying the last inclusion by $g$ on both left and right, we get $Hg \subseteq gH$.

Thus, for all $g \in G$ we have $gH \subseteq Hg$ and $Hg \subseteq gH$, and therefore $gH = Hg$. □

20.3. Applications of the conjugation criterion.

**Theorem 20.3.** Let $G$ and $G'$ be groups and $\varphi : G \to G'$ a homomorphism. Then $\text{Ker} (\varphi)$ is a normal subgroup of $G$. 
Proof. Let $H = \text{Ker}(\varphi)$. We already know from Lecture 16 that $H$ is a subgroup of $G$, so it suffices to check normality. We will do this using the conjugation criterion.

So, take any $h \in H$ and $g \in G$. By definition of the kernel we have $\varphi(h) = e'$ (the identity element of $G'$). Hence $\varphi(ghg^{-1}) = \varphi(g)\varphi(h)\varphi(g^{-1}) = \varphi(g)e'\varphi(g)^{-1} = e'$, so $ghg^{-1} \in \text{Ker}(\varphi) = H$. Therefore, $H$ is normal by Theorem 20.2. □

Here are two more examples of application of the conjugation criterion

**Example 5.** Let $A$ and $B$ be any groups and $G = A \times B$ their direct product. Let $\widetilde{A} = \{(a,e_B) : a \in A\} \subseteq G$, the set of elements of $G$ whose second component is the identity element of $B$.

It is not hard to show that $\widetilde{A}$ is a subgroup of $G$ and $\widetilde{A} \cong A$ (one can think of $\widetilde{A}$ as a canonical copy of $A$ in $G$).

We claim that $\widetilde{A}$ is normal in $G$. Indeed, take any $g \in G$ and $h \in A$. Thus, $g = (x,y)$ and $h = (a,e_B)$ for some $a, x \in A$ and $y \in B$. Then $g^{-1} = (x^{-1},y^{-1})$, so $ghg^{-1} = (x,y)(a,e_B)(x^{-1},y^{-1}) = (xax^{-1},ye_By^{-1}) = (xax^{-1},e_B) \in \widetilde{A}$. Thus, $\widetilde{A}$ is normal by Theorem 20.2.

**Example 6.** Let $F$ be a field. Let $G = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : a, b, c \in F, ac \neq 0 \right\}$ and $H = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in F \right\}$

In Lecture 12 we proved that $G$ is a subgroup of $\text{GL}_2(F)$ (so $G$ itself is a group). We also know that $H$ is a subgroup $\text{GL}_2(F)$ (by Homework #7.5); since clearly $H \subseteq G$, it follows that $H$ is a subgroup of $G$.

Using conjugation criterion, it is not difficult to check that $H$ is normal in $G$. 