This Technical Appendix describes all technical details associated with the paper. It is organized in six sections:

1. First-order solution conditional on $k(0)$, which is the zero-order portfolio share invested in domestic assets in each country (so that $k^D(0) = 2k(0) - 1$).

2. Second-order solution conditional on $k^D(1)$

3. First and second order components of Bellman equation

4. Second and third order components of optimal portfolio equations

5. Overall solution method

6. Balance of payments accounting

1 First-order solution conditional on $k(0)$

While the paper describes a general numerical solution, the model is simple enough to allow for an analytical solution of the first-order components of control and state variables conditional on $k(0)$. We first describe the analytical solution and then turn to the numerical solution. The latter has the advantage that it is also applicable to more general structures than the specific model of the paper.
1.1 Analytical solution

The first-order component of the model equations can be computed by log-linearizing around the zero-order (steady state) component of model variables. The latter are \( W(0) = 1/\psi, R(0) = (1-\psi\theta)/(1-\psi), Q(0) = (1-\psi)/\psi \) and \( A(0) = P_F(0) = 1 \). The zero-order components of the logs of model variables are simply the logs of these values. Linearizing around these values delivers the following first-order components of model equations (46)-(54) in Appendix A (all equations other than the Bellman equations that will be discussed separately in section 3 below):

\[
\begin{align*}
    a_{H,t+1}(1) &= \rho a_{H,t}(1) + \epsilon_{H,t+1} \\
    a_{F,t+1}(1) &= \rho a_{F,t}(1) + \epsilon_{F,t+1} \\
    w_{t+1}(1) + p_{t+1}(1) &= (1 - \psi\theta) \left[ k(0)r_{H,t+1}(1) + (1 - k(0))r_{F,t+1}(1) \right] + \psi\theta a_{H,t+1}(1) + (1 - \psi\theta) (w_t(1) + p_t(1)) \\
    w^*_t(1) + p^*_t(1) &= (1 - \psi\theta) \left[ (1 - k(0)) r_{H,t+1}(1) + k(0) r_{F,t+1}(1) \right] + \psi\theta (a_{F,t+1}(1) + p_{F,t+1}(1)) + (1 - \psi\theta) (w^*_t(1) + p^*_t(1)) \\
    a_{H,t}(1) &= \alpha w_t(1) + (1 - \alpha) w^*_t(1) + \lambda (a_{H,t}(1) + (1 - \alpha)p^*_t(1)) \\
    q_{H,t}(1) &= k(0) (w_t(1) + p_t(1)) + (1 - k(0)) (w^*_t(1) + p^*_t(1)) \\
        &+ 2k^A_t(1) \\
    q_{F,t}(1) &= (1 - k(0)) (w_t(1) + p_t(1)) + k(0) (w^*_t(1) + p^*_t(1)) \\
        &- 2k^A_t(1) \\
    E_t(r_{H,t+1}(1) - r_{F,t+1}(1)) &= 0
\end{align*}
\]

The last equation follows from the first-order component of both Home and Foreign portfolio Euler equations. For the asset market clearing conditions (6)-(7) we have used that \( k^H_{H,t}(1) = k^A_t(1) + 0.5k^D_t(1), k^H_{F,t}(1) = k^A_t(1) - 0.5k^D_t(1), k^F_{H,t}(1) = 1 - k^H_{H,t}(1) \) and \( k^F_{F,t}(1) = 1 - k^H_{F,t}(1) \).

The first order components of consumer price indices and asset returns
in equations (57)-(60) of Appendix A are:

\[ p_t(1) = (1 - \alpha) p_{F,t}(1) \]  \hspace{1cm} (9)  
\[ p_t^*(1) = \alpha p_{F,t}(1) \]  \hspace{1cm} (10)  
\[ r_{H,t+1}(1) = \frac{1 - \psi}{1 - \psi \theta} q_{H,t+1}(1) + \frac{\psi (1 - \theta)}{1 - \psi \theta} a_{H,t+1}(1) - q_{H,t}(1) \]  \hspace{1cm} (11)  
\[ r_{F,t+1}(1) = \frac{1 - \psi}{1 - \psi \theta} q_{F,t+1}(1) + \frac{\psi (1 - \theta)}{1 - \psi \theta} (a_{F,t+1}(1) + p_{F,t+1}(1)) - q_{F,t}(1) \]  \hspace{1cm} (12)  

Notice that only the first-order component of the average portfolio share \( k^A_t(1) \) enters these equations, not the first-order component of the difference in portfolio shares, \( k^D_t(1) \). In addition, the first-order component of the average portfolio share enters only through the asset market clearing equations.

It is useful to write variables in terms of averages and differences across countries, with the superscript \( A \) standing for average and superscript \( D \) standing for the difference between countries. We take differences and averages of the sets of equations (1)-(2), (3)-(4), (6)-(7). For (5) substitute \( a_{H,t}(1) = a^A_t(1) + 0.5a^D_t(1), \ a_{F,t}(1) = a^A_t(1) - 0.5a^D_t(1), \ w_t(1) = w^A_t(1) + 0.5w^D_t(1) \) and \( w^A_t(1) = w^A_t(1) - 0.5w^D_t(1) \). Also using (9)-(10), (1)-(8) then becomes

\[ a^A_{t+1}(1) = \rho a^A_t(1) + \epsilon^A_{t+1} \]  \hspace{1cm} (13)  
\[ a^D_{t+1}(1) = \rho a^D_t(1) + \epsilon^D_{t+1} \]  \hspace{1cm} (14)  
\[ w^A_{t+1}(1) + 0.5p_{F,t+1}(1) = (1 - \psi \theta) r^A_{t+1}(1) + \psi \theta (a^A_{t+1}(1) + 0.5p_{F,t+1}(1)) + \psi \theta (w^A_{t+1}(1) + 0.5p_{F,t+1}(1)) \]  \hspace{1cm} (15)  
\[ w^D_{t+1}(1) + (1 - 2\alpha)p_{F,t+1}(1) = (1 - \psi \theta) (2k(0) - 1)r^D_{t+1}(1) + \psi \theta (a^D_{t+1}(1) - p_{F,t+1}(1)) + (1 - \psi \theta) (w^D_{t+1}(1) + (1 - 2\alpha)p_{F,t}(1)) \]  \hspace{1cm} (16)  
\[ a^A_t(1) + 0.5a^D_t(1) = w^A_t(1) + 0.5 (2\alpha - 1) w^D_t(1) + \lambda 2\alpha (1 - \alpha) p_{F,t}(1) \]  \hspace{1cm} (17)  
\[ q^A_t(1) = w^A_t(1) + 0.5p_{F,t}(1) \]  \hspace{1cm} (18)  
\[ q^D_t(1) = (2k(0) - 1) w^D_t(1) + (1 - 2\alpha)(2k(0) - 1)p_{F,t}(1) + 4k^A_t(1) \]  \hspace{1cm} (19)  
\[ E_t r^D_{t+1}(1) = 0 \]  \hspace{1cm} (20)
Taking the average and difference of the asset returns (11)-(9), we have

\[ r_{t+1}^A(1) = \frac{1 - \psi}{1 - \psi \theta} q_{t+1}^A(1) + \frac{\psi (1 - \theta)}{1 - \psi \theta} \left[ a_{t+1}^A(1) + 0.5 p_{F,t+1}(1) \right] - q_t^A(1) \tag{21} \]

\[ r_{t+1}^D(1) = \frac{1 - \psi}{1 - \psi \theta} q_{t+1}^D(1) + \frac{\psi (1 - \theta)}{1 - \psi \theta} \left( a_{t+1}^D(1) - p_{F,t+1}(1) \right) - q_t^D(1) \tag{22} \]

Combining (15), (18) and (21) yields

\[ w_t^A(1) = a_t^A(1) \tag{23} \]

\[ q_t^A(1) = a_t^A(1) + \frac{1}{2} p_{F,t}(1) \tag{24} \]

\[ r_{t+1}^A(1) = (a_{t+1}^A(1) - a_t^A(1)) + \frac{1}{2} (p_{F,t+1}(1) - p_{F,t}(1)) \tag{25} \]

Using (23), it is immediate from (17) that

\[ p_{F,t}(1) = \frac{1}{\lambda 4 \alpha (1 - \alpha)} a_t^D(1) - \frac{(2\alpha - 1)}{\lambda 4 \alpha (1 - \alpha)} w_t^D(1) \equiv p_a a_t^D(1) + p_w w_t^D(1) \tag{26} \]

For now we make the conjecture that the equity price differential is given by

\[ q_t^D(1) = q_a a_t^D(1) + q_w w_t^D(1) \tag{27} \]

This will be verified below, with coefficients \( q_a \) and \( q_w \) to be determined. The excess return (22) then implies

\[ r_{t+1}^D(1) = \frac{1 - \psi}{1 - \psi \theta} \left[ q_a a_{t+1}^D(1) + q_w w_{t+1}^D(1) \right] + \frac{\psi (1 - \theta)}{1 - \psi \theta} \left( (1 - p_a) a_{t+1}^D(1) - p_a w_{t+1}^D(1) \right) - q_a a_t^D(1) - q_w w_t^D(1) \]

\[ = m_1 a_{t+1}^D(1) + m_2 a_t^D(1) + m_3 w_{t+1}^D(1) + m_4 w_t^D(1) \tag{27} \]

where:

\[ m_1 = \frac{1 - \psi}{1 - \psi \theta} q_a + \frac{\psi (1 - \theta)}{1 - \psi \theta} (1 - p_a) \]

\[ m_2 = -q_a \]

\[ m_3 = \frac{1 - \psi}{1 - \psi \theta} q_w - \frac{\psi (1 - \theta)}{1 - \psi \theta} p_w \]

\[ m_4 = -q_w \]

Substituting (27) into (16) we have

\[ w_{t+1}^D(1) = \eta_1 a_{t+1}^D(1) + \eta_2 a_t^D(1) + \eta_3 w_{t+1}^D(1) \]

\[ (28) \]
where:

\[
\eta_1 = \frac{(2k - 1) [(1 - \psi) q_a + \psi (1 - \theta) (1 - p_a)] + (2\alpha - 1) p_a + \psi \theta (1 - p_a)}{1 - (2k - 1) [(1 - \psi) q_w - \psi (1 - \theta) p_w] - (2\alpha - 1) p_w + \psi \theta p_w}
\]

\[
\eta_2 = -\frac{(1 - \psi \theta) [(2k - 1) q_a + (2\alpha - 1) p_a]}{1 - (2k - 1) [(1 - \psi) q_w - \psi (1 - \theta) p_w] - (2\alpha - 1) p_w + \psi \theta p_w}
\]

\[
\eta_3 = \frac{(1 - \psi \theta) [1 - (2k - 1) q_w - (2\alpha - 1) p_w]}{1 - (2k - 1) [(1 - \psi) q_w - \psi (1 - \theta) p_w] - (2\alpha - 1) p_w + \psi \theta p_w}
\]

Substituting (28) into (27), the zero expected excess return equation (20) leads to two restrictions on the parameters:

\[
0 = (\rho m_1 + m_2) + m_3 (\rho \eta_1 + \eta_2) \quad (29)
\]

\[
0 = m_3 \eta_3 + m_4 \quad (30)
\]

(30) does not depend on \(q_a\) and can be used to solve for \(q_w\):

\[
q_w = -\frac{1 - (2\alpha - 1) p_w}{1 - (2\alpha - 1) p_w + \theta p_w} (1 - \theta) p_w
\]

Having solved for \(q_w\), (29) is used to solve for \(q_a\):

\[
q_a = \frac{(1 - \theta)}{1 - (2\alpha - 1) p_w + \theta p_w} \frac{\rho \psi}{(1 - p_a) - (2\alpha - 1) p_w} [(1 - p_a) - (2\alpha - 1) p_w]
\]

\[
+ \frac{(1 - \theta)}{1 - (2\alpha - 1) p_w + \theta p_w} p_a (2\alpha - 1) p_w
\]

\(q_a\) and \(q_w\) are then used to solve for \(m_1, m_2, m_3, m_4\) and \(\eta_1, \eta_2\) and \(\eta_3\).

The zero-order component of the portfolio share, \(k(0)\), does not affect the parameters \(p_a, p_w, q_a, q_w\) or \(m_1, m_2, m_3, m_4\). It therefore does not affect the solution of relative prices of goods and assets. \(k(0)\) does impact the first-order solution of the model in two ways though. First, it affects the solution of the average portfolio share \(k^A_t(1)\), which follows from (19):

\[
k^A_t(1) = k_a q^D_t(1) + k_w w^D_t(1) \quad (31)
\]

where:

\[
k_a = \frac{1}{4} [q_a + (2k(0) - 1) (2\alpha - 1) p_a]
\]

\[
k_w = \frac{1}{4} [q_w - (2k(0) - 1) [1 - (2\alpha - 1) p_w]]
\]
Second, it affects the accumulations of wealth. Substituting (13) into (28) we have
\[ w_{t+1}^D(1) = \eta_1 \epsilon_{t+1}^D + (\rho \eta_1 + \eta_2) a_t^D(1) + \eta_3 w_t^D(1) \] (32)
where
\[ \rho \eta_1 + \eta_2 = \frac{\rho [(2\alpha - 1) + \theta \psi [\lambda 4\alpha (1 - \alpha) - 1] - (1 - \psi \theta) (2\alpha - 1)]}{1 + (\lambda - 1) 4\alpha (1 - \alpha) - \psi \theta (2\alpha - 1)} \]
\[ \eta_3 = \frac{(1 - \psi \theta)}{1 + (\lambda - 1) 4\alpha (1 - \alpha) - \psi \theta (2\alpha - 1)} \]

While \( \rho \eta_1 + \eta_2 \) and \( \eta_3 \) do not depend on \( k(0) \), \( \eta_1 \) does depend on \( k(0) \). The impact of the innovation \( \epsilon_{t+1}^D \) on \( w_{t+1}^D(1) \) therefore depends on \( k(0) \).

Overall we can summarize the first-order solution of all variables other than \( k^D(1) \) as follows. The solution for the control variables is
\[ p_{F,t}(1) = p_a a_t^D(1) + p_w w_t^D(1) \] (33)
\[ q_t^A(1) = q_a a_t^D(1) + q_w w_t^D(1) \] (34)
\[ q_t^A(1) = a_t^A(1) + 0.5 p_a a_t^D(1) + 0.5 p_w w_t^D(1) \] (35)
\[ w_t^A(1) = a_t^A(1) \] (36)
\[ k_t^A = k_a a_t^D(1) + k_w w_t^D(1) \] (37)

The accumulation of the state variables is described by
\[ a_{t+1}^A(1) = \rho a_t^A(1) + \epsilon_{t+1}^A \] (38)
\[ a_{t+1}^D(1) = \rho a_t^D(1) + \epsilon_{t+1}^D \] (39)
\[ w_{t+1}^D(1) = \eta_1 \epsilon_{t+1}^D + (\rho \eta_1 + \eta_2) a_t^D(1) + \eta_3 w_t^D(1) \] (40)

### 1.2 Numerical solution

The solution method described here is the standard first-order solution method that applies more broadly than to the particulars of the model in the paper. The system (1)-(8) consists of 3 state variables and 5 control variables. The vectors of state and control variables are
\[ S_t = \begin{bmatrix} a_t^D & w_t^D & a_t^A \end{bmatrix}' \] (41)
\[ CV_t = \begin{bmatrix} w_t^A & p_{F,t} & k_t^A & q_{H,t} & q_{F,t} \end{bmatrix}' \] (42)
We write the entire vector of model variables as

\[ X_t = \begin{bmatrix} (S_t)' \\ (CV_t)' \end{bmatrix} \]  

(43)

After substituting the expressions for consumer price indices and asset returns, and applying the expectations operator, equations (57)-(60) of Appendix A can be written compactly as

\[ E_t g(X_t, X_{t+1}) = 0 \]  

(44)

The first-order component of model equations follows from a linear expansion around the steady state, which delivers

\[ M_1X_t(1) + M_2E_tX_{t+1}(1) = 0 \Rightarrow E_tX_{t+1}(1) = MX_t(1) \]

where \( M = -(M_2)^{-1}M_1 \).

