Lab 10 - Harmonic Motion and the Pendulum

A body is said to be in a position of stable equilibrium if, after displacement in any direction, a force restores it to that position. If a body is displaced from a position of stable equilibrium and then released, it will generally oscillate about the equilibrium position.

We will consider a system with a restoring force that is proportional to the displacement from the equilibrium position and show that such a restoring force leads to a sinusoidal oscillation. Such a system is called a simple harmonic oscillator and its motion is called a harmonic motion.

The simple harmonic oscillator is of great importance in physics because many more complicated systems can be treated to a good approximation as harmonic oscillators. Examples are the pendulum, a stretched string (indeed all musical instruments), the molecules in a solid, and the atoms in a molecule.

The system that we will be studying in this session is the pendulum: it has a position of stable equilibrium and undergoes a simple harmonic motion for small displacements from the equilibrium position. We will first analyze the motion theoretically before performing the experiments.
A simple pendulum consists of a mass \( m \) suspended from a fixed point by a string of length \( L \). If the mass is pulled aside and released, it will move along the arc of a circle, as shown in Figure [10.1]. Let \( s \) be the distance from the equilibrium position measured along that arc. While the force of gravity, \( mg \), points downward, only its tangential component along the arc, \( F_{\text{tan}} = -mg\sin\theta \), acts to accelerate the mass. The minus sign indicates that \( F_{\text{tan}} \) is a restoring force, i.e. one that points in a direction opposite to that of the displacement \( s \).

From Newton’s second law, \( F_{\text{tan}} = ma \), we then get:

\[
-mg\sin\theta = ma
\]

(10.1)

By definition \( a = \frac{d^2s}{dt^2} \) so \( \frac{d^2s}{dt^2} = -g\sin\theta \). We can easily express \( s \) in terms of \( \theta \) by noting that \( s = L\theta \), hence \( \frac{d^2s}{dt^2} = L\frac{d^2\theta}{dt^2} \) and thus

\[
\frac{d^2\theta}{dt^2} = -\frac{g}{L}\sin\theta
\]

(10.2)

This second order differential equation is the equation of motion for the pendulum. It looks deceptively simple but is actually quite difficult to solve.

For many applications, including ours, a simplified approximation is sufficient. We can obtain the approximate solution by using the series expansion of the sine, (valid when \( \theta \) is measured in radians). From this series expansion, we see that we can approximate \( \sin\theta \) by the angle \( \theta \) itself, as long as we keep \( \theta \) so small that the higher order terms are much smaller than \( \theta \). In other words, when \( \theta \ll 1 \), \( \sin\theta \approx \theta \). This is called the small angle approximation.

Using the small angle approximation, Equation (10.2) becomes

\[
\frac{d^2\theta}{dt^2} = -\frac{g}{L}\theta
\]

(10.3)

This is the equation of motion of a simple harmonic oscillator, describing a simple harmonic motion. It is solved for arbitrary values of the amplitude, \( \theta_{\text{max}} \), and the phase, \( \delta \), by:

\[
\theta = \theta_{\text{max}}\cos(\omega t + \delta)
\]

(10.4)

If you substitute Equation (10.4) into Equation (10.3), you will find that the angular frequency, \( \omega \), is not arbitrary. It must have the value

\[
\omega = \sqrt{\frac{g}{L}}
\]

(10.5)

The cosine oscillates between -1 and 1; hence, according to Equation (10.4), the angle \( \theta \) will oscillate over time between \(-\theta_{\text{max}} \) and \( \theta_{\text{max}} \). The time to execute one complete cycle is called the period \( T \) and is related to \( \omega \) by

\[
T = \frac{2\pi}{\omega}
\]

(10.6)
NOTE: What is not in an equation is often as revealing as what is in it. Equation (10.6) does not contain the mass of the pendulum. This means that the period is independent of the mass. That can be true only if the mass on the left side of Equation (10.1) cancels the mass on the right side. You might think that this is obvious, but it is not. The mass on the left side is a measure of the resistance that the pendulum bob offers to acceleration, which we call the inertial mass. The mass on the right side is a measure of the gravitational force acting on the bob, and we call it the gravitational mass. While one would expect the period to remain unchanged if one replaces the bob with one of the same material but a different mass, it cannot be taken for granted that the ratio of inertial to gravitational mass is the same for all materials. Newton already worried about this point and did an experiment to uncover a possible difference, but found none. Modern experiments have established that changing the material of the bob changes the ratio of two kinds of mass by no more than 1 part in 1011. It was this observation that led to Einstein’s general theory of relativity.

