APPENDIX C
ACCURACY OF MEASUREMENTS AND TREATMENT OF EXPERIMENTAL UNCERTAINTY

“A measurement whose accuracy is unknown has no use whatever. It is therefore necessary to know how to estimate the reliability of experimental data and how to convey this information to others.”

–E. Bright Wilson, Jr., An Introduction to Scientific Research

Our mental picture of a physical quantity is that there exists some unchanging, underlying value. It is through measurements we try to find this value. Experience has shown that the results of measurements deviate from these “true” values. The purpose of this Appendix is to address how to use measurements to best estimate the “true” values and how to estimate how close the measured value is likely to be to the “true” value. Our understanding of experimental uncertainty (i.e., errors) is based on the mathematical theory of probability and statistics, so the Appendix also includes some ideas from this subject. This Appendix also discusses the notation that scientists and engineers use to express the results of such measurements.

Accuracy and Precision

In common usage, “accuracy” and “precision” are synonyms. To scientists and engineers, however, they refer to two distinct (yet closely related) concepts. When we say that a measurement is “accurate”, we mean that it is very near to the “true” value. When we say that a measurement is “precise”, we mean that it is very reproducible (or as you will learn later the associated random and systematic errors are small). [Of course, we want to make accurate AND precise measurements.] Associated with each of these concepts is a type of error.

In this course you will be required to compare the measured values with true or theoretical values of the same quantity. In your previous science courses you perhaps were asked to calculate the percent difference between these two values. Percent difference between the true and the measured values $A_{\text{true}}$ and $A_{\text{measured}}$ is defined as

$$\text{Percent difference} = \left| \frac{A_{\text{true}} - A_{\text{measured}}}{A_{\text{true}}} \right| \times 100\%$$

As you can see from the discussion earlier, the percent difference tells us how accurate your measurement was but it has no indication on how reproducible (precise) it is. From the statistics discussion below you will learn that the percent difference is incomplete absent an expected deviation for comparison because a single measurement can end up being accurate just by chance. And even then the magnitude of accuracy remains a subjective measure because being within 1% of the true value is not telling us much about the quality of the experiment. In other words, we must be able to compare the accuracy of the result with the precision of the experiment. Therefore, where possible, you will be required to calculate or estimate (if detail calculations do not yield a substantial gain in understanding) the precision of your experiment. There are two types of errors that will tell you how precise your measurement was.

Random errors are caused by fluctuations in the very quantities that we are measuring. You could have a well calibrated pressure gauge, but if the pressure is fluctuating, your reading of the gauge, while perhaps accurate, would be imprecise (not very reproducible).

Systematic errors are due to problems with the technique or measuring instrument. For example, as many of the rulers found in labs have worn ends, length measurements could be
wrong. One can make very precise (reproducible) measurements that are quite inaccurate (far from the true value).

Through careful design and attention to detail, we can usually eliminate (or correct for) systematic errors. Using the worn ruler example above, we could either replace the ruler or we could carefully determine the “zero offset” and simply add it to our recorded measurements.

Random errors, on the other hand, are less easily eliminated or corrected. We usually have to rely upon the mathematical tools of probability and statistics to help us determine the “true” value that we seek. Using the fluctuating gauge example above, we could make a series of independent measurements of the pressure and take their average as our best estimate of the true value.

**Probability**

Scientists base their treatment of random errors on the theory of probability. We do not have space or time for a lengthy survey of this fundamental subject, but can only touch on some highlights. Probability concerns random events (such as the measurements described above). To some events we can assign a theoretical, or *a priori*, probability. For instance, the probability of a “perfect” coin landing heads or tails is $\frac{1}{2}$ for each of the two possible outcomes; the *a priori* probability of a “perfect” die falling with a particular one of its six sides uppermost is $\frac{1}{6}$.