We diagonalize the matrix \( M' \):

\[ M' = EV\Omega EV^{-1} \]

where \( EV \) contains the eigenvectors of \( M' \) and \( \Omega \) is a diagonal matrix with the corresponding eigenvalues. Using the property that \( (EV^{-1})' = (EV')^{-1} \) it follows that

\[ M = (EV')^{-1}\Omega EV' \]

We define

\[ \tilde{X}_t(1) = EV'X_t(1) \]

so that the first-order component of the model becomes

\[ E_t\tilde{X}_{t+1}(1) = \Omega \tilde{X}_t(1) \]

The system is well defined when there are as many zero and explosive eigenvalues as there are control variables (that is 5). We set the corresponding elements of \( \tilde{X}_t(1) \) to zero. Let \( EV'(subs) \) denote the rows of \( EV' \) corresponding to the zero or explosive eigenvalues. The first-order component of control variables as a function of state variables is then solved from \( EV'(subs)X_t(1) = 0 \), which gives

\[ CV_t(1) = -(EV'(sub, 4 : 8))^{-1}EV'(sub, 1 : 3)S_t(1) \equiv \tilde{E}V \cdot S_t(1) \]  

(45)
In particular, we will use the following notation for the solution of goods and asset prices

\[
p_{F,t}(1) = p_{s}S_{t}(1) \quad (46)
\]
\[
q_{H,t}(1) = q_{s}^{H}S_{t}(1) \quad (47)
\]
\[
q_{F,t}(1) = q_{s}^{F}S_{t}(1) \quad (48)
\]

The accumulation of the first-order component of the state variables can be described as

\[
S_{t+1}(1) = N_{1}S_{t}(1) + N_{2}\epsilon_{t+1} \quad (49)
\]
\[
\epsilon_{t+1} = \begin{bmatrix} \epsilon_{t+1}^{H} & \epsilon_{t+1}^{F} \end{bmatrix}
\]

This can be derived as follows. Let $B_{1}$ be a 3x8 matrix that extracts the rows of the model equations corresponding to the accumulation of the state variables: the dynamics of the home wealth (3) and the dynamics of both productivity levels, (1)-(2). Without the expectation operator applied to those equations we have

\[
B_{1}M_{1}X_{t}(1) + B_{1}M_{2}X_{t+1}(1) = B_{3}\epsilon_{t+1} \quad (50)
\]

where:

\[
B_{3} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}
\]

Using (45) we write:

\[
X_{t}(1) = \begin{bmatrix} S_{t}(1) \\ CV_{t}(1) \end{bmatrix} = \begin{bmatrix} I \\ EV \end{bmatrix} S_{t}(1) \equiv B_{2}S_{t}(1) \quad (51)
\]

where $I$ is a 3x3 matrix. Proceeding similarly for $X_{t+1}(1)$, we rewrite (50) as:

\[
B_{1}M_{1}B_{2}S_{t}(1) + B_{1}M_{2}B_{2}S_{t+1}(1) = B_{3}\epsilon_{t+1}
\]

which leads to (49) with

\[
N_{1} = -[B_{1}M_{2}B_{2}]^{-1}B_{1}M_{1}B_{2} \quad \quad N_{2} = [B_{1}M_{2}B_{2}]^{-1}B_{3}
\]
2 Second order solution conditional on $k^D(1)$

2.1 Second-order component of model equations

We now describe the numerical solution of the second-order component of model variables conditional on the first-order component of the portfolio difference, $k^D(1)$. Applying equation (6) of the paper, the second order component of the model $Etg(X_t, X_{t+1}) = 0$ is equal to

$$M_1X_t(2) + M_2E_tX_{t+1}(2) + E_tO_2 = 0$$

(52)

where $E_tO_2$ contains the product of first-order components of model variables. These multiply second-order derivatives of the model equations at the steady state. Let $E_tO_2(i)$ be the $i$th element of $E_tO_2$, corresponding to equation $i$ of the model. We have

$$E_tO_2(i) = \frac{1}{2}X_t(1)'M_{3,i}X_t(1) + \frac{1}{2}E_tX_{t+1}(1)'M_{4,i}X_{t+1}(1) + X_t(1)'M_{5,i}E_tX_{t+1}(1)$$

(53)

where $M_{3,i}$ is the second-order derivative of equation $i$ with respect to $X_t$ and $M_{4,i}, M_{5,i}$ are similarly defined.

While $k^D(1)$ does not enter the first-order components of model equations, it does enter the second-order component through $E_tO_2$. We solve the second-order component of model variables conditional on a conjectured solution for $k^D(1)$, which is $k^D(1) = k_sS_t(1)$. The $M_{3,i}, M_{4,i}$ and $M_{5,i}$ matrices then depend on $k_s$. Rather than numerically recomputing these second-order derivatives for each value of $k_s$ we proceed as follows. Portfolio shares enter through wealth accumulation and asset-market clearing conditions. Using these equations ((48)-(49) and (51)-(52))
in Appendix A of the paper), and focusing on the second-order components that depend on \(k_s\), we have

\[
w_{t+1}(2) = \frac{1 - \psi \theta}{2} k_s S_t(1) \left( r_q (q_{H,t+1}(1) - q_{F,t+1}(1)) - (q_{H,t}(1) - q_{F,t}(1)) \right) + \text{other}
\]

\[
w_{t+1}(2)^* = -\frac{1 - \psi \theta}{2} k_s S_t(1) \left( r_q (q_{H,t+1}(1) - q_{F,t+1}(1)) - (q_{H,t}(1) - q_{F,t}(1)) \right) + \text{other}
\]

\[
q_{H,t}(2) = -\frac{1}{2} k_s S_t(1) \left[ (2\alpha - 1) p_{F,t}(1) - w^D_t(1) \right] + \text{other}
\]

\[
q_{F,t}(2) = \frac{1}{2} k_s S_t(1) \left[ (2\alpha - 1) p_{F,t}(1) - w^D_t(1) \right] + \text{other}
\]

where “other” stands for all the other second-order terms that do not depend on \(k_s\) and \(r_q = (1 - \psi)/(1 - \psi \theta)\). Starting from the second-order derivatives of the model equations at \(k_s = 0\), these equations allow us to analytically adjust the second-order derivatives \(M_{3,i}, M_{4,i}\) and \(M_{5,i}\) as a function of \(k_s\).

Substituting the solution of the first-order components of model variables, described by (51) and (49), into the expression for \(E_t O_2(i)\), we have

\[
E_t O_2(i) = \frac{1}{2} S_t(1)' B_2' M_{3,i} B_2 S_t(1) + \frac{1}{2} S_t(1)' N_1' B_2' M_{4,i} B_2 N_1 S_t(1)
\]

\[
+ \frac{1}{2} E_t \epsilon_{t+1}' N_2' B_2' M_{5,i} B_2 N_2 \epsilon_{t+1} + S_t(1)' B_2' M_{5,i} B_2 N_1 S_t(1)
\]

This is written in a more compact way as:

\[
E_t O_2(i) = S_t(1)' K_i S_t(1) + \sigma^2 k_i
\]

where:

\[
K_i = \frac{1}{2} B_2' M_{3,i} B_2 + \frac{1}{2} N_1' B_2' M_{4,i} B_2 N_1 + B_2' M_{5,i} B_2 N_1
\]

\[
k_i = \text{trace} \left( \frac{1}{2} N_2' B_2' M_{4,i} B_2 N_2 \right)
\]

This uses that \(\text{var}(\epsilon_{t+1}) = \sigma^2 I\), where \(I\) is a 2 by 2 matrix.

It will be useful to write the quadratic terms in \(S_t(1)\) as a vector. Writing
element $i$ of $S_t(1)$ as $S_{t,i}(1)$, define

$$Y_t(2) = \begin{bmatrix}
S_{t,1}(1)^2 \\
S_{t,1}(1)S_{t,2}(1) \\
S_{t,1}(1)S_{t,3}(1) \\
S_{t,2}(1)S_{t,1}(1) \\
S_{t,2}(1)^2 \\
S_{t,2}(1)S_{t,3}(1) \\
S_{t,3}(1)S_{t,1}(1) \\
S_{t,3}(1)S_{t,2}(1) \\
S_{t,3}(1)^2
\end{bmatrix}$$

$E_tO_2(i)$ can then be written as a linear function of $Y_t$:

$$E_tO_2(i) = (K_i^{vec})'Y_t(2) + \sigma^2k_i$$  \hspace{1cm} (56)

where:

$$K_i^{vec} = \begin{bmatrix}
K_{i,1,1} \\
K_{i,1,2} \\
K_{i,1,3} \\
K_{i,2,1} \\
K_{i,2,2} \\
K_{i,2,3} \\
K_{i,3,1} \\
K_{i,3,2} \\
K_{i,3,3}
\end{bmatrix}$$

where $K_{i,x,y}$ is element $(x,y)$ of matrix $K_i$, and $vec$ denotes the vectorization of a matrix. (52) can then be written in a matrix form as:

$$M_1X_t(2) + M_2E_tX_{t+1}(2) + KY_t(2) + k\sigma^2 = 0$$  \hspace{1cm} (57)

where:

$$K = \begin{bmatrix}
(K_1^{vec})' \\
\vdots \\
(K_8^{vec})'
\end{bmatrix} \ \ \ \ \ \ k = \begin{bmatrix}
k_1 \\
\vdots \\
k_8
\end{bmatrix}$$

To compute the dynamics of $Y_t(2)$, start by writing

$$E_tY_{t+1}(2) = (E_tS_{t+1}(1)S_{t+1}(1))^{vec}$$

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From (49) we write:

\[ E_tS_{t+1}(1)S_{t+1}(1)' = N_1S_t(1)S_t(1)'N_1' + \sigma^2N_2N_2' \]

Write \( N_1 \) as:

\[ N_1 = \begin{bmatrix} n_1' \\ \vdots \\ n_3' \end{bmatrix} \]

where \( n_i' \) is row \( i \) of the matrix \( N_1 \). Then element \((i, j)\) of \( N_1S_t(1)S_t(1)'N_1' \) is equal to

\[ n_i'S_t(1)S_t(1)'n_j = \left[ (n_i' n_j')^{vec} \right]'Y_t(2) \equiv z_{i,j}Y_t(2) \]

We then have

\[ E_tS_{t+1}(1)S_{t+1}(1)' = \begin{bmatrix} z_{1,1}Y_t(2) & \ldots & z_{1,3}Y_t(2) \\ \vdots & \ddots & \vdots \\ z_{3,1}Y_t(2) & \ldots & z_{3,3}Y_t(2) \end{bmatrix} + \sigma^2N_2N_2' \]

Also define

\[ \tilde{n} = (N_2N_2')^{vec} \]

which implies:

\[ (E_tS_{t+1}(1)S_{t+1}(1)')^{vec} = ZY_t(2) + \sigma^2\tilde{n} \quad (58) \]

where

\[ Z = \begin{bmatrix} z_{1,1} \\ \vdots \\ z_{1,3} \\ z_{2,1} \\ \vdots \\ z_{2,3} \\ z_{3,1} \\ \vdots \\ z_{3,3} \end{bmatrix} \]

### 2.2 Second-order solution for the control variables

The preceding analysis allows us to write the second-order component of model equations as

\[ \begin{align*}
0 &= M_1X_t(2) + M_2E_tX_{t+1}(2) + KY_t(2) + k\sigma^2 \\
E_tY_{t+1}(2) &= ZY_t(2) + \sigma^2\tilde{n}
\end{align*} \]
In order to compute the second-order component of control variables we proceed as we did with the first-order solution. Define $M = -M_2^{-1}M_1$ and diagonalize $M'$: $M' = EV\Omega EV^{-1}$. This implies: $M = (EV')^{-1}\Omega EV'$. Define $\tilde{X}_t(2) = EV'X_t(2)$. Then the system becomes

$$
E_t\tilde{X}_{t+1}(2) = \Omega\tilde{X}_t(2) + QY_t + \tilde{k}\sigma^2 \\
E_tY_{t+1}(2) = ZY_t(2) + \sigma^2\bar{n}
$$

where $\tilde{k} = -EV'M_2^{-1}k$ and $Q = -EV'M_2^{-1}K$.

Define the matrix $G$ such that: $GZ = \Omega G = Q$. Specifically, we write

$$
G = \begin{bmatrix}
g_1' \\
\vdots \\
g_8'
\end{bmatrix}
$$

where $g_i'$ is row $i$ of the matrix $G$. Let $q_i'$ be row $i$ of matrix $Q$. Then the row $i$ of $GZ = \Omega G = Q$ becomes

$$
g_i'Z - \lambda_i g_i = q_i'$$

where $\lambda_i$ is the $i$’th eigenvalue on the diagonal of the matrix $\Omega$. It follows that

$$Z'g_i - \lambda_i g_i = q_i \Rightarrow g_i = (Z' - \lambda_i I)^{-1} q_i$$

where $I$ is a 9x9 identity matrix.

The two equations of the system are then combined as:

$$E_t(\tilde{X}_{t+1}(2) - GY_{t+1}(2)) = \Omega(\tilde{X}_t(2) - GY_t(2)) + \tilde{k}\sigma^2$$

where $\tilde{k} = \tilde{k} - G\bar{n}$. We again identify the eigenvalues in $\Omega$ that are zero or explosive (as in the first order solution), and set the corresponding rows of $\tilde{X}_t(2) - GY_t - \tilde{k}\sigma^2$ to zero, where $\tilde{k} = (I - \Omega)^{-1}\tilde{k}$. This gives $EV'(sub)X_t(2) - G(sub,)Y_t(2) - \tilde{k}(sub)\sigma^2 = 0$, so that the second order solution of the control variables is

$$CV_t(2) = -(EV'(sub, 4 : 8))^{-1}EV'(sub, 1 : 3)S_t(2) + (EV'(sub, 4 : 8))^{-1}G(sub,)Y_t(2) + (EV'(sub, 4 : 8))^{-1}\tilde{k}(sub)\sigma^2$$

$$\equiv EVS_t(2) + \tilde{G}Y_t(2) + k_c\sigma^2$$

We write

$$\tilde{G} = \begin{bmatrix}
\hat{g}_1 \\
\vdots \\
\hat{g}_8
\end{bmatrix}$$

where $\hat{g}_i$ is row $i$ of matrix $\tilde{G}$.
where $\hat{g}_i$ is row $i$ of $\hat{G}$. Therefore for control variable $i$ the part of the second-order solution that depends on the product of first-order component of state variables is $\hat{g}_i Y_t(2)$. We can convert this back to matrix form: $\hat{g}_i Y_t(2) = S_t(1)^T \hat{g}_i^m S_t(1)$, where the first three elements of $\hat{g}_i$ make up the first row of $\hat{g}_i^m$, the second three elements make up the second row and the last three elements make up the last row. We continue to use the superscript $m$ below to convert vectors to matrices in this way.

For goods and equity prices we will write this second-order solution as

$$
\begin{align*}
  p_{F,t}(2) &= p_s S_t(2) + S_t(1)^T p_{ss} S_t(1) + k_{p} \sigma^2 \\
  q_{H,t}(2) &= q_{ss}^H S_t(2) + S_t(1)^T q_{ss}^H S_t(1) + k_{q}^H \sigma^2 \\
  q_{F,t}(2) &= q_{ss}^F S_t(2) + S_t(1)^T q_{ss}^F S_t(1) + k_{q}^F \sigma^2
\end{align*}
$$

(60) (61) (62)

Note that $p_{ss}$, $q_{ss}^H$ and $q_{ss}^F$ need not be symmetric. It is ok if the $i,j$ and $j,i$ elements differ, as all that matters is their sum.

### 2.3 Second-order dynamics of the state variables

We now turn to the dynamic process of the second-order components of the state variables. Let again $B_1$ be a matrix that extracts the rows corresponding to the state variable accumulation equations. Without the expectation operator applied to accumulation equations for the state variables, we have

$$
B_1 M_1 X_t(2) + B_1 M_2 X_{t+1}(2) + B_1 O_2 = 0
$$

(63)

Using (59) we write:

$$
X_t(2) = \begin{bmatrix} S_t(2) \\ CV_t(2) \end{bmatrix} = \begin{bmatrix} S_t(2) \\ \hat{E}VS_t(2) + \hat{G}Y_t(2) + k_c \sigma^2 \end{bmatrix}
$$

= $B_2 S_t(2) + \hat{G}Y_t(2) + k_x \sigma^2

(64)

where:

$$
B_2 = \begin{bmatrix} I_{3x3} \\ EV \end{bmatrix} \quad \hat{G} = \begin{bmatrix} 0_{3x9} \\ G \end{bmatrix} \quad k_x = \begin{bmatrix} 0_{3x1} \\ k_c \end{bmatrix}
$$

Substituting (64) into (63) we have

$$
S_{t+1}(2) = N_1 S_t(2) + B_4 Y_t(2) + B_5 Y_{t+1}(2) + B_6 B_1 O_2 + N_6 \sigma^2
$$

(65)
where
\[
B_4 = - (B_1 M_2 B_2)^{-1} B_1 M_1 G
\]
\[
B_5 = - (B_1 M_2 B_2)^{-1} B_1 M_2 G
\]
\[
B_6 = - (B_1 M_2 B_2)^{-1}
\]
\[
N_6 = - (B_1 M_2 B_2)^{-1} B_1 (M_1 + M_2) k_x
\]

First consider the term \(B_6 B_1 O_2\) in (65). Element \(i\) of \(O_2\)
\[
O_2(i) = \frac{1}{2} X_t(1)' M_{3,i} X_t(1) + \frac{1}{2} X_{t+1}(1)' M_{4,i} X_{t+1}(1) + X_t(1)' M_{5,i} X_{t+1}(1) =
\]
\[
= S_t(1)' V_{1,i} S_t(1) + \epsilon'_{t+1} V_{2,i} \epsilon_{t+1} + S_t(1)' V_{3,i} \epsilon_{t+1}
\]
where
\[
V_{1,i} = \frac{1}{2} (B_2' M_{3,i} B_2 + N_1' B_2' M_{4,i} B_2 N_1) + B_2' M_{5,i} B_2 N_1
\]
\[
V_{2,i} = \frac{1}{2} N_2' B_2' M_{4,i} B_2 N_2
\]
\[
V_{3,i} = B_2' M_{5,i} B_2 N_2 + N_1' B_2' M_{4,i} B_2 N_2
\]

Here we used \(X_t(1) = B_2 S_t(1)\) and \(S_{t+1}(1) = N_1 S_t(1) + N_2 \epsilon_{t+1}\).