The speed of the mass is given by \( v = \frac{ds}{dt} \) or, since \( s = L\theta \), by \( v = L\frac{d\theta}{dt} \). If we substitute \( \theta \) from Equation (10.4) we obtain

\[
v(t) = -\theta_{\text{max}} L\omega \sin(\omega t + \delta) = \theta_{\text{max}} L\omega \cos(\omega t + \delta + \pi/2) \tag{10.7}
\]

Comparing Equation (10.7) with Equation (10.4), we see that the speed \( v \) oscillates with the same frequency as the angle \( \theta \) but is out of phase with \( \theta \) by \( \pi/2 \). Clearly this makes sense; the speed is zero when the angle is greatest.

What is the significance of \( \theta_{\text{max}} \) and \( \delta \)? They are not determined by the previous equations but are fixed by the initial conditions. \( \theta_{\text{max}} \) is the maximum value of \( \theta \) during the motion. It is determined by how far the pendulum was pulled back initially. The phase \( \delta \) depends on when the clock was started that is used to measure \( t \).

**INVESTIGATION 1: Mass Independence of Period**

In this activity you will measure the period of a simple pendulum.

The materials you will need are:

- photogate timer and accessory photogate
- large and small support stands
- thread (string), meter stick, protractor
- pendulum string clamp
- brass and aluminum pendulum bobs
- 2 right angle clamps
- long aluminum rod
- short aluminum rod
- digital mass scale
Activity 1-1: Period of Aluminum Bob

If we combine Equations (10.5) and (10.6) we obtain

\[ T = 2\pi \sqrt{\frac{L}{g}} \]  

(10.8)

Note that the mass of the bob does not appear in this expression. To see if this is true, we want to measure the periods of an aluminum and a brass bob having different masses, but having the same length.

To make sure that you start successive measurements with the same amplitude, mount an aluminum rod (a reference bar) at an appropriate distance from the equilibrium position and start each measurement with the bob touching the rod.

Question 1-1: Discuss the conditions that need to be met to test Equation (10.8). Discuss, for example, the initial angle of displacement. What are its limitations? Why should the initial angle always be the same?

Note: To make the pendulums the same length suspend them side by side in the same clamp (held between the clamp and the bar) and adjust their lengths so that the center-of-mass grooves of the two bobs are at exactly the same level. Have your TA check your setup before proceeding!

1. Set the pendulum lengths (measured from the pivot point to the line marking the bob’s center of mass) to about 90 cm. Record the measured length:

   Pendulum length: ________ cm

   Prediction 1-1: What do you predict the period of the pendulum motion to be? Use the local value of the acceleration of gravity \( g = 9.809 \text{ m/s}^2 \). Show your calculation.

2. Move the brass bob out of the way by draping it over the stand, and measure the period of the aluminum bob. To make sure that you always start with the same amplitude, use a reference bar to set the initial angle \( \theta_{\text{max}} \) to 5°. The timing light is best placed at the equilibrium position \( s = 0 \). Align the photogate such that the center of the bob breaks the “beam”.

3. The timer should be in pendulum mode at 0.1 ms resolution. In pendulum mode, three triggers are necessary. The first trigger starts the timer. [In this case, the bob passing
through the photogate the first time causes this first trigger.] The second trigger is ignored by the timer. [The bob swinging back through the photogate causes the second trigger.] The third trigger stops the timer (which corresponds to the bob passing through the photogate a third time – after one complete period). If the memory switch is on, additional triggers will not change the displayed time unless you push the memory switch to READ. Practice releasing the bob a few times.

4. Pull back the bob and release it. Wobbling will result in a poor measurement; Try to minimize the wobbling by holding the bob at its center of mass.