The previous examples illustrate four basic principles about probability:

1. The possible outcomes have to be mutually exclusive. If a coin lands heads, it does not land tails, and *vice versa*.

2. The list of outcomes has to exhaust all possibilities. In the example of the coin we implicitly assumed that the coin neither landed on its edge, nor could it be evaporated by a lightning bolt while in the air, or any other improbable, but not impossible, potential outcome. (And ditto for the die.)

3. Probabilities are always numbers between zero and one, inclusive. A probability of one means the outcome always happens, while a probability of zero means the outcome never happens.

4. When all possible outcomes are included, the sum of the probabilities of each exclusive outcome is one. That is, the probability that *something* happens is one. So if we flip a coin, the probability that it lands heads or tails is $\frac{1}{2} + \frac{1}{2} = 1$. If we toss a die, the probability that it lands with 1, 2, 3, 4, 5, or 6 spots showing is $\frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = 1$.

The mapping of a probability to each possible outcome is called a *probability distribution*. Just as our mental picture of there being a “true” value that we can only estimate, we also envision a “true” probability distribution that we can only estimate through observation.

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1 ... one of a pair of dice.
The most common probability distribution encountered in
the lab is the **Gaussian** distribution (Figure 1). The Gauss-
ian distribution is also known as the **normal** distribution.
You may have heard it called the **bell curve** (it is shaped
somewhat like a fancy bell) when applied to grade distrib-
utions.

The mathematical form of the Gaussian distribution is:

\[
P_G(d) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-d^2/2\sigma^2}
\]  

The Gaussian distribution is ubiquitous because it is the
end result you get if you have a number of processes, each
with their own probability distribution, that “mix together” to yield a final result. A key aspect of
this distribution is that the probability of obtaining a result between \(-\sigma\) and \(\sigma\) is approximately
2 out of 3.

**Statistics**

Measurements of physical quantities are expressed in numbers. The numbers we record are
called **data**, and numbers we compute from our data are called **statistics**. A statistic is by defini-
tion a number we can compute from a set of data.

Perhaps the single most important statistic is the mean or average. Often we will use a “bar”
over a variable (e.g., \(\bar{x}\)) or “angle brackets” (e.g., \(\langle x \rangle\)) to indicate that it is an average. So, if we
have \(N\) measurements \(x_i\) (i.e., \(x_1, x_2, \ldots, x_N\)), the average is given by:

\[
\bar{x} = \langle x \rangle = \frac{1}{N} \sum_{i=1}^{N} x_i
\]  

In the lab, the average of a set of measurements is usually our best estimate of the “true” value
\(^2\)

\[
\bar{x} \approx x
\]

Another useful statistic is given by the **sample standard deviation**:

\[
s_x = \sqrt{\left( \frac{1}{N-1} \right) \sum_{i=1}^{N} (\bar{x} - x_i)^2}
\]

The sample standard deviation is defined as the average amount by which values in a distribution
differ from the mean (ignoring the sign of the difference). The standard deviation is useful
because it gives us a measure of the spread or statistical uncertainty in the measurements.

Modern spreadsheets (such as MS Excel) as well as some calculators (such as HP and TI) also
have built-in statistical functions. For example, **AVERAGE** (Excel) and \(\bar{x}\) (calculator) calculate
the average of a range of cells; whereas **STDEV** (Excel) and \(s_x\) (calculator) calculate the sample
standard deviations.

\(^2\) For these discussions, we will denote the “true” value as a variable without adornment (e.g., \(x\)).
Standard Error

Whenever we report a result, we also want to specify a standard error in such a way as to indicate that we think that there is roughly a $\frac{2}{3}$ probability (defined by the integral under the probability distribution) that the “true” value is within the range of values between our result minus the standard error to our result plus the standard error. In other words, if $\bar{x}$ is our best estimate of the “true” value $x$ and $\sigma_{\bar{x}}$ is our best estimate of the standard error in $\bar{x}$, then there is a $\frac{2}{3}$ probability that:

$$\bar{x} - \sigma_{\bar{x}} \leq x \leq \bar{x} + \sigma_{\bar{x}}$$

When we report results, we use the following notation:

$$\bar{x} \pm \sigma_{\bar{x}}$$

Thus, for example, the electron mass is given in the 2012 Particle Data Group as

$$m_e = (9.10938291 \pm 0.00000040) \times 10^{-31} \text{ kg}$$

By this we mean that the electron mass lies between $9.10938251 \times 10^{-31}$ kg and $9.10938331 \times 10^{-31}$ kg, with a probability of roughly $\frac{2}{3}$. Note that if we report the mean value of several measurements the associated standard error is of the mean not of the sample. We will address the difference in section below.