Let the three rows of the model corresponding to the state accumulation equations (the rows extracted by \(B_1\)) be rows \(a, b\) and \(c\):
\[
B_1 O_2 = \begin{bmatrix}
S_t(1)' V_{1,a} S_t(1) + \epsilon'_{t+1} V_{2,a} \epsilon_{t+1} + S_t(1)' V_{3,a} \epsilon_{t+1} \\
S_t(1)' V_{1,b} S_t(1) + \epsilon'_{t+1} V_{2,b} \epsilon_{t+1} + S_t(1)' V_{3,b} \epsilon_{t+1} \\
S_t(1)' V_{1,c} S_t(1) + \epsilon'_{t+1} V_{2,c} \epsilon_{t+1} + S_t(1)' V_{3,c} \epsilon_{t+1}
\end{bmatrix}
\]

We can then write
\[
B_6 B_1 O_2 = \begin{bmatrix}
S_t(1)' \tilde{V}_{1,1} S_t(1) + \epsilon'_{t+1} \tilde{V}_{1,2} \epsilon_{t+1} + S_t(1)' \tilde{V}_{1,3} \epsilon_{t+1} \\
S_t(1)' \tilde{V}_{2,1} S_t(1) + \epsilon'_{t+1} \tilde{V}_{2,2} \epsilon_{t+1} + S_t(1)' \tilde{V}_{2,3} \epsilon_{t+1} \\
S_t(1)' \tilde{V}_{3,1} S_t(1) + \epsilon'_{t+1} \tilde{V}_{3,2} \epsilon_{t+1} + S_t(1)' \tilde{V}_{3,3} \epsilon_{t+1}
\end{bmatrix}
\]

where
\[
\tilde{V}_{1,1} = B_{6,i,1} V_{1,a} + B_{6,i,2} V_{1,b} + B_{6,i,3} V_{1,c}
\]
\[
\tilde{V}_{1,2} = B_{6,i,1} V_{2,a} + B_{6,i,2} V_{2,b} + B_{6,i,3} V_{2,c}
\]
\[
\tilde{V}_{1,3} = B_{6,i,1} V_{3,a} + B_{6,i,2} V_{3,b} + B_{6,i,3} V_{3,c}
\]
where \( B_{6,x,y} \) is the element \((x, y)\) of matrix \( B_6 \).

We now turn to the term \( B_5 Y_{t+1} \) in (65):

\[
B_5 Y_{t+1} (2) = B_5 \left( S_{t+1}(1) S_{t+1} (1) \right) \text{vec}
\]

Using (49) we have:

\[
S_{t+1}(1) S_{t+1} (1)' = N_1 S_t (1) S_t (1)' N_1' + N_2 \epsilon_{t+1} \epsilon_{t+1}' N_2' + N_1 S_t (1) \epsilon_{t+1} N_2 + N_2 \epsilon_{t+1} S_t (1)' N_1'
\]

We have already derived that

\[
(N_1 S_t (1) S_t (1)' N_1') \text{vec} = ZY_t (2)
\]

We can similarly derive that

\[
(N_2 \epsilon_{t+1} \epsilon_{t+1}' N_2') \text{vec} = \tilde{Z} Y_t^{eps}
\]

where

\[
Y_t^{eps} = \begin{bmatrix}
\epsilon_H^2 \\
\epsilon_H \epsilon_F \\
\epsilon_F \epsilon_H \\
\epsilon_F^2
\end{bmatrix}
\]

and

\[
\tilde{Z} = \begin{bmatrix}
\tilde{z}_{1,1} \\
.. \\
\tilde{z}_{1,3} \\
\tilde{z}_{2,1} \\
.. \\
\tilde{z}_{2,3} \\
\tilde{z}_{3,1} \\
.. \\
\tilde{z}_{3,3}
\end{bmatrix}
\]

with

\[
\tilde{z}_{i,j} = \left( (\bar{n}_i \bar{n}_j') \text{vec} \right)'
\]

where \( \bar{n}_i' \) is row \( i \) of \( N_2 \).

Next turn to \( N_1 S_t (1) \epsilon_{t+1} N_2' \). Element \((i, j)\) of \( N_1 S_t (1) \epsilon_{t+1} N_2' \) is equal to \((n_i' \text{ row } i \text{ of } N_1 \text{ and } \bar{n}_j' \text{ row } j \text{ of } N_2)\):

\[
n_i' S_t (1) \epsilon_{t+1} \bar{n}_j = S_t (1)' n_i \bar{n}_j' \epsilon_{t+1}
\]
Similarly, element \((i, j)\) of \(N_2\epsilon_{t+1}S'_t(1)N'_1\) is equal to:

\[
\bar{n}'_i\epsilon_{t+1}S'(1)'n_j = \epsilon'_{t+1}\bar{n}_i\bar{n}'_jS'(1) = S'(1)'n_j\bar{n}'_i\epsilon_{t+1}
\]

So the element \((i, j)\) of \(N_1S_t(1)\epsilon'_{t+1}N'_2 + N_2\epsilon_{t+1}S'(1)'N'_1\) is:

\[
S_t(1)'[n_i\bar{n}'_j + n_j\bar{n}'_i] \epsilon_{t+1}
\]

Therefore row \(i\) (out of the three rows) of

\[
B_5 (N_1S_t(1)\epsilon'_{t+1}N'_2 + N_2\epsilon_{t+1}S'(1)'N'_1)^{vec}
\]

is written as:

\[
B_{5,i,1} S_t(1)'[n_1\bar{n}'_i + n_1\bar{n}'_i] \epsilon_{t+1} + B_{5,i,2} S_t(1)'[n_1\bar{n}'_2 + n_2\bar{n}'_1] \epsilon_{t+1} + B_{5,i,3} S_t(1)'[n_1\bar{n}'_3 + n_3\bar{n}'_1] \epsilon_{t+1} + B_{5,i,4} S_t(1)'[n_2\bar{n}'_1 + n_1\bar{n}'_2] \epsilon_{t+1} + B_{5,i,5} S_t(1)'[n_2\bar{n}'_2 + n_2\bar{n}'_2] \epsilon_{t+1} + B_{5,i,6} S_t(1)'[n_2\bar{n}'_3 + n_3\bar{n}'_2] \epsilon_{t+1} + B_{5,i,7} S_t(1)'[n_3\bar{n}'_1 + n_1\bar{n}'_3] \epsilon_{t+1} + B_{5,i,8} S_t(1)'[n_3\bar{n}'_2 + n_2\bar{n}'_3] \epsilon_{t+1} + B_{5,i,9} S_t(1)'[n_3\bar{n}'_3 + n_3\bar{n}'_3] \epsilon_{t+1}
\]

where \(B_{5,x,y}\) is the element of \(B_5\) on the \(x\)th row and the \(y\)th column. This is written in a more compact way as:

\[
S_t(1)'\bar{N}_{5,i}\epsilon_{t+1}
\]

where

\[
\bar{N}_{5,i} = \sum_{v=1}^{3} \sum_{w=1}^{3} B_{5,i,v,w}^m (n_v\bar{n}'_w + n_w\bar{n}'_v)
\]

where \(B_{5,i,v,w}^m\) is element \((v, w)\) of matrix \(B_{5,i}^m\), where \(B_{5,i}^m\) is:

\[
B_{5,i}^m = \begin{pmatrix}
B_{5,i,1} & B_{5,i,2} & B_{5,i,3} \\
B_{5,i,4} & B_{5,i,5} & B_{5,i,6} \\
B_{5,i,7} & B_{5,i,8} & B_{5,i,9}
\end{pmatrix}
\]

Here \(B_{5,i}^m\) is the matrix form associated with row \(i\) of \(B_5\). Similarly write

\[
\bar{N}_{3,i} = (B_4 + B_5\bar{Z})_i^m
\]

\[
\bar{N}_{4,i} = (B_5\bar{Z})_i^m
\]

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These are the matrix form of row $i$ of the respective matrices.

Putting all these steps together, (65) becomes

$$
S_{t+1}(2) = N_1 S_t(2) + 
\left( \begin{array}{c}
S_t(1)' N_{3,1} S_t(1) + e_{t+1}' N_{4,1} e_{t+1} + S_t(1)' N_{5,1} e_{t+1} \\
S_t(1)' N_{3,2} S_t(1) + e_{t+1}' N_{4,2} e_{t+1} + S_t(1)' N_{5,2} e_{t+1} \\
S_t(1)' N_{3,3} S_t(1) + e_{t+1}' N_{4,3} e_{t+1} + S_t(1)' N_{5,3} e_{t+1}
\end{array} \right) + N_6 \sigma^2 (66)
$$

where

$$
\begin{align*}
N_{3,i} &= \bar{N}_{3,i} + \bar{V}_{i,1} \\
N_{4,i} &= \bar{N}_{4,i} + \bar{V}_{i,2} \\
N_{5,i} &= \bar{N}_{5,i} + \bar{V}_{i,3}
\end{align*}
$$

This describes the dynamics of the second-order components of the state variables.

### 2.4 Expected portfolio return

We finally derive the second-order component of the expected portfolio return, $E_t r_{H,t+1}^p (2)$, which is needed when computing the second-order component of the Bellman equation in section 3. We obtain the second-order component of $r_{H,t+1}^p$ from the second-order component of equation (61) in Appendix A of the paper. This gives

$$
r_{H,t+1}^p (2) = \frac{1}{2} k(0) (r_{H,t+1}(1) - r_{F,t+1}(1))^2 (67)
$$

Start with the expectation of the last two terms in (67). As the expected excess return is zero to a first order and the portfolio shares are known at time $t$, we have

$$
E_t(r_{H,t+1}(1) - r_{F,t+1}(1)) k_{H,t}^H(1) = k_{H,t}^H E_t(r_{H,t+1}(1) - r_{F,t+1}(1)) = 0
$$

From the first-order solution the first-order component of the excess return is proportional to the innovation $\epsilon_{t+1}^D$ since the expected excess return is zero.
to a first order. We write the first-order component of the excess return as
\[ r_{H,t+1}(1) - r_{F,t+1}(1) = r_{DE}e_{t+1}^D. \]
Therefore
\[ E_t(r_{H,t+1}(1) - r_{F,t+1}(1))^2 = 2r_{DE}^2\sigma^2 \]

As shown later (in section 4), the expected excess return is also zero to a
second order, so the expectation of the second term in (67) becomes:
\[ E_t(r_{F,t+1}(2) - r_{H,t+1}(2) - \tau) = -\tau \]
and (67) is written in expected terms as:
\[ E_t r_{t+1}^{p,H}(2) = E_t r_{H,t+1}(2) + p_t(2) - E_t p_{t+1}(2) - (1-k(0))\tau + k(0)(1-k(0))r_{DE}^2\sigma^2 \]

(68)

Turning to the consumer prices, the second-order component of the home
CPI (equation 57 in Appendix A of the paper) is
\[ p_t(2) = (1 - \alpha)p_{F,t}(2) - \frac{1}{2}(\lambda - 1)\alpha (1 - \alpha) p_{F,t}(1)^2 \]
\[ \equiv \tilde{p}sS_t(2) + S_t(1)'\tilde{p}ssS_t(1) + \tilde{k}_p\sigma^2 \]
where we used (60) and:
\[ \tilde{p}s = (1 - \alpha)p_s \]
\[ \tilde{p}ss = (1 - \alpha)p_{ss} - \frac{1}{2}(\lambda - 1)\alpha (1 - \alpha) p'_sp_s \]
\[ \tilde{k}_p = (1 - \alpha)k_p \]
Using (66) this implies:
\[ E_t p_{t+1}(2) = \tilde{p}sN_1S_t(2) + S_t(1)'\tilde{p}ssS_t(1) + \tilde{p}\sigma^2 \]
where:
\[ \tilde{p}ss = N'_1\tilde{p}ssN_1 + \sum_{v=1}^3 \tilde{p}s(v)N_{3,v} \]
\[ \tilde{p} = \tilde{p}sN_6 + \tilde{k}_p + \tilde{p} \]
\[ \tilde{p} = \text{trace} \left[ \sum_{v=1}^3 \tilde{p}s(v)N_{4,v} + N'_2\tilde{p}ssN_2 \right] \]
We next turn to $E_t r_{H,t+1}(2)$. Using equation (59) of Appendix A of the paper, the expected second order component of the Home return is

$$E_t r_{H,t+1}(2) = -q_{H,t}(2) + r_q E_t q_{H,t+1}(2) + r_a E_t a_{H,t+1}(2) + \frac{1}{2} r_{qq} E_t q_{H,t+1}(1)^2 + \frac{1}{2} r_{aa} E_t a_{H,t+1}(1)^2 + r_{qa} E_t q_{H,t+1}(1) a_{H,t+1}(1)$$

(69)

where $r_q = (1 - \psi)(1 - \psi \theta)^{-1}$, $r_a = 1 - r_q$ and $r_{qq} = r_{aa} = -r_{qa} = r_q(1 - r_q)$. Consider the last three terms of (69) first. We can simply substitute the first-order results. Using $q_{H,t+1}(1) = q_s^H S_t(1)$, we have

$$E_t q_{H,t+1}(1)^2 = E_t S_{t+1}(1)' (q_s^H(1))' q_s^H S_{t+1}(1) = S_t(1)' N_1'(q_s^H)' q_s^H N_1 S_t(1) + e_1 \sigma^2$$

where $e_1 = \text{trace} \left[ N_2'(q_s^H)' q_s^H N_2 \right]$. Similarly, writing $a_{H,t+1}(1) = a_s^H S_t(1) + a_E^H \epsilon_{t+1}$, where $a_s^H = (0.5 \rho, 0, \rho)$ and $a_E^H = (1, 0)$, we have

$$E_t a_{H,t+1}(1)^2 = S_t(1)' (a_s^H)' a_s^H S_t(1) + e_3 \sigma^2$$

where $e_3 = \text{trace} \left[ (a_E^H)' a_E^H \right]$. Finally:

$$E_t q_{H,t+1}(1) a_{H,t+1}(1) = E_t (q_s^H N_1 S_t(1) + q_s^H N_2 \epsilon_{t+1}) (a_s^H S_t(1) + a_E^H \epsilon_{t+1})$$

$$= S_t(1)' N_1'(q_s^H)' a_s^H S_t(1) + e_2 \sigma^2$$

where $e_2 = \text{trace} \left[ N_2'(q_s^H)' a_s^H \right]$.

Next consider the first three terms of (69), using (61) and (66):

$$q_{H,t}(2) = q_s^H S_t(2) + S_t(1)' q_{ss}^H S_t(1) + k^H \sigma^2$$

$$E_t q_{H,t+1}(2) = q_s^H N_1 S_t(2) + S_t(1)' \tilde{q}_{ss} S_t(1) + \tilde{q} \sigma^2$$

where:

$$\tilde{q}_{ss} = N_1' q_{ss}^H N_1 + \sum_{v=1}^{3} q_s^H(v) N_3(v)$$

$$\tilde{q} = q_s^H N_6 + k^H + \tilde{q}$$

$$\tilde{q} = \text{trace} \left[ \sum_{v=1}^{3} q_s^H(v) N_4(v) + N_2' q_{ss}^H N_2 \right]$$
The last elements is: $E_t a_{H,t+1}(2) = a_s^H S_t(2)$.