5. Measure the period (three times) for the aluminum bob and put your results in Table 10.1.

<table>
<thead>
<tr>
<th>Trial</th>
<th>Aluminum Time (s)</th>
<th>Brass Time (s)</th>
</tr>
</thead>
<tbody>
<tr>
<td># 1</td>
<td></td>
<td></td>
</tr>
<tr>
<td># 2</td>
<td></td>
<td></td>
</tr>
<tr>
<td># 3</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 10.1: Period of Pendulum

**Activity 1-2: Period of Brass Bob**

1. Hang the bobs side by side once again and see if they are still the same length. If the length has changed, you may need to measure the length again and repeat your measurements. Move the aluminum bob aside so you can measure the period of the brass bob.

2. Measure the period of the brass pendulum (again, three times) and enter your data into Table 10.1.

**Question 1-2:** What are your experimental errors in the pendulum measurement? Discuss them and make estimates here:

3. Average your measurements of the period for each mass and find the *standard error*. [See Appendix D for more information.]

   Aluminum pendulum period (mean): ___________ s   Standard error ___________ s

   Brass pendulum period (mean): ___________ s   Standard error ___________ s
**Question 1-3:** Is there a significant difference between the two periods? How do you know?

**Question 1-4:** How well do your measurements for the period agree with Prediction 11? Explain any differences.

---

**INVESTIGATION 2: Dependence on Angle and Length**

**Activity 2-1: Dependence on Angle**

The simple theory says the period does not depend on the amplitude $\theta_{\text{max}}$, but that statement is based on the assumption that $\theta$ is small. In this experiment we want to investigate the dependence of the period on the initial angle $\theta$. We want to see if there is a dependence on the magnitude of the initial angle and to see the effect of assuming small angles in this measurement.

**Do not allow the bobs to strike the photogate, ESPECIALLY DURING THE LARGER OSCILLATIONS!!**

1. You want to use the brass bob and measure the period $T$ for several initial angles $\theta$ as shown in Figure [10.1]. Make sure that you have aligned the protractor properly with the bob in its equilibrium position.

2. Use $L \approx 60$ cm and $\theta = 5^\circ, 10^\circ, 20^\circ, 30^\circ, 40^\circ$. For each angle, measure $T$ three times and make sure the results are consistent. If not, re-measure until they are and average the results.

Length of Pendulum: ____________ cm

<table>
<thead>
<tr>
<th>Angle</th>
<th>$T_1$</th>
<th>$T_2$</th>
<th>$T_3$</th>
<th>$\bar{T}$</th>
<th>$\sigma_T$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$5^\circ$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$10^\circ$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$20^\circ$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$30^\circ$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$40^\circ$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
**Question 2-1:** Describe what you observe in the raw data. Is the period constant as the angle increases?

3. Use Excel to plot your average period, $\overline{T}$, versus $\theta_{\text{max}}$. Your plot should show one half of a shallow parabola opening upward.

4. **Print** one copy of your graph.

**Question 2-2:** Note that in this experiment we have had you repeat measurements. Why do you think this is so? In general how well have your measurements agreed?

**Activity 2-2: Dependence on Pendulum Length**

1. You will now measure $T$ for several lengths, say 30, 60, and 90 cm for one angle: $\theta_{\text{max}} = 5^\circ$.

**Question 2-3:** What criteria should be used when determining what initial angle to use for this experiment? Discuss these criteria.

2. Take three trials at each length and enter your data into the following table. Average your results. It is not necessary for you to have the lengths set right at 30, 60, and 90 cm. Set it close and measure the actual length.