Significant Figures

In informal usage the least significant digit implies something about the precision of the measurement. For example, if we measure a rod to be 101.3 mm long but consider the result accurate to only $\pm 0.5$ mm, we round off and say, “The length is 101 mm.” That is, we believe the length lies between 100 mm and 102 mm, and is closest to 101 mm. The implication, if no error is stated explicitly, is that the uncertainty is 1/2 of one digit, in the place following the last significant digit. This implication is very useful in estimating the error of a single measurement on digital devices such as timers, voltage generators etc.

Zeros to the left of the first non-zero digit do not count in the tally of significant figures. If we say $U = 0.001325$ Volts, the zero to the left of the decimal point, and the two zeros between the decimal point and the digits 1325 merely locate the decimal point; they do not indicate precision. [The zero to the left of the decimal point is included because decimal points are small and hard to see. It is just a visual clue—and it is a good idea to provide this clue when you write down numerical results in a laboratory!] The voltage $U$ has thus been stated to four (4), not seven (7), significant figures. An easy rule to apply is to re-write the number in scientific notation by eliminating all possible zeroes into the exponent. We could bring this out more clearly by writing either $U = 1.325 \times 10^{-3}$ V, or $U = 1.325$ mV.

Propagation of Errors

More often than not, we want to use our measured quantities in further calculations. The question that then arises is: How do the errors “propagate”? In other words: What is the standard error in a particular calculated quantity given the standard errors in the input values?

Before we answer this question, we want to introduce a new term. The relative error of a quantity $Q$ is simply its standard error, $\sigma_Q$, divided by the absolute value of $Q$. For example, if a length is known to $49 \pm 4$ cm, we say it has a relative error of $4/49 = 0.082$. It is often useful
to express such fractions in percent. In this case we would say that we had a relative error of 8.2%.

We will simply give the formulae for propagating errors as the derivations are a bit beyond the scope of this exposition. The final results of these derivations are summarized on the last page of this Appendix in an easy to look up table.

**Standard Error in the Mean**

Suppose that we make two independent measurements of some quantity: $x_1$ and $x_2$. Our best estimate of $x$, the “true value”, is given by the mean, $\bar{x} = \frac{1}{2}(x_1 + x_2)$, and our best estimate of the standard error in $x_1$ and in $x_2$ is given by the sample standard deviation, $\sigma_{x_1} = \sigma_{x_2} = s_x = \sqrt{\frac{1}{2}[(x_1 - \bar{x})^2 + (x_2 - \bar{x})^2]}$. Note that $s_x$ is not our best estimate of $\sigma_{\bar{x}}$, the standard error in $\bar{x}$.

We must use the propagation of errors formulas to get $\sigma_{\bar{x}}$. Now, $\bar{x}$ is not exactly in one of the simple forms where we have a propagation of errors formula. However, we can see that it is of the form of a constant, $1/2$, times something else, $(x_1 + x_2)$, and so:

$$\sigma_{\bar{x}} = \frac{1}{\sqrt{2}} s_x$$

The “something else” is a simple sum of two quantities with known standard errors ($s_x$) and we do have a formula for that:

$$\sigma_{x_1+x_2} = \sqrt{\sigma_{x_1}^2 + \sigma_{x_2}^2} = \sqrt{s_x^2 + s_x^2} = \sqrt{2} s_x$$