Putting all our results together, (68) becomes:

$$E_t r_{t+1}^{p,H} = r_s S_t(2) + r_s(1)^t S_t(1) + \hat{\sigma}^2$$  \hspace{1cm} (70)

where

$$r_s = -q_s^H + r_q q_s N_1 + r_a a_s^H + \bar{p}_s - \bar{p}_s N_1$$

$$\hat{r} = -k_q + r_q q + \frac{1}{2} r_{qq} e_1 + \frac{1}{2} r_{aa} e_3 + r_{qa} e_2$$

$$+ \hat{k}_p - \hat{p} + k(0) (1 - k(0)) r_{DE}^2 - (1 - k(0)) \frac{\tau}{\sigma^2}$$

$$r_{ss} = -q_{ss}^H + r_q q_{ss} + \frac{1}{2} r_{qq} N_1' (q_s^H)' q_s^H N_1$$

$$+ r_{qa} N_1' (q_s^H)' a_s^H + \frac{1}{2} r_{aa} (a_s^H)' a_s^H + \bar{p}_{ss} - \bar{p}_{ss}$$

3 First and second-order components of Bellman equation

3.1 Second order Taylor expansion

The Bellman equation for the Home country is listed in equation (55) of Appendix A and repeated here for convenience:

$$e^{v(0) + v(1) + v(2) + f_H(S_t)}$$

$$= \beta(1 - \psi) E_t e^{v(0) + v(1) + v(2) + f_H(S_{t+1}) + (1 - \gamma) r_{t+1}^{p,H}}$$

$$+ \beta \psi E_t e^{(1 - \gamma) r_{t+1}^{q,H}}$$  \hspace{1cm} (71)

We only list the zero, first and second-order components of the constant term $v$ since higher order components will not matter for the analysis. We will write the first and second-order derivatives of $f_H$ at $S = 0$ as respectively $H_{1,H}$ and $H_{2,H}$.

Taking a second-order Taylor expansion of the left hand side of (71) around $S = 0$ and $v(1) = v(2) = 0$, we get

$$e^{v(0) + v(1) + v(2) + f_H(S_t)}$$

$$= e^{v(0)} [1 + v(1) + v(2) + H_{1,H} S_t]$$

$$+ \frac{1}{2} e^{v(0)} [v(1) + v(2) + H_{1,H} S_t]^2 + S_t^2 H_{2,H} S_t]$$
Similarly, the first term on the right-hand side of (71) is expanded around $S = 0, v(1) = v(2) = 0$ and $r_{t+1}^p = \bar{r}$. Denote $r_{t+1}^p = r_{t+1}^p - \bar{r}$. A second-order expansion then gives

$$e^{v(0)+v(1)+v(2)+f_H(S_{t+1})+(1-\gamma)r_{t+1}^p}$$

$$= e^{v(0)+(1-\gamma)^{p}} \left[ 1 + v(1) + v(2) + H_{1,H}S_{t+1} + (1-\gamma)r_{t+1}^p \right]$$

$$+ \frac{1}{2} e^{v(0)+(1-\gamma)^{p}} \left[ \left[ v(1) + v(2) + H_{1,H}S_{t+1} + (1-\gamma)r_{t+1}^p \right]^2 \right]$$

The last term on the right-hand side of (71) is expanded as

$$e^{(1-\gamma)r_{t+1}^p} = e^{(1-\gamma)^{p}} \left[ 1 + (1-\gamma)r_{t+1}^p \right]$$

$$+ \frac{1}{2} e^{(1-\gamma)^{p}} \left[ (1-\gamma)r_{t+1}^p \right]^2$$

Combining the terms of order zero we get:

$$e^{v(0)} = \frac{\beta\psi e^{(1-\gamma)^{p}}}{1 - \beta (1 - \psi) e^{(1-\gamma)^{p}}}$$

It is convenient to substitute this result into the remaining terms of the second-order expansion of the Bellman equation, which gives

$$[v(1) + v(2) + H_{1,H}S_{t}] + \frac{1}{2} \left[ [v(1) + v(2) + H_{1,H}S_{t}]^2 + S_{t+1}'H_{2,H}S_{t} \right]$$

$$= (1 - \psi') E_t \left[ \left[ v(1) + v(2) + H_{1,H}S_{t+1} + (1-\gamma)r_{t+1}^p \right]^2 \right]$$

$$+ \frac{1}{2} \left[ \left[ v(1) + v(2) + H_{1,H}S_{t+1} + (1-\gamma)r_{t+1}^p \right]^2 \right]$$

$$+ \psi' E_t \left[ (1-\gamma)r_{t+1}^p + \frac{1}{2} \left[ (1-\gamma)r_{t+1}^p \right]^2 \right]$$

where:

$$\psi' = 1 - \beta \left(1 - \psi\right) \exp \left[(1-\gamma)\bar{r}\right]$$

(73)
3.2 First order terms

Focusing on the first-order terms in (72), we have

\[ v(1) + H_{1,H}S_t(1) = (1 - \psi') E_t \left[ v(1) + H_{1,H}S_{t+1}(1) + (1 - \gamma)r_{t+1}^{p,H}(1) \right] + \psi' E_t (1 - \gamma)r_{t+1}^{p,H}(1) \]

\[ = (1 - \psi') (H_{1,H}N_1S_t(1) + v(1)) + (1 - \gamma)r_sS_t(1) \]

where we used (49) and the first order equivalent of (70), namely: \( E_t r_{t+1}^{p,H}(1) = r_sS_t(1) \). This clearly implies that:

\[ v(1) = 0 \]

\[ H_{1,H} = (1 - \gamma)r_s(I - (1 - \psi')N_1)^{-1} \quad (74) \]

where \( I \) is a 3x3 identity matrix.

3.3 Second order terms

Now take the second-order terms in (72).

\[ H_{1,H}S_t(2) + \frac{1}{2} S_t(1)' (H_{2,H} + H_{1,H}' H_{1,H}) S_t(1) + v(2) \]

\[ = (1 - \psi') E_t \left( H_{1,H}S_{t+1}(2) + v(2) + \frac{1}{2} S_{t+1}(1)' H_{2,H} S_{t+1}(1) \right) + (1 - \gamma) E_t r_{t+1}^{p,H}(2) \]

\[ + \frac{1}{2} (1 - \psi') E_t \left( H_{1,H} S_{t+1}(1) + (1 - \gamma)r_{t+1}^{p,H}(1) \right)^2 + \frac{1}{2} \psi' E_t \left[ (1 - \gamma)r_{t+1}^{p,H}(1) \right]^2 \]

Using (70) and (66) the second order terms become:

\[ H_{1,H}S_t(2) + \frac{1}{2} S_t(1)' (H_{2,H} + H_{1,H}' H_{1,H}) S_t(1) + \psi' v_2 \]

\[ = (1 - \psi') H_{1,H} N_1 S_t(2) + (1 - \psi') H_{1,H} N_0 \sigma^2 + S_t(1)' F_1 S_t(1) + f_1 \sigma^2 \]

\[ + \frac{1}{2} (1 - \psi') S_t(1)' N_1' H_{2,H} N_1 S_t(1) + f_2 \sigma^2 \]

\[ + (1 - \gamma)r_sS_t(2) + (1 - \gamma)S_t(1)' r_{ss}S_t(1) + (1 - \gamma)\hat{r} \sigma^2 \]

\[ + \frac{1}{2} (1 - \psi') E_t S_{t+1}(1)' H_{1,H}' H_{1,H} S_{t+1}(1) \]

\[ + \frac{1}{2} (1 - \gamma)^2 E_t \left( r_{t+1}^{p,H}(1) \right)^2 + (1 - \psi')(1 - \gamma) E_t S_{t+1}(1)' H_{1,H}' r_{t+1}^{p,H}(1) \]

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where

\[ F_1 = (1 - \psi') \sum_{v=1}^{3} H_{1,H}(v) N_{3,v} \]

\[ f_1 = (1 - \psi') \text{trace} \left[ \sum_{v=1}^{3} H_{1,H}(v) N_{4,v} \right] \]

\[ f_2 = \frac{1}{2} (1 - \psi') \text{trace} [N_{2}' H_{2,H} N_2] \]

The first-order component of the portfolio return is

\[ r_{t+1}^{p,H}(1) = k(0) r_{H,t+1}(1) + (1 - k(0)) r_{F,t+1}(1) + p_t(1) - p_{t+1}(1) \]  

(76)

Using

\[ r_{H,t+1}(1) = -q_{H,t}(1) + r_q q_{H,t+1}(1) + r_a a_{H,t+1}(1) \]

\[ r_{F,t+1}(1) = -q_{F,t}(1) + r_q q_{F,t+1}(1) + r_a a_{F,t+1}(1) + r_a P_{F,t+1}(1) \]

the first-order component of the portfolio return can be written as

\[ r_{t+1}^{p,H}(1) = r_s S_t(1) + r_E \epsilon_{t+1} \]  

(77)

where \( r_s \) is as in (70) and:

\[ r_E = k(0) r_q q_{s,H}^N N_2 + (1 - k(0)) \left( r_q q_{s,F}^N N_2 + r_a p_s N_2 \right) + \tilde{r} - (1 - \alpha) p_s N_2 \]

with

\[ \tilde{r} = \left[ k(0) r_a (1 - k(0)) r_a \right] \]

Using the first order solution for \( r_{t+1}^{p,H}(1) \) the last three terms in (75) become:

\[ \frac{1}{2} (1 - \psi') S_t(1)' H_{1,H}' H_{1,H} S_{t+1}(1) \]

\[ \frac{1}{2} (1 - \gamma) E_t \left( r_{t+1}^{p,H}(1) \right)^2 + (1 - \psi')(1 - \gamma) E_t S_{t+1}(1)' H_{1,H}' r_{t+1}^{p,H}(1) \]

\[ = \frac{1}{2} (1 - \psi') S_t(1)' N_1' H_{1,H}' H_{1,H} N_1 S_t(1) + f_3 \sigma^2 + 0.5 (1 - \gamma)^2 r_E r_E \sigma^2 \]

\[ + (1 - \psi')(1 - \gamma) S_t(1)' N_1' H_{1,H}' r_s S_t(1) + f_4 \sigma^2 \]

\[ + \frac{1}{2} (1 - \gamma)^2 S_t(1)' r_s' r_s S_t(1) \]
where

\[ f_3 = \frac{1}{2} (1 - \psi') \text{trace} \left[ N_2' H_{1,H} H_{1,H} N_2 \right] \]
\[ f_4 = (1 - \psi')(1 - \gamma) \text{trace} \left[ N_2' H_{1,H} r_E \right] \]

Using these results along with (74), (75) becomes:

\[
\frac{1}{2} S_t(1)' \left( H_{2,H} + H_{1,H}' H_{1,H} \right) S_t(1) + \psi' v(2) = \begin{bmatrix} (1 - \psi') N_1' H_{2,H} N_1 + 2 F_1 + 2 (1 - \gamma) r_{ss} + (1 - \psi') N_1' H_{1,H}' H_{1,H} N_1 + 2 (1 - \psi')(1 - \gamma) N_1' H_{1,H} r_s + (1 - \gamma)^2 r_s' r_s \\
+ [(1 - \psi') H_{1,H} N_6 + f_1 + f_2 + f_3 + f_4 + (1 - \gamma) \hat{r} + 0.5 (1 - \gamma)^2 r_E r_E\end{bmatrix} S_t(1) \]

This implies:

\[
\psi' v(2) = \begin{bmatrix} (1 - \psi') H_{1,H} N_6 + f_1 + f_2 + f_3 + f_4 \\
+ (1 - \gamma) \hat{r} + 0.5 (1 - \gamma)^2 r_E r_E\end{bmatrix} \sigma^2 \]

and:

\[
H_{2,H} = (1 - \psi') N_1' H_{2,H} N_1 + H_3 \tag{78}
\]

where

\[
H_3 = -H_{1,H}' H_{1,H} + 2 F_1 + 2 (1 - \gamma) r_{ss} + (1 - \psi') N_1' H_{1,H}' H_{1,H} N_1 + 2 (1 - \psi')(1 - \gamma) N_1' H_{1,H} r_s + (1 - \gamma)^2 r_s' r_s
\]

To solve for \( H_{2,H} \) from (78) we write it in vector notation:

\[
H_{2,H}^{\text{vec}} = \begin{bmatrix} H_{2,H,1,1} \\
H_{2,H,1,2} \\
H_{2,H,1,3} \\
H_{2,H,2,1} \\
H_{2,H,2,2} \\
H_{2,H,2,3} \\
H_{2,H,3,1} \\
H_{2,H,3,2} \\
H_{2,H,3,3} \end{bmatrix}
\]

where \( H_{2,H,x,y} \) is element \((x, y)\) of matrix \( H_{2,H} \), and \( \text{vec} \) denotes the vectorization of a matrix.
The element \((i,j)\) of \(N_1^tH_{2,H}N_1\) is

\[
\hat{n}_i^tH_{2,H}\hat{n}_j = \left( (\hat{n}_i^t\hat{n}_j)^{vec} \right)^tH_{2,H}^{vec} \equiv n_{i,j}H_{2,H}^{vec}
\]

where \(\hat{n}_i\) is column \(i\) of matrix \(N_1\). We write

\[
\hat{N} = \begin{bmatrix}
n_{1,1} \\
\vdots \\
n_{1,3} \\
\vdots \\
n_{3,1} \\
\vdots \\
n_{3,3}
\end{bmatrix}
\]

It then follows from (78) that

\[
H_{2,H}^{vec} = (1 - \psi')\hat{N}H_{2,H}^{vec} + H_{3}^{vec}
\]

which implies:

\[
H_{2,H}^{vec} = (I - (1 - \psi')\hat{N})^{-1}H_{3}^{vec} \tag{79}
\]

where \(I\) is a 9x9 identity matrix.

### 4 Second and third-order components of the optimal portfolio equations

The Euler equations for optimal portfolio choice are used to solve for the difference in portfolio shares. Using (73) and \(v(1) = 0\), the Home and Foreign portfolio Euler equations (equations (53) and (54) in Appendix A of the paper) become

\[
E_t \left[ (1 - \psi')e^{\psi_2 + \psi_3 + f_H(S_{t+1}) + \psi_f} \right] e^{-\gamma r_{t+1}^H + r_{H,t+1}^H} \tag{80}
\]

and

\[
E_t \left[ (1 - \psi')e^{\psi_2 + \psi_3 + f_F(S_{t+1}) + \psi_f} \right] e^{-\gamma r_{t+1}^F + r_{F,t+1}^F} - \tau
\]

and

\[
E_t \left[ (1 - \psi')e^{\psi_2 + \psi_3 + f_F(S_{t+1}) + \psi_f} \right] e^{-\gamma r_{t+1}^F + r_{F,t+1}^F} - \tau
\]
where elements of $v$ higher than third order are omitted as they are not relevant for the analysis of second and third-order terms that follows.

A first order expansion of either relation shows that the expected excess Return is zero to a first order:

$$E_t (r_{H,t+1}(1) - r_{F,t+1}(1)) = 0$$

### 4.1 Second order component of optimal portfolio equations

The second-order component of the Home portfolio Euler equation (80) is

$$0 = E_t(r_{H,t+1}(2) - r_{F,t+1}(2) + \tau) + \frac{1}{2}E_t (r_{H,t+1}(1))^2 - \frac{1}{2}E_t (r_{F,t+1}(1))^2$$

$$-\gamma_{t}E_t r_{t+1}^p r_{t+1}^H(1)(r_{H,t+1}(1) - r_{F,t+1}(1)) +$$

$$(1 - \psi')E_t H_{1,H} S_{t+1}(1)(r_{H,t+1}(1) - r_{F,t+1}(1))$$

Since $r_{H,t+1}^H = r_{H,t+1} + p_t - p_{t+1}$ and $r_{F,t+1}^H = r_{F,t+1} + p_t - p_{t+1}$, it follows that

$$\frac{1}{2}E_t (r_{H,t+1}(1))^2 - \frac{1}{2}E_t (r_{F,t+1}(1))^2$$

$$= \frac{1}{2}E_t (r_{H,t+1}(1))^2 - \frac{1}{2}E_t (r_{F,t+1}(1))^2 + E_t(p_t(1) - p_{t+1}(1))(r_{H,t+1}(1) - r_{F,t+1}(1))$$

so that the second-order component of the Home portfolio Euler equation becomes

$$0 = E_t(r_{H,t+1}(2) - r_{F,t+1}(2) + \tau) + \frac{1}{2}E_t (r_{H,t+1}(1))^2 - \frac{1}{2}E_t (r_{F,t+1}(1))^2$$

$$+ E_t (p_t(1) - p_{t+1}(1) - \gamma r_{t+1}^p r_{t+1}^H(1) + (1 - \psi')H_{1,H} S_{t+1}(1))(r_{H,t+1}(1) - r_{F,t+1}(1))$$

Following similar steps for the Foreign optimal portfolio condition we get

$$0 = E_t(r_{H,t+1}(2) - r_{F,t+1}(2) - \tau) + \frac{1}{2}E_t (r_{H,t+1}(1))^2 - \frac{1}{2}E_t (r_{F,t+1}(1))^2$$

$$+ E_t (p_t^*(1) - p_{t+1}^*(1) - \gamma r_{t+1}^p r_{t+1}^F(1) + (1 - \psi')H_{1,F} S_{t+1}(1))(r_{H,t+1}(1) - r_{F,t+1}(1))$$

