Homework #1. Due on Thursday, September 1st in class

Reading:
1. For this assignment: Online lectures 1, 2 and the beginning of 3.
2. For next week’s classes: Online lectures 3, 4 and 5 and Sections 1.1 and 1.2 of the book.

Online lectures are currently posted on last semester’s webpage

[http://people.virginia.edu/~mve2x/3354_Spring2016](http://people.virginia.edu/~mve2x/3354_Spring2016)

Problems:

**Problem 1:** Let \( R \) be a commutative ring with 1. Prove the following equalities using only the ring axioms and results proved in class or online lectures.

(a) \((-xy) = (-x)y\) for all \( x, y \in R \)
(b) \((-1)(-1) = 1\)
(c) \((-x)(-y) = xy\) for all \( x, y \in R \)
(d) \(x(y - z) = xy - xz\) for all \( x, y, z \in R \)

**Hint:** Additive cancellation law (proved in lecture 1) can be used to solve many questions of this type as follows. Suppose that we want to prove inequality of the form \( a = b \). By additive cancellation law, if we prove that \( a + c = b + c \) for some \( c \in R \), we can conclude that \( a = b \). Note that the implication would work for any \( c \), so \( c \) is for us to choose. The idea is to choose \( c \) in such a way that both expressions \( a + c \) and \( b + c \) can be simplified (using ring axioms) so that after simplification it becomes easy to prove that \( a + c = b + c \).

Recall that by definition \( x - y = x + (-y) \).

**Problem 2:** Let \( F \) be a field, and suppose that \( xy = 0 \) for some \( x, y \in F \). Prove that \( x = 0 \) or \( y = 0 \). **Hint:** Consider two cases: \( x = 0 \) (in this case there is nothing to prove) and \( x \neq 0 \). Recall that in a field every nonzero element has multiplicative inverse.

**Note:** If \( F \) was only assumed to be a commutative ring with unity, the above assertion would have been false in general. Can you think of an example?

**Problem 3:** Let \( R \) be an ordered ring and \( x, y, z \in R \). Prove that

(a) If \( x > y \), then \( x + z > y + z \)
(b) If \( x > y \) and \( z > 0 \), then \( xz > yz \)
(c) If \( x > y \) and \( z < 0 \), then \( xz < yz \)
Note: You may use freely standard properties of ring operations (addition, subtraction and multiplication). However, all statement involving inequalities must be deduced directly from the axioms.

Problem 4: Prove by induction that the following equalities hold for any \( n \in \mathbb{N} \):

(a) \( 1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6} \)

(b) \( a + ar + ar^2 + \ldots + ar^{n-1} = a \frac{1-r^n}{1-r} \) where \( a, r \in \mathbb{R} \) and \( r \neq 1 \)

Problem 5: Consider the following “proof” by induction: For each \( n \in \mathbb{N} \) let \( P(n) \) be the statement

\[ \sum_{i=0}^{n} 2^i = 2^{n+1}. \] (***)

Claim: \( P(n) \) is true for all \( n \in \mathbb{N} \).

Proof: “\( P(n-1) \Rightarrow P(n) \).” Assume that \( P(n-1) \) is true for some \( n \in \mathbb{N} \). Then \( \sum_{i=0}^{n-1} 2^i = 2^n \). Adding \( 2^n \) to both sides, we get \( \sum_{i=0}^{n} 2^i = 2^n + 2^n \), whence \( \sum_{i=0}^{n} 2^i = 2^{n+1} \), which is precisely \( P(n) \). Thus, \( P(n) \) is true.

By the principle of mathematical induction, \( P(n) \) is true for all \( n \). \( \square \)

(a) Show that the statement \( P(n) \) is false (it is actually false for any \( n \)).

(b) Explain why the above “proof” does not contradict the principle of mathematical induction, that is, find a mistake in the above “proof” (Hint: the mistake is in the general logic).

Problem 6: In online lecture 3 it is proved that for every \( n \in \mathbb{N} \) there exist \( a_n, b_n \in \mathbb{Z} \) such that \( (1 + \sqrt{2})^n = a_n + b_n \sqrt{2} \). Moreover, it is shown that such \( a_n \) and \( b_n \) satisfy the following recursive relations: \( a_1 = b_1 = 1 \) and \( a_{n+1} = a_n + 2b_n, b_{n+1} = a_n + b_n \) for all \( n \in \mathbb{N} \).

(a) Use the above recursive formulas and mathematical induction to prove that \( a_n^2 - 2b_n^2 = (-1)^n \) for all \( n \in \mathbb{N} \).

(b) Prove that for all \( n \in \mathbb{N} \) there exist \( c_n, d_n \in \mathbb{Z} \) such that \( (1 + \sqrt{3})^n = c_n + d_n \sqrt{3} \).

(c) (bonus) Find a simple formula relating \( c_n \) and \( d_n \) (similar to the one in (a)) and prove it.