So we get the desired result for two measurements:

$$\sigma_{\bar{x}} = \frac{1}{\sqrt{2}} s_x$$

By taking a second measurement, we have reduced our standard error by a factor of $1/\sqrt{2}$. You can probably see now how you would go about showing that adding third, $x_3$, changes this factor to $1/\sqrt{3}$. The general result (for $N$ measurements) for the standard error in the mean is:

$$\sigma_{\bar{x}} = \frac{1}{\sqrt{N}} s_x$$

(5)

As mentioned above in the section Standard Error, the standard error of the mean determines the 2/3 probability range for the mean value of our measurements. Excel does not have a built-in function for the standard deviation of the mean, however you can use the function **COUNT** to calculate the error of the mean. For example for rows A1 through A10 standard error of the mean equals $\text{STDEV(A1:A10)}/\text{SQRT(COUNT(A1:A10))}$

**Examples**

**Example-1**: What do I write on the “±” line in my lab report?

Most quantities you will be calculating and reporting in this lab are written in the following format:

Value : ___________ ± ___________ units

What are you expected to write for the error?

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3 From per centum, Latin for “by the hundred”.

4 Important Note: These expressions assume that the deviations are small enough for us to ignore “higher order” terms and that there are no correlations between the deviations of any of the quantities $x$, $y$, etc.
• If the quantity was measured several times you should at the minimum report the standard error in the mean.

• If the measured quantity was measured only once, you can always report the error in the significant figure of the measuring device. If a ruler, for example, has the smallest division of 1 mm than the measured value is known at best with a 0.5 mm precision. If a voltage generator digital display reads the output voltage as 1.0 V then one can assume that the standard error of that measurement is 0.05 V etc. etc. Note that the significant error of the measuring device is always there even if you took a sample of measurements. In most cases however, significant figure error would be much smaller than the sample standard deviation fluctuations of the sample and can be easily neglected. In cases where it is not true, the largest of the two should be recorded.

• If the measured quantity was not measured directly but rather calculated using a graph (for example velocity from position vs. time plot) the Data Studio, the program we use to display these graphs, offers variety of pre-packaged statistics to choose from. One can either use the statistics feature or the fit feature. Both tools readily report associated errors. However, be very careful with either feature and try to understand the limits of applicability of offered statistics (a bit more on the subject later).

• A lot of times you will find yourself not being quite comfortable reporting, for the most part, the significant figure error as not being conservative. For example you measured the length of a pendulum string once and reported the standard deviation as 0.5 mm. However while doing the measurement you clearly saw that holding the ruler quite vertical was challenging and the measuring technique could easily double the reported error. And it is OK to double it as long as you justify in your report why you have done so.

• Finally some quantities are derived from other measured values. Measured values have errors associated with them (calculated with any of the methods described above). In such cases you should apply the propagation of error technique and apply the formulas summarized in this Appendix. We will give you some examples of how this technique can be applied below.

Example-2: What should my answer to ”Discuss the agreement” kind of question be?

After you reported the final measured value with associated error you will be asked to discuss the agreement between the measurement and the theory or two (or more) measurements. The answer ”Yes, two quantities agree well” or ”No, two quantities do not agree well” is not an acceptable answer. Reporting just the percent difference (as you hopefully learned from this Appendix) is not very useful either and should be resorted to cases where you have exhausted all other possibilities of estimating the associated error. Most, if not all, measured values will have errors associated with it. You should use your knowledge of statistics and probabilities to justify your answer.

For example a theoretical value of a resonance frequency in the system was calculated to be 600 Hz. You measured the same frequency experimentally by finding the maximum of the system’s amplitude to be 630 Hz. You also noticed that the system’s resonance was not very well pronounced and within 30 Hz around your best estimate of 630 Hz you could not see a detectable change in the amplitude. You would probably report your findings as follows