Taking the difference between (81) and (82) we get

$$0 = 2\tau - E_t ((p_{t+1}(1) - p_t(1)) - (p_{t+1}(1)^* - p_t^*(1))(r_{H,t+1}(1) - r_{F,t+1}(1))$$

$$- \gamma E_t (r_{t+1}^p r_{t+1}^H(1) - r_{t+1}^p r_{t+1}^F(1))(r_{H,t+1}(1) - r_{F,t+1}(1))$$

$$+ (1 - \psi')E_t (H_{1,H} - H_{1,F}) S_{t+1}(1)(r_{H,t+1}(1) - r_{F,t+1}(1))$$

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Since the first-order components of Home and Foreign portfolio returns are
\[ r_{t+1}^{p,H} = k(0)(r_{H,t+1}(1) - r_{F,t+1}(1)) + r_{F,t+1}(1) + p_t(1) - p_{t+1}(1) \]
\[ r_{t+1}^{p,F} = -k(0)(r_{H,t+1}(1) - r_{F,t+1}(1)) + r_{H,t+1}(1) + p_t^*(1) - p_{t+1}^*(1) \]
we have
\[ r_{t+1}^{p,H} - r_{t+1}^{p,F} = (2k(0) - 1)(r_{H,t+1}(1) - r_{F,t+1}(1)) + (p_t(1) - p_{t+1}(1)) - (p_t^*(1) - p_{t+1}^*(1)) \]
Use that the first-order solution of the return differential is \( r_{H,t+1}(1) - r_{F,t+1}(1) = r_t \epsilon_{t+1} \), where
\[ r_t = \begin{bmatrix} r_{DE} & -r_{DE} \end{bmatrix} \]
Also using (74), the second-order component of the difference between the Home and Foreign portfolio Euler equations then becomes
\[ 0 = 2\tau + (\gamma - 1)E_t \left( (p_{t+1}(1) - p_t(1)) - (p_t^*(1) - p_{t+1}^*(1)) \right) (r_{H,t+1}(1) - r_{F,t+1}(1)) - \gamma(2k(0) - 1)\text{var}(r_{H,t+1}(1) - r_{F,t+1}(1)) + (1 - \psi'')\sigma^2(H_{1,H} - H_{1,F})N_2r'_\epsilon \]
We can then solve for \( k(0) \) as
\[
k(0) = \frac{1}{2} + \frac{\tau}{2 \gamma \text{var}(r_{H,t+1}(1) - r_{F,t+1}(1))} + \frac{1}{2} \frac{\gamma - 1}{\gamma} \frac{E_t \left( (p_{t+1}(1) - p_t^*(1)) \right) (r_{H,t+1}(1) - r_{F,t+1}(1))}{\text{var}(r_{H,t+1}(1) - r_{F,t+1}(1))} + \frac{1}{2} \frac{(1 - \psi'')\sigma^2(H_{1,H} - H_{1,F})N_2r'_\epsilon}{\gamma \text{var}(r_{H,t+1}(1) - r_{F,t+1}(1))} \tag{83} \]
One can also think of this as a solution of the zero-order component of the difference in portfolio shares, which is \( 2k(0) - 1 \).

### 4.2 Second-order expected excess return

The solution of \( k(0) \) is based on the difference between the second-order components of the Home and Foreign portfolio Euler equations. Given the solution for \( k(0) \) we now return to the second-order component of the Home
portfolio Euler equation (81) in order to solve for the second-order component of the expected excess return. We start by writing

\[
\frac{1}{2} E_t(r_{H,t+1}(1))^2 - \frac{1}{2} E_t(r_{F,t+1}(1))^2 = \\
\frac{1}{2} E_t(r_{H,t+1}(1) + r_{F,t+1}(1))(r_{H,t+1}(1) - r_{F,t+1}(1)) = \\
E_t r_{t+1}^A(r_{H,t+1}(1) - r_{F,t+1}(1)) = \\
E_t \left( a_{t+1}^A(1) + \frac{1}{2} p_{F,t+1}(1) \right) (r_{H,t+1}(1) - r_{F,t+1}(1)) \\
- \left( a_t^A(1) + \frac{1}{2} p_{F,t}(1) \right) E_t (r_{H,t+1}(1) - r_{F,t+1}(1)) = \\
\frac{1}{2} E_t p_{F,t+1}(1) (r_{H,t+1}(1) - r_{F,t+1}(1))
\]

where we used (25), the fact that the first-order expected excess return is zero and that \( \epsilon_{t+1}^D \) is uncorrelated with \( \epsilon_{t+1}^A \). In addition we write

\[
E_t \epsilon_{t+1}^{p,H}(1)(r_{H,t+1}(1) - r_{F,t+1}(1)) = \\
k(0) \text{var}(r_{H,t+1}(1) - r_{F,t+1}(1)) + E_t r_{F,t+1}(1)(r_{H,t+1}(1) - r_{F,t+1}(1)) \\
+ E_t (p_t(1) - p_{t+1}(1))(r_{H,t+1}(1) - r_{F,t+1}(1))
\]

Using \( r_{F,t+1}(1) = r_{t+1}^A(1) - 0.5 r_{t+1}^D(1) = a_{t+1}^A(1) - a_t^A(1) + 0.5(p_{F,t+1}(1) - p_{F,t}(1)) - 0.5 r_{t+1}^D(1) \), we get:

\[
E_t r_{F,t+1}(1)(r_{H,t+1}(1) - r_{F,t+1}(1)) = \\
- \frac{1}{2} \text{var}(r_{H,t+1}(1) - r_{F,t+1}(1)) + \frac{1}{2} E_t p_{F,t+1}(1)(r_{H,t+1}(1) - r_{F,t+1}(1))
\]

The expected product of the portfolio return and excess return is then

\[
E_t \epsilon_{t+1}^{p,H}(1)(r_{H,t+1}(1) - r_{F,t+1}(1)) = \\
\frac{2k(0) - 1}{2} \text{var}(r_{H,t+1}(1) - r_{F,t+1}(1)) \\
+ \frac{2\alpha - 1}{2} E_t p_{F,t+1}(1)(r_{H,t+1}(1) - r_{F,t+1}(1))
\]

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Using these results, (81) becomes

\[
0 = E_t(r_{H,t+1}(2) - r_{F,t+1}(2) + \tau) + \\
\left(1 - \gamma \right) \frac{2\alpha - 1}{2} E_t p_{F,t+1}(1)(r_{H,t+1}(1) - r_{F,t+1}(1)) \\
- \gamma \frac{2k(0) - 1}{2} var(r_{H,t+1}(1) - r_{F,t+1}(1)) + \\
(1 - \psi') H_{1,H} E_t S_{t+1}(1)(r_{H,t+1}(1) - r_{F,t+1}(1))
\]

Using (83) we get

\[
E_t(r_{H,t+1}(2) - r_{F,t+1}(2)) = -\frac{1}{2}(1 - \psi')(H_{1,H} + H_{1,F})\sigma^2 N_2 r'_e 
\]  

(84) shows that the expected excess return is zero to a second order. This can be seen as follows. Because of symmetry, the first two elements of $H_{1,H}$ are equal to minus the first two elements of $H_{1,F}$ as they multiply cross-country differentials. The last element of $H_{1,H}$ is the same as the last element of $H_{1,F}$ as they apply to the worldwide shock. The first two elements of $H_{1,H} + H_{1,F}$ are then zero. Writing $w_{DE}$ as the coefficient multiplying $e_{t+1}^D$ in the first-order solution for $w_{t+1}^D$, we have

\[
N_2 r'_e = \begin{bmatrix}
1 \\
\frac{1}{2} w_{DE} \\
\frac{1}{2} w_{DE}
\end{bmatrix}
\begin{bmatrix}
r_{DE} \\
-1 \frac{1}{2} w_{DE} \\
r_{DE}
\end{bmatrix}
= r_{DE} \begin{bmatrix}
2 \\
2 w_{DE} \\
0
\end{bmatrix}
\]

Since the first two elements of $H_{1,H} + H_{1,F}$ are zero, it follows that $(H_{1,H} + H_{1,F})N_2 r'_e = 0$ and therefore $E_t(r_{H,t+1}(2) - r_{F,t+1}(2)) = 0$.

4.3 Third order component of Home’s optimal portfolio equation

We will denote $\dot{x} = x - x(0)$ for any variable $x$. A third-order Taylor expansion of the left-hand side of (80), treating the sum of terms in the exponentials
as one variable, is equal to (ignoring the multiplication constant $e^{(1-\gamma)\tau}$)

$$(1 - \psi') \left( f_H(S_{t+1}) - \gamma \hat{r}^p_{t+1} + \hat{r}_{H,t+1} + \hat{p}_t - \hat{p}_{t+1} + v(2) + v(3) \right)$$

$$+ \frac{1}{2} (1 - \psi') \left( f_H(S_{t+1}) - \gamma \hat{r}^p_{t+1} + \hat{r}_{H,t+1} + \hat{p}_t - \hat{p}_{t+1} + v(2) + v(3) \right)^2$$

$$+ \frac{1}{6} (1 - \psi') \left( f_H(S_{t+1}) - \gamma \hat{r}^p_{t+1} + \hat{r}_{H,t+1} + \hat{p}_t - \hat{p}_{t+1} + v(2) + v(3) \right)^3$$

$$+ \psi' \left( -\gamma \hat{r}^p_{t+1} + \hat{r}_{H,t+1} + \hat{p}_t - \hat{p}_{t+1} \right)$$

$$+ \frac{1}{2} \psi' \left( -\gamma \hat{r}^p_{t+1} + \hat{r}_{H,t+1} + \hat{p}_t - \hat{p}_{t+1} \right)^2$$

$$+ \frac{1}{6} \psi' \left( -\gamma \hat{r}^p_{t+1} + \hat{r}_{H,t+1} + \hat{p}_t - \hat{p}_{t+1} \right)^3$$

The right hand side of (80) is the same except that $\hat{r}_{H,t+1}$ is replaced by $\hat{r}_{F,t+1} - \tau$. Combining both sides of (80) we get

$$0 = E_t(\hat{r}_{H,t+1} - \hat{r}_{F,t+1} + \tau)$$

$$+ \frac{1}{2} E_t \left( (\hat{r}_{H,t+1})^2 - (\hat{r}_{F,t+1} - \tau)^2 \right)$$

$$+(1 - \psi')E_t(\hat{r}_{H,t+1} - \hat{r}_{F,t+1} + \tau) \left( f_H(S_{t+1}) + v(2) + v(3) \right)$$

$$+ E_t(\hat{r}_{H,t+1} - \hat{r}_{F,t+1} + \tau)(-\gamma \hat{r}^p_{t+1} + \hat{p}_t - \hat{p}_{t+1})$$

$$+ O_3$$

(85)

where

$$O_3 = \frac{1}{6} E_t \left( (\hat{r}_{H,t+1})^3 - (\hat{r}_{F,t+1})^3 \right) +$$

$$+ \frac{1}{2} E_t \left( (\hat{r}_{H,t+1})^2 - (\hat{r}_{F,t+1})^2 \right) \left( -\gamma \hat{r}^p_{t+1} + \hat{p}_t - \hat{p}_{t+1} + (1 - \psi') f_H(S_{t+1}) \right)$$

$$+ \frac{1}{2} (1 - \psi')E_t(\hat{r}_{H,t+1} - \hat{r}_{F,t+1}) \left( f_H(S_{t+1}) - \gamma \hat{r}^p_{t+1} + \hat{p}_t - \hat{p}_{t+1} \right)^2$$

$$+ \frac{1}{2} \psi' E_t(\hat{r}_{H,t+1} - \hat{r}_{F,t+1}) \left( -\gamma \hat{r}^p_{t+1} + \hat{p}_t - \hat{p}_{t+1} \right)^2$$

(86)

In the expression for $O_3$ we have omitted $\tau$, $v(2)$ and $v(3)$ since when multiplied with other terms they lead to fourth and higher order terms.
The third-order component of (85) is equal to

\[ 0 = E_t(r_{H,t+1}(3) - r_{F,t+1}(3)) + O_3 \]  
\[ + c\text{ov}_t \left( r_{H,t+1} - r_{F,t+1}, r_{t+1}^A + (1 - \psi')f_H(S_{t+1}) - \gamma r_{t+1}^{pH} + p_t - p_{t+1} \right) \]  
\[ + \tau E_t \left( r_{t+1}^A (1) + (1 - \psi')H_{1,H}S_{t+1}(1) - \gamma r_{t+1}^{pH}(1) + p_t(1) - p_{t+1}(1) \right) \]  

where \( c\text{ov}_t \) is defined as \( c\text{ov}_t(x, y) = E_t x(1)y(2) + E_t x(2)y(1) \) and \( O_3 \) is equal to (86) after replacing each variable with its first-order component. \( v(2) \) drops out as it only enters the third-order component of (85) when multiplied with the first-order expected excess return, which is zero.

We now simplify the cubic terms in (86) by substituting the first-order solution of all variables. Using that \( r_{H,t+1}(1) = r_{t+1}^A(1) + 0.5r_{t+1}^D(1) \) and \( r_{F,t+1}(1) = r_{t+1}^A(1) - 0.5r_{t+1}^D(1) \), we have

\[ E_t \left( (r_{H,t+1}(1))^3 - (r_{F,t+1}(1))^3 \right) = E_t \left( \frac{1}{4}(r_{t+1}^D(1))^3 + 3(r_{t+1}^A(1))^2r_{t+1}^D(1) \right) \]

Since \( r_{t+1}^D(1) = r_{DE}\epsilon_{t+1}^D \), the first term of the above expression is zero because \( E_t(\epsilon_{t+1}^D)^3 = 0 \). Therefore

\[ E_t \left( (r_{H,t+1}(1))^3 - (r_{F,t+1}(1))^3 \right) = 3E_t(r_{t+1}^A(1))^2r_{t+1}^D(1) \]  

Define

\[ l_{t+1}(1) = r_{t+1}^A(1) - \gamma r_{t+1}^{pH}(1) + p_t(1) - p_{t+1}(1) \]

Then (86) becomes

\[ O_3 = \frac{1}{2}(1 - \psi')E_t(r_{H,t+1}(1) - r_{F,t+1}(1))(l_{t+1}(1) + H_{1,H}S_{t+1}(1))^2 \]  
\[ + \frac{1}{2}\psi'E_t(r_{H,t+1}(1) - r_{F,t+1}(1))(l_{t+1}(1))^2 \]  

(89)

We can simplify this further by using the second-order component of Home’s optimal portfolio equation, equation (81), which we write as

\[ E_t(r_{H,t+1}(2) - r_{F,t+1}(2)) + \]  
\[ E_t(r_{H,t+1}(1) - r_{F,t+1}(1))(l_{t+1}(1) + (1 - \psi')H_{1,H}S_{t+1}(1)) = 0 \]

We have shown that \( E_t(r_{H,t+1}(2) - r_{F,t+1}(2)) = 0 \). After substituting the first order solution for the variables we then must have

\[ r_{DE}E_t(l_{t+1}(1) + (1 - \psi')H_{1,H}S_{t+1}(1))\epsilon_{t+1}^D = -\tau \]
The variables in the big parentheses depend on $S_t(1)$, $\epsilon^A_{t+1}$ and $\epsilon^D_{t+1}$:

\[ l_{t+1}(1) + (1 - \psi')H_{1,H}S_{t+1}(1) = A_H S_t(1) + B_H \epsilon^D_{t+1} + C_H \epsilon^A_{t+1} \] (90)

As $E_t \epsilon^D_{t+1} = E_t \epsilon^A_{t+1} \epsilon^D_{t+1} = 0$ we get:

\[ 2B_H r_D \sigma^2 = -\tau \] (91)

as $E_t (\epsilon^D_{t+1})^2 = 2\sigma^2$.