**NOTE:** You can use the “brass” data from Activity 12 for the “90 cm” column and the “5°” data from Activity 21.
<table>
<thead>
<tr>
<th>Trial</th>
<th>Period $T$ (s) for each length $L$ (cm)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Nominal length:</td>
</tr>
<tr>
<td></td>
<td>30 cm</td>
</tr>
<tr>
<td>Actual length:</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
</tr>
</tbody>
</table>

3. Plot the square of the period, $T^2$, vs. the pendulum length, $L$, using Excel. Fit a straight line to the data using the LINEST function. To do so, highlight a 2x2 cell area. Select Insert → Function → Linest. In the pop-up menu enter corresponding columns for the $x - y$ axes and set Statistics to TRUE (1). Since LINEST is an array function, you must hold down "Ctrl" + "Shift" + "Enter" keys to fill the cells properly. The upper row of the 2x2 cell area should be filled with the values of the linear fit parameters and the bottom row with the values of uncertainty of the linear fit parameters.

4. Print a copy for your report.

**Question 2-4:** From your fit to $T^2$ vs. $L$, calculate $g$ and $\sigma_g$. Show your work.

---

**INVESTIGATION 3: Conservation of Energy**

As you have seen in previous labs, the total energy, i.e. the sum of potential and kinetic energy, should be a constant. On the other hand, both kinetic and potential energies change with time. To check the conversion of potential energy into kinetic energy, we want to measure the maximum speed $v_{\text{max}}$ of the bob as it passes through the equilibrium position (see Figure 10.2). Conservation of energy ($|\Delta KE| = |\Delta PE|$) requires that:

$$\frac{1}{2}mv^2 = mg\Delta h = mg(h_2 - h_1)$$

(10.9)

**Activity 3-1: Measurement of Speed**

To measure the maximum speed of the bob, we will measure how long it takes the bob to pass between two photogate timers. We can measure the time very accurately, but we must also know the distance between the photogates, $\Delta s$. 

Prediction 3-1: We claim that a distance of about $\Delta s = 7\ \text{cm}$ between timers is better than, say, 3 cm or 20 cm. Discuss with your lab partners why this is so. Think about possible uncertainties in the measurement. [Things to consider: Which distance is best for determining the velocity at the bottom of the arc? Which distance minimizes the uncertainty in knowing the distance between the photogate timers? Which are the most important?] Discuss.

1. Make $L$, the length of the pendulum, about 80 cm or so.

   Length of pendulum ___________ cm

2. In order to measure the bob speed, we must know the distance between the photogate light signals. Experience has taught us that it does not work to simply measure the distance between where the light emitting diodes appear to be. [The light beams diverge a bit.]. We must measure the effective width of the photogate. Wrap aluminum foil on one end of the clear plastic ruler and move the ruler along a fixed rod to determine the point at which each photogate is triggered. You can see a reflection of the red LED on the bottom side of the photogates on the table on another piece of aluminum foil that you lay flat on the table. **Your TA may have to show you how to do this.** Do this three times and average your results.

3. If you use a sufficiently large amplitude (large $L$ and $\theta$), you may assume that the average speed between the photogates (which is what you measure) is the same as the maximum speed that you need in Equation (10.9).

4. Set the photogate timer to pulse mode. Look at the diagram in Figure 10.2. Measure the height $h_1$ of the bob at equilibrium with the meter stick and write the value in Table 10.3. Make sure the photogate timers are aligned the same way. **Be very careful to not let the bob hit anything!**
Table 10.2: Determination of Photogate Effective Width

<table>
<thead>
<tr>
<th>Trial</th>
<th>Left Side</th>
<th>Right Side</th>
<th>Effective Width, $\Delta s_{\text{eff}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Average, $\overline{\Delta s_{\text{eff}}}$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 10.3: Measurement of $T$ for Heights and Comparison of Energy

| Height \( h_2 \) | | |
| Height \( h_1 \) | | |
| Height \( h \) | | |
| Trial #1 $\Delta t$ | | |
| Trial #2 $\Delta t$ | | |
| Trial #3 $\Delta t$ | | |
| Average, $\overline{\Delta t}$ | | |
| Speed $v = \frac{\Delta s_{\text{eff}}}{\Delta t}$ | | |
| $|\Delta KE|$ | | |
| $|\Delta PE|$ | | |

**Question 3-1:** How well does the change in kinetic energy, $|\Delta KE|$, agree with the change in potential energy, $|\Delta PE|$? Discuss any differences in terms of your experimental errors.
Please clean up your lab area!
Turn off the photogate timers.