\[ f_{\text{measured}} = 630 \pm 30 \text{ Hz} \quad \text{and} \quad f_{\text{theory}} = 600 \text{ Hz} \]
The two values are within one standard deviation of one another and thus there is a \( \approx 66\% \) chance that the value you measured is the true value. Such an agreement (when the values are 1 or less standard deviation apart) is traditionally considered quite satisfactory. However two values being separated by more than one standard deviation does not negate your effort and make your measurement invalid. In science we talk about the measurement being unlikely to represent the true value when the two are 3 (three) or more standard deviations apart (thus making the probability of the measurement being the true value 1\% or lower). On the other hand a large discrepancy (over 1 standard deviation) between the theoretical and experimental values is quite often a clue that your experiment was not conducted correctly and there are some systematic errors in your procedure. Most interesting however is the case when you can, to the best of your knowledge, eliminate and correct for such systematic errors while the two values still remain few standard deviations apart. Such a situation may mean that the underlying theory is incorrect and needs revising. Thus having a reliable error associated with your measurement may indicate the presence of earlier undetected or not understood physical phenomena which is an exciting opportunity for every scientist.

**Example-3**: Two simple and (hopefully) intuitive examples of most commonly used propagation of error formulae.

Two most commonly used formulae of propagation of error in our lab are propagation of error of the sum (difference) of two quantities and propagating the error of the ratio (product) of two quantities. If we refer to the Summary of Statistical Formulae in this Appendix we can see that the propagation of error formulae for sum and the ratio have very similar, let’s call it a “funny” sum, form

\[
\text{Some error}_{\text{final}} = \sqrt{(\text{Some error}_1)^2 + (\text{Some error}_2)^2}
\]

The only difference is that in case of the sum (difference) ”Some error” is the actual value of the standard error of the sum (difference), while in case of ratio (product) ”Some error” is the relative error of the ratio (product). A simple mnemonic rule might help you memorize these formulae. The error of the sum is the ”funny” sum of the errors. And the error of the ratio when written as the ratio (aka relative error), is the ”funny” sum of the ratios.

We can apply these simple rules to the following example. Imagine we have measured number of radioactive decays for two similar time intervals and the numbers we calculated were \( N_1 = 100 \pm 1 \) and \( N_2 = 101 \pm 1 \). Clearly both measurements are within one standard deviation of each other and we are convinced that these are good measurements of the same radioactive decay. \( N_1 \) is known with standard deviation of \( \sigma_{N_1} = 1 \) and relative error of \( \sigma_{N_1}^{\text{relative}} = \sigma_{N_1}/N_1 = 1\% \), and similarly \( \sigma_{N_2} = 1 \) and relative error (to a very good approximation) of \( \sigma_{N_2}^{\text{relative}} = \sigma_{N_2}/N_2 = 1\% \).

Now let’s calculate the difference and the ratio of \( N_2 \) and \( N_1 \). Using the ”funny” sum for the difference we get

\[
N_2 - N_1 = 1.0 \quad \text{and} \quad \sigma_{N_2-N_1} = \sqrt{\sigma_{N_1}^2 + \sigma_{N_2}^2} = \sqrt{1 + 1} \approx 1.4
\]

or

\[
N_2 - N_1 = 1.0 \pm 1.4
\]

Similarly

\[
N_2/N_1 = 1.01 \quad \text{and} \quad \frac{\sigma_{N_2}/N_1}{N_2/N_1} = \sqrt{(\sigma_{N_2}/N_1)^2 + (\sigma_{N_2}/N_2)^2} = \sqrt{1\% + 1\%} \approx 1.4\%
\]
therefore
\[ \sigma_{N_2/N_1} = 1.4\% \text{ of } (N_2/N_1) = 0.014 \approx 0.01 \]
or
\[ N_2/N_1 = 1.01 \pm 0.01 \]