Next use the fact that

\[ H_{1,H} S_{t+1}(1) = H_{1,H} N_1 S_t(1) + H_{1,H} N_2 \epsilon_{t+1} =
\]

\[ f_{HS} S_t(1) + f_{HD} \epsilon_{t+1} + f_{HA} \epsilon^A_{t+1} \] (92)

where:

\[ f_{HS} = H_{1,H} N_1 \]
\[ f_{HD} = H_{1,H,1} + H_{1,H,2} w_D \]
\[ f_{HA} = H_{1,H,3} \]

where we used:

\[ H_{1,H} N_2 \epsilon_{t+1} = 
\]

\[ = [ H_{1,H,1} H_{1,H,2} H_{1,H,3} ] 
\]

\[ [ \begin{array}{cc} 1 & -1 \\ w_D & -w_D \\ 0.5 & 0.5 \end{array} ] 
\]

\[ [ \begin{array}{c} \epsilon_{Ht+1} \\ \epsilon_{Ft+1} \end{array} ] 
\]

\[ = [ H_{1,H,1} H_{1,H,2} H_{1,H,3} ] 
\]

\[ [ \begin{array}{c} \epsilon^D_{t+1} \\ w_D \epsilon^D_{t+1} \\ \epsilon^A_{t+1} \end{array} ] 
\]

Substituting (90) and (92) into (89), we get (recall that $E_t (\epsilon^D_{t+1})^3 = E_t (\epsilon^A_{t+1})^3 = E_t (\epsilon^A_{t+1})^2 \epsilon^D_{t+1} = E_t \epsilon^A_{t+1} (\epsilon^D_{t+1})^2 = E_t \epsilon^A_{t+1} \epsilon^D_{t+1} = 0$)

\[ O_3 = 2\sigma^2 r_D (B_H A_H + \psi'(1 - \psi') f_{HD} f_{HS}) S_t(1) \] (93)

Using (93) the third order expansion of the Home optimal portfolio (87) is written as

\[ 0 = E_t (r_{H,t+1}(3) - r_{F,t+1}(3)) 
\]

\[ + 2\sigma^2 r_D (B_H A_H + \psi'(1 - \psi') f_{HD} f_{HS}) S_t(1) 
\]

\[ + \sum_{t=1}^{T} \left( r_{H,t+1} - r_{F,t+1} + (1 - \psi') f_H(S_{t+1}) - \gamma p_{t+1} + p_t - p_{t+1} \right) 
\]

\[ + \tau E_t \left( \epsilon^A_{t+1} (1 + (1 - \psi') H_{1,H} S_{t+1}(1) - \gamma p_{t+1}(1) + p_t(1) - p_{t+1}(1) \right) 
\]

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Focusing for a moment on the last term, using (90), (91) and \( E_t \epsilon^*_t = E_t \epsilon^*_{t+1} = 0 \), we write:

\[
\tau E_t \left( r^A_{t+1}(1) + (1 - \psi')H_{1,H}S_{t+1}(1) - \gamma r^p_{t+1}(1) + p_t(1) - p_{t+1}(1) \right) = \\
\tau A_H S_t(1) = -2B_Hr_D\sigma^2 A_H S_t(1)
\]

The third-order component of the Home optimal portfolio condition then simplifies to

\[
0 = E_t(r_{H,t+1}(3) - r_{F,t+1}(3)) + 2\sigma^2 r_D \psi'(1 - \psi') f_H f_H S_t(1) \\
+ c \text{cov}_t \left( r_{H,t+1} - r_{F,t+1}, r^A_{t+1} + (1 - \psi') f_H(S_{t+1}) - \gamma r^p_{t+1} - p_{t+1} \right) (94)
\]

where \( p_t \) is preset and can be omitted from \( c \text{cov}_t \).

### 4.4 Combining the Home and Foreign optimal portfolio equations

Following similar steps for the Foreign country and writing (analogous to (92) for the Home country)

\[
H_{1,F} S_{t+1}(1) = f_{FS} S_t(1) + f_{FD} \epsilon^D_{t+1} + f_{FA} \epsilon^A_{t+1}
\]

the third-order component of the optimal portfolio equation for the Foreign country is

\[
0 = E_t(r_{H,t+1}(3) - r_{F,t+1}(3)) + 2\sigma^2 r_D \psi'(1 - \psi') f_{FD} f_{FS} S_t(1) \\
+ c \text{cov}_t \left( r^D_{t+1}, r^A_{t+1} + (1 - \psi') f_F(S_{t+1}) - \gamma r^p_{t+1} - p^*_t \right)
\]

where \( r^D_{t+1} = r_{H,t+1} - r_{F,t+1} \).

Taking the difference between the third-order component of the Home and Foreign portfolio Euler equations we have

\[
0 = 2\sigma^2 r_D \psi'(1 - \psi')(f_H f_H - f_{FD} f_{FS}) S_t(1) \\
+ c \text{cov}_t \left( r^D_{t+1}, (1 - \psi') (f_H(S_{t+1}) - f_F(S_{t+1})) - \gamma (r^p_{t+1} - r^p_{t+1}) - (p_{t+1} - p^*_t) \right) (96)
\]
The second-order component of Home and Foreign portfolio returns are

\[
 r_{t+1}^{p,H}(2) = \frac{1}{2}k(0)(1 - k(0))(r_{t+1}^{D}(1)) + r_{t+1}^{D}(1)k_{H,t+1}(1)
\]

\[
 r_{t+1}^{p,F}(2) = \frac{1}{2}k(0)(1 - k(0))(r_{t+1}^{D}(1)) - r_{t+1}^{D}(1)k_{F,t+1}(1)
\]

(97)

(98)

The difference is

\[
 r_{t+1}^{p,H}(2) - r_{t+1}^{p,F}(2) = \frac{1}{2}k(0)(1 - k(0))(r_{t+1}^{D}(1)) + r_{t+1}^{D}(1)k_{H,t+1}(1)
\]

\[
 r_{t+1}^{p,F}(2) = \frac{1}{2}k(0)(1 - k(0))(r_{t+1}^{D}(1)) - r_{t+1}^{D}(1)k_{F,t+1}(1)
\]

The first-order component of the portfolio return difference is

\[
 r_{t+1}^{p,H}(1) - r_{t+1}^{p,F}(1) = \frac{1}{2}k(0)(1 - k(0))(r_{t+1}^{D}(1)) + r_{t+1}^{D}(1)k_{H,t+1}(1)
\]

Substituting the first and second-order components of portfolio return differences into (96), we have

\[
 0 = 2\sigma^2 r_{DE} \psi(1 - \psi')(f_{HD}f_{HS} - f_{FD}f_{FS})S_t(1)
\]

\[
 + (1 - \psi')\text{cov}_t(r_{t+1}^{D}, f_H(S_{t+1}) - f_F(S_{t+1}))
\]

\[
 - \gamma(2k(0) - 1)v\bar{r}_t(r_{t+1}^{D})
\]

\[
 + (\gamma - 1) \text{cov}_t(r_{t+1}^{D}, p_t - p_{t+1}^{*})
\]

\[
 - \gamma k_{t+1}(1)\text{var}(r_{t+1}^{D}(1))
\]

(99)

where \(v\bar{r}_t(x) = 2E_t(x_1x_2)\) and \(\text{var}(r_{t+1}^{D}(1)) = 2r_{DE}^2\).

We can use this to solve for \(k_{t+1}^{D}(1)\):

\[
 k_{t+1}^{D}(1) = \frac{-2k(0) - 1}{\text{var}(r_{t+1}^{D}(1))} + \gamma - 1 \frac{\text{cov}_t(r_{t+1}^{D}, p_t - p_{t+1}^{*})}{\text{var}(r_{t+1}^{D}(1))}
\]

\[
 + (1 - \psi') \frac{\text{cov}_t(r_{t+1}^{D}, f_H(S_{t+1}) - f_F(S_{t+1}))}{\text{var}(r_{t+1}^{D}(1))}
\]

\[
 + 2\sigma^2 r_{DE} \psi(1 - \psi')(f_{HD}f_{HS} - f_{FD}f_{FS})S_t(1)
\]

(100)

This corresponds to equation (43) of the paper, where \(er_{t+1} = r_{t+1}^{D}\) and using that from (92) and (95)

\[
 E_t (f_{H,t+1}(1)^2 - f_{F,t+1}(1)^2) r_{t+1}^{D}(1) = 4r_{DE}^2 (f_{HS}f_{HD} - f_{FS}f_{FD})
\]

where \(f_{H,t+1}(1) = H_{1,H}S_{t+1}(1)\) and \(f_{F,t+1}(1) = H_{1,F}S_{t+1}(1)\).
4.5 Computing third-order expectations

Let $R_1$ and $R_2$ denote, respectively, the first and second-order components of $r^D_{t+1}$. Similarly, let $F_1$ and $F_2$ denote the first and second-order components of $f_H(S_{t+1}) - f_F(S_{t+1})$ and $P_1$, $P_2$ the first and second-order components of $p_{t+1} - p^*_t$. In order to evaluate (100) we then need to compute $E_t R_1 R_2$, $E_t R_1 F_2$, $E_t R_1 P_2$, $E_t R_2 F_1$ and $E_t R_2 P_1$.

The computation of these terms uses the second-order solution to the model. Let’s start with the first and second-order terms of $r^D_{t+1}$. We know that the first order term is $r^D_{DEt+1}$. Using the definitions of returns (equations (59)-(60) in Appendix A of the paper), the second-order components are

$$r_{H,t+1}(2) = -q_{H,t}(2) + r_q q_{H,t+1}(2) + (1 - r_q) a_{H,t+1}(2) + \frac{1}{2} r_{qq} \left[ (q_{H,t+1}(1))^2 + (a_{H,t+1}(1))^2 - 2 q_{H,t+1}(1)a_{H,t+1}(1) \right]$$

$$r_{F,t+1}(2) = -q_{F,t}(2) + r_q q_{F,t+1}(2) + (1 - r_q) (a_{F,t+1}(2) + p_{F,t+1}(2)) + \frac{1}{2} r_{qq} \left[ (q_{F,t+1}(1))^2 + (a_{F,t+1}(1))^2 + (p_{F,t+1}(1))^2 + 2 a_{F,t+1}(1)p_{F,t+1}(1) \right] - 2 q_{F,t+1}(1)a_{F,t+1}(1) - 2 q_{F,t+1}(1)p_{F,t+1}(1)$$

where $r_q = (1 - \psi)(1 - \psi \theta)^{-1}$ and $r_{qq} = r_q (1 - r_q)$. The second-order solution for the relative price of Foreign goods and equity prices are (60)-(62).

The second order components in $r_{H,t+1}$ and $r_{F,t+1}$ take four different forms: (i) quadratic in $S_t(1)$, (ii) proportional to $\sigma^2$, (iii) quadratic in the innovations $\epsilon_{t+1}$ and (iv) product of $S_t(1)$ and innovations. We focus on the difference between $r_{H,t+1}(2)$ and $r_{F,t+1}(2)$. We know that the expected value of this difference is zero. We can then ignore (i) and (ii) as they are known at time $t$. We can also ignore (iii). The expected value of those terms in $R_2$ is zero. When multiplying them with first-order terms later on, we can therefore ignore them to the extent that the first-order terms are linear in the state space (the expected product of those terms in $R_2$ times $S_t(1)$ remains zero.) To the extent that the first order terms are linear in innovations, we can ignore them as well since the expectation of any cubic form of innovations is zero. The only relevant second-order terms are therefore ones that are products of the state space at time $t$ and innovations at $t + 1$. We will therefore focus on those terms only.
Using (60)-(62) the second-order component of the excess return is

\[ R_2 = r_{H, t+1}(2) - r_{F, t+1}(2) = \]
\[- \left[ (q^H - q^F) S_t(2) + S_t(1)' (q^H - q^F) S_t(1) + (k^H - k^F) \sigma^2 \right] \]
\[ + r_q \left[ (q^H - q^F) S_{t+1}(2) + S_{t+1}(1)' (q^H - q^F) S_{t+1}(1) + (k^H - k^F) \sigma^2 \right] \]
\[ + (1 - r_q) \left[ a_{H, t+1}(2) - a_{F, t+1}(2) - p_S S_{t+1}(2) - S_{t+1}(1)' p_S S_{t+1}(1) - k_p \sigma^2 \right] \]
\[ + \frac{1}{2} \tau^{qq} \left[ \begin{array}{c}
(q_{H, t+1}(1))^2 + (a_{H, t+1}(1))^2 - 2q_{H, t+1}(1)a_{H, t+1}(1) \\
-(q_{F, t+1}(1))^2 - (a_{F, t+1}(1))^2 - (p_{F, t+1}(1))^2 - 2a_{F, t+1}(1)p_{F, t+1}(1) \\
+2q_{F, t+1}(1)a_{F, t+1}(1) + 2q_{F, t+1}(1)p_{F, t+1}(1)
\end{array} \right] \]

The productivity terms are exactly first-order by assumption, hence \( a_{H, t+1}(2) = a_{F, t+1}(2) = 0 \). Using (49), (66) and

\[ a_{H, t+1}(1) = \rho a_{H, t}(1) + \epsilon_{H, t+1} = a_s^H S_t(1) + a_E^H \epsilon_{t+1} \]
\[ a_{F, t+1}(1) = \rho a_{F, t}(1) + \epsilon_{F, t+1} = a_s^F S_t(1) + a_E^F \epsilon_{t+1} \]

the terms in the expression for \( r_{H, t+1}(2) - r_{F, t+1}(2) \) that involve the product of \( S_t(1) \) and model innovations are

\[ [r_q (q^H - q^F) - (1 - r_q) p_S] \left( \begin{array}{c}
S_t(1)' N_{5,1} \epsilon_{t+1} \\
S_t(1)' N_{5,2} \epsilon_{t+1} \\
S_t(1)' N_{5,3} \epsilon_{t+1}
\end{array} \right) \]
\[ + 2S_t(1)' N_1' [r_q (q^H - q^F) - (1 - r_q) p_S] N_2 \epsilon_{t+1} \]
\[ + \frac{1}{2} \tau^{qq} \left[ \begin{array}{c}
2S_t(1)' N_1' \left[ (q^H)' q^H - (q^F)' q^F - (p_S)' p_S + (q^F)' p_S + (p_S)' q^F \right] N_2 \epsilon_{t+1} \\
+ 2S_t(1)' \left[ (a_s^H)' a_s^H - (a_s^F)' a_s^F \right] \epsilon_{t+1} \\
- 2S_t(1)' N_1' \left[ (q^H)' a_s^H - (q^F)' a_s^F \right] a_s^F \epsilon_{t+1} \\
- 2S_t(1)' \left[ (a_s^H)' q^H - (a_s^F)' q^F \right] q^F \epsilon_{t+1}
\end{array} \right] \]

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Another way to write these terms is

\[
S_t(1)' \left[ \sum_{v=1}^{3} \left[ r_q(q_{s,v}^H - q_{s,F}^F) - (1 - r_q) p_{s,v} \right] N_{5,v} \right] \epsilon_{t+1} \\
+ 2S_t(1)'N_1' \left[ r_q(q_{ss}^H - q_{ss}^F) - (1 - r_q) p_{ss} \right] N_2\epsilon_{t+1} \\
+ r_{qq}S_t(1)'N_1' \left[ (q_s^H)'q_s^H - (q_s^F)'q_s^F - (p_s)'p_s + (q_s^F)'p_s + (p_s)'q_s^F \right] N_2\epsilon_{t+1} \\
- r_{qq}S_t(1)' \left[ (a_s^H)'a_s^H - (a_s^F)'a_s^F \right] \epsilon_{t+1} \\
- r_{qq}S_t(1)' \left[ (q_s^H)'a_s^H - (q_s^F)'a_s^F \right] \epsilon_{t+1} \\
- r_{qq}S_t(1)' \left[ (a_s^F)'p_s N_2\epsilon_{t+1} \right] - r_{qq}S_t(1)'N_1' (p_s)'a_s^F \epsilon_{t+1}
\]

where \( q_{s,x}^H \) is \( x \)th element of the vector \( q_s^H \) and similarly for \( p_{s,x} \). We can write these terms in a compact way as \( S_t(1)'\bar{M}_{t+1} \). Since \( R_1 = r_{DE}\epsilon_{t+1} = r_e\epsilon_{t+1} \), it follows that

\[
E_tR_1R_2 = \sigma^2 S_t(1)'\bar{M}_{t+1} \epsilon_{t+1}
\]

(101)

We next turn to \( P_2 \), the second-order component of \( p_{t+1} - p_{t+1}^* \). We have

\[
p_{t+1}(2) = (1 - \alpha)p_{F,t+1}(2) - \frac{1}{2} \alpha (1 - \alpha) (\lambda - 1) (p_{F,t+1}(1))^2
\]

(102)

\[
p_{t+1}^*(2) = \alpha p_{F,t+1}(2) - \frac{1}{2} \alpha (1 - \alpha) (\lambda - 1) (p_{F,t+1}(1))^2
\]

(103)

Combining this with (60) we have

\[
P_2 = p_{t+1}(2) - p_{t+1}^*(2) = (1 - 2\alpha) (p_sS_{t+1}(2) + S_{t+1}(1)'p_{ss}S_{t+1}(1) + k_p\sigma^2)
\]
Using (49) and (66) this becomes

\[
P_2 = (1 - 2\alpha)p_s N_1 S_t(2) + (1 - 2\alpha)p_s N_0 \sigma^2
+ (1 - 2\alpha)S_t(1)' \left( \sum_{v=1}^{3} p_{s,v} N_{3,v} \right) S_t(1)
+ (1 - 2\alpha)\epsilon'_{t+1} \left( \sum_{v=1}^{3} p_{s,v} N_{4,v} \right) \epsilon_{t+1}
+ (1 - 2\alpha)S_t(1)' \left( \sum_{v=1}^{3} p_{s,v} N_{5,v} \right) \epsilon_{t+1}
+ (1 - 2\alpha)S_t(1)' N'_{1} p_{ss} N_1 S_t(1)
+ (1 - 2\alpha)\epsilon'_{t+1} N'_{2} p_{ss} N_2 \epsilon_{t+1}
+ (1 - 2\alpha)2S_t(1)' N'_{1} p_{ss} N_2 \epsilon_{t+1}
+ (1 - 2\alpha)k_p \sigma^2
\]