We applied the formulae correctly but the result, at first glance, seems to be absurd. How could the difference of \( N_2 \) and \( N_1 \) be so less precise than their ratio while being of the same magnitude? The result is the consequence of our "funny" sum rule. In case of the difference the absolute value of standard error was 1.4 while in case of the ratio the relative error was 1.4 (\%) and applied to the same value made the error hundred times smaller. On the other hand the result is quite intuitive. Imagine that the value of \( N_2 \) was measured to be 102 (we have a 2/3 chance of getting such a value in a subsequent measurement). Such one standard deviation change in \( N_2 \) will result in \( N_2 - N_1 = 2.0 \) a nearly one standard deviation change in the difference \((1+1)\) which is what we intuitively expect from properly administered propagation of error. This deviation in \( N_2 \) is a 100\% change of the value of \( N_2 - N_1 \) and that is why the standard error of the difference is so large. The same one standard deviation in \( N_2 \) will lead to \( N_2/N_1 = 1.02 \) which is also a nearly one standard deviation change of the final result \((1.01+0.01)\) - a consistency we expect from our methods. But we only had a relatively small \((\approx 1\%)\) change in \( N_2 \) and that is why the standard error of the ratio is so small.

**Example-4:** Using (correctly and incorrectly) the *Data Studio* features to extract the error and propagating it to the final result.

As mentioned in Example-1 *Data Studio* can be really helpful in calculating the experimental errors. Imagine that we inelastically collided a cart with another cart and recorded the velocity of the carts before, during and after the collision. A typical graph of such a process is shown in Figure 2. In order to check the so-called conservation of linear momentum law we would need to measure the velocity of the cart right before and right after the collision. Had our track been absolutely frictionless the velocities before and after the collision would have been constant, one could easily apply the statistics feature of *Data Studio* to extract the mean and the standard deviation of an uniformly distributed value of velocity around some constant value. As can be clearly seen from the graph, due to some frictional forces along the track our velocities do not stay constant. Applying the sample standard deviation equations to non-uniformly distributed sample would be incorrect. The mean value of the sample distributed along the sloped line would overestimate the value of velocity before the collision, and it would also overestimate the value of the standard deviation of the mean and make our experiment less precise (and less accurate as well).

We can remedy the situation by using the fit feature of our software. Instead of wrongly assuming that the velocities were constant, we can make a more physical assumption that the force of friction was constant and therefore our velocity vs. time graph is a straight line right to the point in time before the carts collided. We can apply a linear fit to our velocity graph and extract the parameters of the best fit (including their errors)

\[ v = m \cdot t + b \]

where \( v \) is the velocity, \( t \) is time, \( m \pm \sigma_m \) is the slope (and error) of our line and \( b \pm \sigma_b \) is the y-axis intercept and its error. The values of the slope and intercept errors will be reported next to the values of the parameters. On the graph we can identify the time before the collision \((t_b)\) quite accurately and plug this value of time to extract the value of velocity right before the collision.

\[ v_b = m \cdot t_b + b \]
and using propagation of error formulae

\[ \sigma_{v_b} = \sqrt{(\sigma_m \cdot t_b)^2 + (\sigma_b)^2} \]

In our example we can identify from the graph \( t_b = 0.7 \text{ s} \). Therefore the value of velocity right before the collision is \( v_b = -0.1004 \cdot 0.7 + 0.4702 = 0.3999 \) and \( \sigma_{v_b} = \sqrt{(0.0094 \cdot 0.7)^2 + (0.0042)^2} = 0.0078 \)

\[ v_b = 0.3999 \pm 0.0078 \]

Also note that if we blindly applied the statistics feature to the data points of velocity before the collision (0.461, 0.45, 0.439, 0.431, 0.42, 0.409, and 0.401) we would have obtained \( \bar{v} = 0.4301 \pm 0.0082 \). The two values are 2.6 standard deviations apart and the latter value would introduce a large systematic error in our final calculation.

We will apply the fit method to obtain both more accurate and precise confirmations of the conservation laws several times in our labs.