Recall that \( P_1 = (1 - 2\alpha)p_s [N_1 S_t(1) + N_2 \epsilon_{t+1}] \). Using these results and focusing on the terms in the cross product of \( S_t(1) \) and \( \epsilon_{t+1} \) we have

\[
E_t R_1 P_2 = 2(1 - 2\alpha)\sigma^2 S_t(1)' N'_{1} p_{ss} N_2 r'_{\epsilon}
+ (1 - 2\alpha)\sigma^2 S_t(1)' \left( \sum_{v=1}^{3} p_{s,v} N_{5,v} \right) r'_{\epsilon}
\quad \tag{104}
\]

\[
E_t R_2 P_1 = (1 - 2\alpha)\sigma^2 S_t(1)' M N'_{2} p'_{s}
\quad \tag{105}
\]

Finally consider \( F_2 \), the second-order component of \( f_H(S_{t+1}) - f_F(S_{t+1}) \). We have:

\[
F_2 = (H_{1,H} - H_{1,F}) S_{t+1}(2) + \frac{1}{2} S_{t+1}(1)' (H_{2,H} - H_{2,F}) S_{t+1}(1)
\]
Using (49) and (66) this becomes

\[
F_2 = (H_{1,H} - H_{1,F})N_1S_t(2) + (H_{1,H} - H_{1,F})N_6\sigma^2 \\
+ S_t(1)\left( \sum_{v=1}^{3} (H_{1,H,v} - H_{1,F,v})N_{3,v} \right) S_t(1) \\
+ \epsilon'_t \left( \sum_{v=1}^{3} (H_{1,H,v} - H_{1,F,v})N_{4,v} \right) \epsilon_{t+1} \\
+ S_t(1)\left( \sum_{v=1}^{3} (H_{1,H,v} - H_{1,F,v})N_{5,v} \right) \epsilon_{t+1} \\
+ \frac{1}{2} S_t(1)'N'_1((H_{2,H} - H_{2,F})N_1S_t(1) \\
+ \frac{1}{2} \epsilon_{t+1} N'_2(H_{2,H} - H_{2,F})N_2\epsilon_{t+1} \\
+ \frac{1}{2} S_t(1)'N'_1[(H_{2,H} - H_{2,F}) + (H_{2,H} - H_{2,F})']N_2\epsilon_{t+1}
\]

Recall that \( F_1 = (H_{1,H} - H_{1,F}) [N_1S_t(1) + N_2\epsilon_{t+1}] \). Using these results and focusing on the terms in the cross product of \( S_t(1) \) and \( \epsilon_{t+1} \), it follows that

\[
E_t R_1 F_2 = \frac{1}{2} \sigma^2 S_t(1)' N'_1 [(H_{2,H} - H_{2,F}) + (H_{2,H} - H_{2,F})'] N_{2}'\epsilon' \\
+ \sigma^2 S_t(1)' \left( \sum_{v=1}^{3} (H_{1,H,v} - H_{1,F,v})N_{5,v} \right) \epsilon_{t+1}' \tag{106}
\]

\[
E_t R_2 F_1 = \sigma^2 S_t(1)' \tilde{M} N'_2(H_{1,H} - H_{1,F})' \tag{107}
\]

To summarize, we have

\[
E_t R_1 R_2 = \sigma^2 r, S_t(1) \\
E_t R_1 P_2 = (1 - 2\alpha)\sigma^2 r \left[ 2N'_2(p_{ss})' N_1 + \left( \sum_{v=1}^{3} p_{s,s,v}N_{5,v} \right) \right] S_t(1) \\
E_t R_2 P_1 = (1 - 2\alpha)\sigma^2 p_{s}s N_2 \tilde{M}' S_t(1) \\
E_t R_1 F_2 = \sigma^2 r \left[ \frac{1}{2} N'_2 [(H_{2,H} - H_{2,F}) + (H_{2,H} - H_{2,F})'] N_1 \right] S_t(1) \\
E_t R_2 F_1 = \sigma^2 (H_{1,H} - H_{1,F}) N_2 \tilde{M}' S_t(1)
\]
Using this (100) becomes

\[ k_{t+1}^D(1) = k_s S_t(1) \]  

where

\[ 2\gamma r_{DE}^2 k_s = (1 - \psi') r_\epsilon \frac{1}{2} N_2' [(H_{2,H} - H_{2,F}) + (H_{2,H} - H_{2,F})'] N_1 \]

\[ + (1 - \psi') r_\epsilon \left( \sum_{i=1}^{3} (H_{1,H,w} - H_{1,F,w}) N_{5,w} \right)' \]

\[ + (1 - \psi')(H_{1,H} - H_{1,F}) N_2 \bar{M}' \]

\[ - 2\gamma (2k(0) - 1) r_\epsilon \bar{M}' \]

\[ + (\gamma - 1)(1 - 2\alpha) 2r_s N_2' (p_{ss})' N_1 + (\gamma - 1)(1 - 2\alpha) r_\epsilon \left( \sum_{i=1}^{3} p_{s,v} N_{5,v} \right)' \]

\[ + (\gamma - 1)(1 - 2\alpha) p_s N_2 \bar{M}' \]

\[ + 2r_{DE} \psi'(1 - \psi')(f_{HD} f_{HS} - f_{FD} f_{FS}) \]

(109)

where we used \( \text{var}(r_{t+1}^D) = 2\sigma^2 r_{DE}^2 \).

### 4.6 Third-order component of expected excess return

We have already shown that the first and second-order components of the expected excess return are zero. We now turn to computing the third-order component of the expected excess return. We start by taking the sum of the third-order component of the Home portfolio Euler equation in (94) and its foreign equivalent:

\[ 2E_t(r_{H,t+1}(3) - r_{F,t+1}(3)) = -2\sigma^2 r_{DE} \psi'(1 - \psi')(f_{HD} f_{HS} + f_{FD} f_{FS}) S_t(1) \]

\[ - \text{cov}_t \left( r_{t+1}^D, 2r_{t+1}^A + (1 - \psi')(f_H(S_{t+1}) + f_F(S_{t+1})) \right) \]

\[ + \text{cov}_t \left( r_{t+1}^D, \gamma \left( r_{t+1}^{p,H} + r_{t+1}^{p,F} \right) + (p_{t+1} + p_{t+1}^*) \right) \]

(110)

Taking the sum of (97)-(98) we write:

\[ r_{t+1}^{p,H}(2) + r_{t+1}^{p,F}(2) = r_{H,t+1}(2) + r_{F,t+1}(2) - (1 - k(0)) 2\tau \]

\[ + p_t(2) + p_t^*(2) - p_{t+1}(2) - p_{t+1}^*(2) \]

\[ + k(0)(1 - k(0))(r_{t+1}^D(1))^2 + r_{t+1}^D(1) \left( k_{H,t}^H(1) - k_{F,t}^F(1) \right) \]
Similarly:
\[ r_{t+1}^{p,H} + r_{t+1}^{p,F} = r_{H,t+1}(1) + r_{F,t+1}(1) + p_t(1) + p_t^*(1) - p_{t+1}(1) - p_{t+1}^*(1) \]

Using these results we write:
\[ \text{cov}_t \left( r_{t+1}^D, r_{t+1}^{p,H} + r_{t+1}^{p,F} \right) = 2\text{cov}_t(r_{t+1}^D, r_{t+1}^A) \]
\[ -\text{cov}_t(r_{t+1}^D, p_{t+1} + p_{t+1}^*) + 2k_t^{A}(1)\text{var}(r_{t+1}^D) \]

where we used \( E_t(r_{t+1}^D)^3 = 0 \). Substituting this result into (110) yields:
\[ E_t(r_{H,t+1}(3) - r_{F,t+1}(3)) = -\sigma^2 r_{DE}^2 \psi'(1 - \psi') (f_{HD}f_{HS} + f_{FD}f_{FS}) S_t(1) \]
\[ - (1 - \gamma) \text{cov}_t \left( r_{t+1}^D, r_{t+1}^A \right) + 0.5 (1 - \gamma) \text{cov}_t \left( r_{t+1}^D, p_{t+1} + p_{t+1}^* \right) \]
\[ -0.5(1 - \psi')\text{cov}_t \left( r_{t+1}^D, f_H(S_{t+1}) + f_F(S_{t+1}) \right) \]
\[ +\gamma k_t^{A}(1)\text{var}(r_{t+1}^D) \]

(111)

The last term can also be written as \( 2\sigma^2 r_{DE}^2 \gamma k_s^A S_t(1) \) where we have written the first-order solution of the average portfolio share as \( k_t^{A}(1) = k_s^A S_t(1) \).

The term involving the consumer price indexes is also relatively easy to compute from (102)-(103):
\[ \text{cov}_t \left( r_{t+1}^D, p_{t+1} + p_{t+1}^* \right) = \frac{1}{1 - 2\alpha} (E_t R_1 P_2 + E_t R_2 P_1) \]
\[ -\alpha(1 - \alpha)(\lambda - 1)2\sigma^2 r_{2}^N p_s^p N_1 S_t(1) \]

The two terms in (111) left to compute are then \( \text{cov}_t(r_{t+1}^D, r_{t+1}^A) \) and \( \text{cov}_t(r_{t+1}^D, f_H(S_{t+1}) + f_F(S_{t+1})) \). Let \( R_1^A \) and \( R_2^A \) denote the first and second-order component of \( r_{t+1}^A \) and \( F_1^A \) and \( F_2^A \) be the first and second-order components of \( 0.5 [f_H(S_{t+1}) + f_F(S_{t+1})] \). We therefore need to compute \( E_t R_1 R_2^A, E_t R_2 R_1^A, E_t R_1 F_2^A \) and \( E_t R_2 F_1^A \).

In the second-order terms again only products of \( S_t(1) \) and \( \epsilon_{t+1} \) are relevant. These terms in \( R_2^A \) are computed analogously to those for \( R_2 \) computed above and can be summarized as \( S_t(1)' \mathbf{M}^A \epsilon_{t+1} \), where
\[ \bar{M}^A = 0.5 \left[ \sum_{v=1}^3 \left[ r_q q^H_{s,v} + q^F_{s,v} \right] N_{5,v} \right] + N'_1 \left[ r_q (q^H_{ss} + q^F_{ss}) + (1 - r_q) p_{ss} \right] N_2 \\
+ 0.5 \rho_{qq} N'_1 \left[ (q^H_s q^H_s + q^F_s q^F_s + (p_s)' p_s - (q^F_s)' p_s) n_2 \left( a^H_E + a^F_E \right) \right] N_2 \\
+ 0.5 \rho_{qq} \left[ (a^H_s q^H_s + a^F_s q^F_s) n_2 \right] N_2 \\
- 0.5 \rho_{qq} N'_1 \left[ (q^H_s a^H_E + a^F_s) q^F_s \right] N_2 \\
+ 0.5 \rho_{qq} (a^F_s)' p_s N_2 + 0.5 \rho_{qq} N'_1 (p_s)' a^E_E \]

Therefore

\[ E_t R_1 R^A = \sigma^2 S_t (1)' \bar{M}^A n_e' \] (112)

The innovation component of \( R^A_1 \) is \( n_e' \epsilon_{t+1} \), where

\[ n_e^A = 0.5 \rho_q (q^H_s + q^F_s) n_2 + 0.5(1 - r_q) p_{ss} n_2 + 0.5(1 - r_q) \left( a^H_E + a^F_E \right) \]

Therefore

\[ E_t R_2 R^A = \sigma^2 S_t (1)' \bar{M} (n_e^A)' \] (113)

Analogous to (106) and (107) we have

\[ E_t R_1 F^A = \frac{1}{4} \sigma^2 S_t (1)' N'_1 \left[ (H_{2,H} + H_{2,F}) + (H_{2,H} + H_{2,F})' \right] N_{2} n_e' \\
+ 0.5 \sigma^2 S_t (1)' \left( \sum_{v=1}^3 (H_{1,H,v} + H_{1,F,v}) N_{5,v} \right) n_e' \]

\[ E_t R_2 F^A = 0.5 \sigma^2 S_t (1)' \bar{M} N'_2 (H_{1,H} + H_{1,F})' \]

Substituting these results into (111) we have

\[ E_t (r_{H,t+1} - r_{F,t+1}(3)) = \sigma^2 r_3 S_t (1) \] (114)
where

\[
\begin{align*}
    r_3 &= -r_{DE} \psi'(1 - \psi') (f_{HD} f_{HS} + f_{FD} f_{FS}) - (1 - \gamma) \left[ r_{\epsilon} (\tilde{M}^A)' + r_{\epsilon}^{A^2} \tilde{M}' \right] \\
    &\quad + (1 - \gamma) r_{\epsilon} N'_2 p_s' s N_1 + 0.5 (1 - \gamma) r_{\epsilon} \left( \sum_{v=1}^{3} p_{s,v} N_{5,v} \right)' \\
    &\quad + 0.5 (1 - \gamma) p_s N_2 \tilde{M}' - (1 - \gamma) \alpha (1 - \alpha)(\lambda - 1) r_{\epsilon} N'_2 p'_s p_s N_1 \\
    &\quad - (1 - \psi') \frac{1}{4} r_{\epsilon} N'_2 \left[ (H_{2,H} + H_{2,F}) + (H_{2,H} + H_{2,F})' \right] N_1 \\
    &\quad - (1 - \psi') 0.5 r_{\epsilon} \left( \sum_{v=1}^{3} (H_{1,H,v} + H_{1,F,v}) N_{5,v} \right)' \\
    &\quad - (1 - \psi') 0.5 (H_{1,H} + H_{1,F}) N_2 \tilde{M}' \\
    &\quad + 2 r_{DE}^2 \gamma k_s^A
\end{align*}
\]

5 Solution method

The numerical solution proceeds as follows. Conditional on \(k(0)\) we obtain the first-order solution summarized by (45) and (49). The first-order solution also gives us \(H_{1,H}\) from (74), which is based on the first-order component of the Bellman equation. \(H_{1,F}\) follows by symmetry. We use these results to compute a new value for \(k(0)\) from (83). This procedure therefore yields a mapping of \(k(0)\) into itself. This mapping is non-linear. The resulting fixed point problem is solved numerically. At this point we have solved for \(k(0)\) as well as the first-order component of all variables other than \(k_{D,t}\).