**Example-5**: Calculating and propagating statistical errors of multiple measurements.

We can measure the gravitational acceleration \( g \) near the Earth’s surface by dropping a mass in a vertical tube from which the air has been removed. Since the distance of fall \( (D) \), time of fall \( (t) \) and \( g \) are related by \( D = \frac{1}{2} gt^2 \), we have \( g = \frac{2D}{t^2} \). So we see that we can determine \( g \) by simply measuring the time it takes an object to fall a known distance. We hook up some photogates\(^5\) to a timer so that we measure the time from when we release a ball to when it gets to the photogate. We very carefully use a ruler to set the distance \( (D) \) that the ball is to fall to 1.800 m. We estimate that we can read our ruler to within 1 mm. We drop the ball ten times and get the following times \( (t_i) \): 0.6053, 0.6052, 0.6051, 0.6050, 0.6052, 0.6054, 0.6053, 0.6047, 0.6048, and 0.6055 seconds. The average of these times \( (\bar{t}) \) is 0.605154 seconds. Our best estimate of \( g \) is then

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\(^5\) A device with a light source and detector that changes an output when something comes between the source and detector.
\( g_{\text{exp}} = \frac{2D}{\bar{t}^2} = 9.83039 \text{ m/s}^2 \)

This is larger than the “known local” value of 9.809 m/s\(^2\) by 0.0215 m/s\(^2\) (0.2\%). We do expect experimental uncertainties to cause our value to be different, but the question is: Is our result consistent with the “known value”, within experimental errors? To check this we must estimate our standard error.

**VERY IMPORTANT NOTE:** Do NOT round off intermediate results when making calculations. Keep full “machine precision” to minimize the effects of round-off errors. Only round off final results and use your error estimate to guide you as to how many digits to keep.

Our expression for \( g \) is, once again\(^6\), not precisely in one of the simple propagation of errors forms and so we must look at it piecemeal. This time we will not work it all out algebraically, but will instead substitute numbers as soon as we can so that we can take a look at their effects on the final standard error.

What are our experimental standard errors? We’ve estimated that our standard error in the distance (\( \sigma_D \)) is 1 mm (hence a relative error, \( \sigma_D/|D| \), of 0.000556 or 0.0556\%). From our time data we calculate the sample standard deviation \( s_t \) to be 0.000259 seconds. Recall that this is not the standard error in the mean (our best estimate of the “true” time for the ball to fall), it is the standard error in any single one of the time measurements \( t_i \). The standard error in the mean is \( s_t \) divided by the square root of the number of samples (10): \( \sigma_{\bar{t}} = s_t / \sqrt{10} = 0.0000819 \text{ seconds} \) (for a relative error, \( \sigma_{\bar{t}}/\bar{t} \), of 0.000135 or 0.0135\%).

We see that the relative error in the distance measurement is quite a bit larger than the relative error in the time measurement and so we might assume that we could ignore the time error (essentially treating the time as a constant). However, the time enters into \( g \) as a square and we expect that that makes a bigger contribution than otherwise. So we don’t (yet) make any such simplifying assumptions.

We see that our estimate of \( g \) (which we denote by \( g_{\text{exp}} \)) is of the form of a constant (2) times something else (\( D/\bar{t}^2 \)) and so:

\[
\sigma_{g_{\text{exp}}} = |2|\sigma_{D/\bar{t}^2}
\]

\( D/\bar{t}^2 \) is of the form of a simple product of two quantities (\( D \) and \( \bar{t}^2 \)) and so:

\[
\sigma_{D/\bar{t}^2}/|D/\bar{t}^2| = \sqrt{(\sigma_D/D)^2 + (\sigma_{\bar{t}}/\bar{t})^2}
\]

Now we are getting somewhere as we have

\[
\sigma_{D/|D|} = 0.000556
\]

We need only find \( \sigma_{\bar{t}}/\bar{t}^2 \). Since \( \bar{t}^2 \) is of the form of a quantity raised to a constant power and so:

\[
\sigma_{\bar{t}}/\bar{t}^2 = |2|\sigma_{\bar{t}/|\bar{t}|} = 0.000271
\]

Now we can see the effect of squaring \( \bar{t} \): Its contribution to the standard error is doubled. Consider the two terms under the square root:

\[
(\sigma_D/D)^2 = 3.09 \times 10^{-7} \quad \text{and} \quad (\sigma_{\bar{t}}/\bar{t})^2 = 0.733 \times 10^{-7}
\]

\(^6\) Refer to the discussion of the standard error in the mean.
Now we can see that, even though the time enters as a square, we would have been justified in ignoring its contribution to the standard error in \( g \). Plugging the numbers back in, we finally get \( \sigma_{\text{exp}} = 0.00608 \text{ m/s}^2 \). Note again, just like discussed in Example-3, the use of the relative errors in case of the ratios, products and exponents makes the calculations very similar to calculating the errors for sums and differences. Basically you will only need to remember ”one formula” and where the absolute or relative errors are appropriate. Following the standard notation we can write

\[
g = 9.83039 \pm 0.00608 \text{ m/s}^2
\]

At this point we can round off our final result to a more tangible form since there is no particular advantage in keeping 6 significant figures in the value which is already uncertain in the 4th significant figure.

\[
g = 9.830 \pm 0.006 \text{ m/s}^2
\]

We see that our result is 3.5 standard deviations larger than the “known value”. While not totally out of the question, it is still very unlikely that our value is accurate, and so we need to look for the source of the problem. In this case we find that the ruler is one of those with a worn end. We carefully measure the offset and find the ruler to be 5.0 mm short. Subtracting this changes \( D \) to 1.795 m and \( g_{\text{exp}} \) to 9.803 m/s\(^2\), well within our experimental error.

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\[^7\] The implied error in our measurement of the offset (0.05 mm) is much smaller than the error in the original \( D \) and so we can afford to ignore its contribution to the standard error in \( g_{\text{exp}} \).
Summary of Statistical Formulae

Sample mean: \( \bar{x} = \frac{1}{N} \sum_{i=1}^{N} x_i \)
(best estimate of the “true” value of \( x \), using \( N \) measurements)

Sample standard deviation: \( s_x = \sqrt{\left( \frac{1}{N-1} \right) \sum_{i=1}^{N} (\bar{x} - x_i)^2} \)
(best estimate of error in any single measurement, \( x_i \))

Standard error of the mean: \( \sigma_{\bar{x}} = \frac{1}{\sqrt{N}} s_x \)
(best estimate of error in determining the population mean, \( \bar{x} \))

Summary of Error Propagation Formulae

<table>
<thead>
<tr>
<th>Functional form</th>
<th>Error propagation formula</th>
</tr>
</thead>
</table>
| 1. \( f(x) = Cx \) | \( \sigma_f = |C| \sigma_x \)
|                 | \( \sigma_f / f = \sigma_x / x \) |
| 2. \( f(x,y) = x \pm y \) | \( \sigma_f = \sqrt{\sigma_x^2 + \sigma_y^2} \) |
| 3. \( f(x,y) = x \times y \) or \( x/y \) | \( \sigma_f / |f| = \sqrt{(\sigma_x/x)^2 + (\sigma_y/y)^2} \) |
| 4. \( f(x) = x^a \) | \( \sigma_f / |f| = |a| \sigma_x / |x| \) |
| 5. \( f(x) = \ln(x) \) | \( \sigma_f = \sigma_x / |x| \) |
| 6. \( f(x) = e^x \) | \( \sigma_f / |f| = \sigma_x \) |
| 7. \( f(x) = Cx^a y^b \) | \( \sigma_f / |f| = \sqrt{\left( \frac{a \sigma_x}{x} \right)^2 + \left( \frac{b \sigma_y}{y} \right)^2} \) |
| and the general form: | |
| 8. \( f(x,y,...) \) | \( \sigma_f^2 = \left( \frac{\partial f}{\partial x} \right)^2 \sigma_x^2 + \left( \frac{\partial f}{\partial y} \right)^2 \sigma_y^2 + \ldots \) |

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8 These expressions assume that the deviations are small enough for us to ignore “higher order” terms and that there are no correlations between the deviations of any of the quantities \( x, y \), etc.