Next, we conjecture a solution for the first-order component of \(k_{D,t}\): \(k_{D,t}^P(1) = k_s S_t(1)\). This affects the second-order component of model equations (see the discussion in section 2.1). Conditional on this first order solution of \(k_{D,t}^P\) we then obtain the second-order solution summarized by (59) and (66). The second-order solution is then used to compute \(H_{2,H}\) from (79), which is based on the second-order component of the Bellman equation. \(H_{2,F}\) follows by symmetry. We then use these results to solve for \(k_s\) in (109). This then leads to a fixed point problem in \(k_s\), which is solved numerically. At this point we have solved for the first-order components of both \(k_{D,t}^P\) and \(k_{D,t}^A\), which yields the first-order components of all portfolio shares.
6 Balance of payments accounting

6.1 Definitions

At the end of period \( t \) home agents have a nominal wealth of \( (1 - \psi) W_t P_t \) (measured in terms of the Home good) invested in equities, and Foreign agents have a nominal wealth \( (1 - \psi) W^*_t P^*_t \) invested. The nominal value of the various holdings of equities, as well as the quantity of shares held by each agent, is outlined in the table below:

<table>
<thead>
<tr>
<th></th>
<th>Nominal value</th>
<th>Quantity of shares</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Home agents’ wealth</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>in Home equity</td>
<td>((1 - \psi) W_t P_t)</td>
<td>( G^H_{H,t} = \frac{(1 - \psi) k^H_{H,t} W_t P_t}{Q_{H,t}} )</td>
</tr>
<tr>
<td>in Foreign equity</td>
<td>((1 - \psi) (1 - k^H_{H,t}) W_t P_t)</td>
<td>( G^H_{F,t} = \frac{(1 - \psi)(1 - k^H_{H,t}) W_t P_t}{Q_{F,t}} )</td>
</tr>
<tr>
<td><strong>Foreign agents’ wealth</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>in Home equity</td>
<td>((1 - \psi) W^<em>_t P^</em>_t)</td>
<td>( G^F_{H,t} = \frac{(1 - \psi)(1 - k^F_{H,t}) W^<em>_t P^</em><em>t}{Q</em>{H,t}} )</td>
</tr>
<tr>
<td>in Foreign equity</td>
<td>((1 - \psi) (1 - k^F_{H,t}) W^<em>_t P^</em>_t)</td>
<td>( G^F_{F,t} = \frac{(1 - \psi)(1 - k^F_{F,t}) W^<em>_t P^</em><em>t}{Q</em>{F,t}} )</td>
</tr>
</tbody>
</table>

The home net foreign asset position is the difference between gross foreign assets, \( GA_t \), and gross foreign liabilities, \( GL_t \):

\[
GA_t = (1 - \psi) (1 - k^H_{H,t}) W_t P_t \tag{115}
\]

\[
GL_t = (1 - \psi) (1 - k^F_{F,t}) W^*_t P^*_t \tag{116}
\]

\[
NFA_t = GA_t - GL_t \tag{117}
\]

\[
= (1 - \psi) \left[ (1 - k^H_{H,t}) W_t P_t - (1 - k^F_{F,t}) W^*_t P^*_t \right]
\]

We next turn to trade flows. The value of Home exports during period \( t \) (measured in terms of Home good) is simply the consumption of Home goods by Foreign agents:

\[
X^H_t = (1 - \alpha) (P_t^*)^\lambda \psi W^*_t
\]

Similarly for the value of Foreign exports:

\[
X^F_t = (1 - \alpha) (P_{F,t})^{1 - \lambda} (P_t)^\lambda \psi W_t
\]

so the Home trade balance is

\[
TB_t = X^H_t - X^F_t = (1 - \alpha) \psi \left[ (P_t^*)^\lambda W^*_t - (P_{F,t})^{1 - \lambda} (P_t)^\lambda W_t \right]
\]
Note that it is also the value of output minus consumption:

\[ TB_t = A_{H,t} - \psi P_t W_t = -P_{F,t} A_{F,t} + \psi P^*_t W^*_t \]

which can also be written as:

\[ 2TB_t = (A_{H,t} - P_{F,t} A_{F,t}) - \psi (P_t W_t - P^*_t W^*_t) \]  

(118)

We now look at international factor payments in period \( t \). The quantity of foreign shares owned by home residents at the beginning of the period is \( G^H_{F,t-1} \). Each share receives \((1 - \theta) A_{F,t}\) in terms of Foreign goods, so the payment in terms of Home goods is:

\[ GD^H_t = G^H_{F,t-1} (1 - \theta) P_{F,t} A_{F,t} \]  

(119)

Similarly the dividend payments of the Home country are:

\[ GD^F_t = G^F_{F,t-1} (1 - \theta) A_{H,t} \]  

(120)

The net dividend income of the home country is then:

\[ ND_t = GD^H_t - GD^F_t = (1 - \theta) \left[ G^H_{F,t-1} P_{F,t} A_{F,t} - G^F_{H,t-1} A_{H,t} \right] \]  

(121)

The current account is the sum of the trade balance and net dividend income:

\[ CA_t = TB_t + ND_t \]  

(122)

The capital gain on Home gross foreign assets in period \( t \) is

\[ GK^H_t = G^H_{F,t-1} [Q_{F,t} - Q_{F,t-1}] \]  

(123)

Similarly the capital gain on Home gross foreign liabilities is

\[ GK^F_t = G^F_{H,t-1} [Q_{H,t} - Q_{H,t-1}] \]  

(124)

and the net capital gain is:

\[ NK_t = GK^H_t - GK^F_t = G^H_{F,t-1} [Q_{F,t} - Q_{F,t-1}] - G^F_{H,t-1} [Q_{H,t} - Q_{H,t-1}] \]  

(125)

Recalling that positions are measured at the end of the periods, the Home gross asset position change between period \( t - 1 \) and \( t \) is the sum of gross financial outflows, \( GF^H_t \), and capital gains, \( GK^H_t \):

\[ GA_t - GA_{t-1} = GF^H_t + GK^H_t \]
Similarly for the Home gross liability position:

\[ GL_t - GL_{t-1} = GF^H_t + GK^F_t \]

In net terms we have:

\[
NFA_t - NFA_{t-1} = (GF^H_t - GF^F_t) + (GK^H_t - GK^F_t) = NF_t + NK_t
\]

where \( NF_t \) stands for net financial flows (net capital outflows). In addition, the net financial flows have to match the current account:

\[ NF_t = CA_t \]

This is simply a consequence of the dynamics of the net foreign assets reflecting the trade balance, net dividend income and net capital gains:

\[ NFA_t = NFA_{t-1} + TB_t + ND_t + NK_t \quad (127) \]

We define the passive portfolio share of Home equity as the share when the quantities of assets are held at the zero-order levels, and asset prices take their actual level:

\[
\begin{align*}
{k^H_{H,p}}_{t} & = \frac{Q_{H,t}k(0)}{Q_{H,t}k(0) + Q_{F,t}(1-k(0))} \\
{k^F_{H,p}}_{t} & = \frac{Q_{H,t}(1-k(0))}{Q_{H,t}(1-k(0)) + Q_{F,t}k(0)}
\end{align*}
\]

### 6.2 First-order components

We focus on the first-order components of all balance of payments variables. Some key zero-order components are:

\[
\begin{align*}
GA(0) & = GL(0) = (1 - \psi)(1 - k(0)) \quad W(0) = \frac{1 - \psi}{\psi}(1 - k(0)) \\
GD^H(0) & = GD^F(0) = (1 - k(0))(1 - \theta) \quad , \quad GK^H(0) = GK^F(0) = 0 \\
X^H(0) & = X^F(0) = (1 - \alpha)
\end{align*}
\]
We scale all variables by the zero-order component of GDP, which is 1, so they can all be interpreted as percentage of GDP. The resulting variables are indicated with lower case letters. The asset positions (115)-(117) are:

\[ ga_t (1) = \frac{1 - \psi}{\psi} [(1 - k (0)) (w_t (1) + p_t (1)) - k^H_{H,t} (1)] \]

\[ gl_t (1) = \frac{1 - \psi}{\psi} [(1 - k (0)) (w^*_t (1) + p^*_t (1)) - k^F_{F,t} (1)] \]

\[ nfa_t (1) = \frac{1 - \psi}{\psi} \left[ (1 - k (0)) (w_t (1) - w^*_t (1)) + (1 - k (0)) (p_t (1) - p^*_t (1)) - 2k^A_t (1) \right] \]

where we use \( k^A_t (1) = 0.5 (k^H_{H,t} (1) + k^F_{F,t} (1)) \).

The trade flows and trade balance are:

\[ x^{H}_t (1) = (1 - \alpha) [w^*_t (1) + \lambda p^*_t (1)] \]

\[ x^{F}_t (1) = (1 - \alpha) [w_t (1) + \lambda p_t (1) + (1 - \lambda) p_{F,t} (1)] \]

\[ tb_t (1) = (1 - \alpha) \left[ (1 - k (0)) (w_t (1) - w^*_t (1)) + (1 - k (0)) (p_t (1) - p^*_t (1)) - (1 - \lambda) p_{F,t} (1) \right] \]

From (118) we also have:

\[ tb_t (1) = \frac{1}{2} (a_{H,t} (1) - p_{F,t} (1) - a_{F,t} (1)) \]

\[ -\frac{1}{2} (p_t (1) + w_t (1) - p^*_t (1) - w^*_t (1)) \]

The dividend flows are:

\[ gd^H_{t} (1) = (1 - \theta) [(1 - k (0)) [p_{F,t} (1) + a_{F,t} (1) + w_{t-1} (1) + p_{t-1} (1) - q_{F,t-1} (1)] - k^H_{H,t-1} (1)] \]

\[ gd^F_{t} (1) = (1 - \theta) [(1 - k (0)) [a_{H,t} (1) + w^*_t (1) + p_{t-1} (1) - q_{H,t-1} (1)] - k^F_{F,t-1} (1)] \]

\[ nd_t (1) = (1 - \theta) (1 - k (0)) \left[ \frac{p_{F,t} (1) - (a_{H,t} (1) - a_{F,t} (1))}{(1 - k (0)) + (p_{t-1} (1) - p^*_t (1)) + (q_{H,t-1} (1) - q_{F,t-1} (1))} \right] \]

\[ - (1 - \theta) 2k^A_{t} (1) \]
The current account is

$$ca_t (1) = tb_t (1) + nd_t (1)$$

The capital gains are

$$gk_t^H (1) = \frac{1 - \psi}{\psi} (1 - k (0)) \Delta q_{F,t} (1)$$
$$gk_t^F (1) = \frac{1 - \psi}{\psi} (1 - k (0)) \Delta q_{H,t} (1)$$
$$nk_t (1) = \frac{1 - \psi}{\psi} (1 - k (0)) [\Delta q_{F,t} (1) - \Delta q_{H,t} (1)]$$

where $\Delta x_t = x_t - x_{t-1}$. The gross financial flows (gross outflows and gross inflows) are

$$gf_t^H (1) = \Delta g_{a_t} (1) - gk_t^H (1) = \frac{1 - \psi}{\psi} [(1 - k (0)) [\Delta w_t (1) + \Delta p_t (1) - \Delta q_{F,t} (1)] - \Delta k^H_{H,t} (1)]$$
$$gf_t^F (1) = \Delta g_{l_t} (1) - gk_t^F (1) = \frac{1 - \psi}{\psi} [(1 - k (0)) [\Delta w_t^* (1) + \Delta p_t^* (1) - \Delta q_{H,t} (1)] - \Delta k^F_{F,t} (1)]$$

We can also check that

$$ca_t (1) = gf_t^H (1) - gf_t^F (1)$$

### 6.3 Financial flows and valuation effects

The changes in positions can be decomposed between financial flows and capital gains:

$$\Delta g_{a_t} (1) = gf_t^H (1) + gk_t^H (1)$$
$$\Delta g_{l_t} (1) = gf_t^F (1) + gk_t^F (1)$$
$$\Delta nf_{a_t} (1) = ca_t (1) + nk_t (1)$$

The net valuation effect can be split between real exchange rate movements (that is movements in the relative price of Foreign goods) and equity
price changes. For this purpose we write the Foreign equity price in terms of Foreign goods, denoted $q_{F,t}^*$. This gives

$$nk_t(1) = \frac{1 - \psi}{\psi} (1 - k(0)) \begin{bmatrix} -\Delta q_{H,t} (1) & \Delta q_{F,t}^* (1) & \Delta p_{F,t} (1) \end{bmatrix}$$

The passive portfolio shares are:

$$k_{H,p}^H (1) = k_{H,t}^F (1) = k_t (1) = k(0) (1 - k(0)) [q_{H,t} (1) - q_{F,t} (1)]$$

### 6.4 The drivers of capital flows

The changes in Home gross foreign assets and liabilities are written as:

$$\Delta GA_t = (1 - \psi) \left[ (1 - k_{H,t-1}^H) \Delta (W_t P_t) - W_t P_t \Delta k_{H,t}^H \right]$$
$$\Delta GL_t = (1 - \psi) \left[ (1 - k_{F,t-1}^F) \Delta (W_t^* P_t^*) - W_t^* P_t^* \Delta k_{F,t}^F \right]$$

The changes of invested wealth stem from savings (labor income plus dividend income minus consumption) and capital gains on Home and Foreign equity. Using the dynamics of home wealth we get the relation for the Home country:

$$(1 - \psi) \Delta (W_t P_t) = S_t + (1 - \psi) W_{t-1} P_{t-1} \left[ k_{H,t-1}^H \frac{\Delta Q_{H,t}}{Q_{H,t-1}} + (1 - k_{H,t-1}^H) \frac{\Delta Q_{F,t}}{Q_{F,t-1}} \right]$$

where home savings are:

$$S_t = \theta A_{H,t} - \psi W_t P_t$$
$$+ (1 - \theta) (1 - \psi) W_{t-1} P_{t-1} \left[ k_{H,t-1}^H \frac{A_{H,t}}{Q_{H,t-1}} + (1 - k_{H,t-1}^H) \frac{P_{F,t} A_{F,t}}{Q_{F,t-1}} \right]$$

Using the dynamics of Foreign wealth we get the relation for the Foreign country:

$$(1 - \psi) \Delta (W_t^* P_t^*) = S_t^* + (1 - \psi) W_{t-1}^* P_{t-1}^* \left[ (1 - k_{F,t-1}^F) \frac{\Delta Q_{H,t}}{Q_{H,t-1}} + k_{F,t-1}^F \frac{\Delta Q_{F,t}}{Q_{F,t-1}} \right]$$

where foreign savings are:

$$S_t^* = \theta P_{F,t} A_{F,t} - \psi W_t^* P_t^*$$
$$+ (1 - \theta) (1 - \psi) W_{t-1}^* P_{t-1}^* \left[ (1 - k_{F,t-1}^F) \frac{A_{H,t}}{Q_{H,t-1}} + k_{F,t-1}^F \frac{P_{F,t} A_{F,t}}{Q_{F,t-1}} \right]$$

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We can show that $S_t + S^*_t = 0$.

Gross financial outflows are the change in gross assets minus capital gains on Home agents’ holding of Foreign equity:

$$GF_t^H = \Delta GA_t - GK_t^H$$

$$= (1 - k_{H,t-1}^H) S_t - (1 - \psi) W_t P_t \Delta k_{H,t}^H$$

$$+ k_{H,t-1}^H (1 - k_{H,t-1}^H) (1 - \psi) W_{t-1} P_{t-1} \left[ \frac{\Delta Q_{H,t}}{Q_{H,t-1}} - \frac{\Delta Q_{F,t}}{Q_{F,t-1}} \right]$$

(134)

Similarly for gross financial inflows:

$$GF_t^F = \Delta GL_t - GK_t^F$$

$$= (1 - k_{F,t-1}^F) S^*_t - (1 - \psi) W_t^* P_t^* \Delta k_{F,t}^F$$

$$- k_{F,t-1}^F (1 - k_{F,t-1}^F) (1 - \psi) W_{t-1}^* P_{t-1}^* \left[ \frac{\Delta Q_{H,t}}{Q_{H,t-1}} - \frac{\Delta Q_{F,t}}{Q_{F,t-1}} \right]$$

(135)

The first-order components of (132)-(133) are

$$s_t (1) = \theta a_{H,t} (1) - (w_t (1) + p_t (1))$$

$$+ (1 - \theta) \left[ (w_{t-1} (1) + p_{t-1} (1)) + k (0) (a_{H,t} (1) - q_{H,t-1} (1)) \right]$$

$$s^*_t (1) = \theta (a_{F,t} (1) + p_{F,t} (1)) - (w^*_t (1) + p^*_t (1))$$

$$+ (1 - \theta) \left[ (w^*_{t-1} (1) + p^*_{t-1} (1)) + (1 - k (0)) (a_{H,t} (1) - q_{H,t-1} (1)) \right.$$  

$$+ k (0) (a_{F,t} (1) + p_{F,t} (1) - q_{F,t-1} (1))$$

Note that $s^*_t (1) = -s_t (1)$, hence:

$$s_t (1) = \frac{1}{2} (s_t (1) - s^*_t (1))$$

$$= \frac{1}{2} \theta [a_{H,t} (1) - a_{F,t} (1) - p_{F,t} (1)]$$

$$- \frac{1}{2} (w_t (1) - w^*_t (1) + p_t (1) - p^*_t (1))$$

$$+ \frac{1 - \theta}{2} \left[ w_{t-1} (1) - w^*_{t-1} (1) + p_{t-1} (1) - p^*_t (1) + (2k (0) - 1) [a_{H,t} (1) - a_{F,t} (1) - p_{F,t} (1)] \right.$$  

$$- (2k (0) - 1) (q_{H,t-1} (1) - q_{F,t-1} (1))$$

(136)
Using (131), the first-order components of (134)-(135) are written as:

\[ g_{f_t}^H (1) = (1 - k(0)) s_t (1) - \frac{1 - \psi}{\psi} (\Delta k_{H,t}^H (1) - \Delta k_t^p (1)) \quad (137) \]

\[ g_{f_t}^F (1) = -(1 - k(0)) s_t (1) + \frac{1 - \psi}{\psi} (\Delta k_{H,t}^F (1) - \Delta k_t^p (1)) \quad (138) \]

where we used \( \Delta k_{H,t}^F (1) + \Delta k_{F,t}^F (1) = 0 \). The net portfolio flows are

\[ n_{f_t} (1) = 2 (1 - k(0)) s_t (1) - 2 \frac{1 - \psi}{\psi} (\Delta k_t^A (1) - \Delta k_t^p (1)) \]

we can check that \( n_{f_t} (1) = c a_t (1) \).

### 6.5 Components of external adjustment

The rates of return on Home and Foreign equity reflect a capital gain and a dividend return:

\[ R_{H,t+1} = 1 + \frac{Q_{H,t+1} - Q_{H,t}}{Q_{H,t}} + D_{H,t+1} \]

\[ R_{F,t+1} = 1 + \frac{Q_{F,t+1} - Q_{F,t}}{Q_{F,t}} + D_{F,t+1} \]

where:

\[ D_{H,t+1} = (1 - \theta) \frac{A_{H,t+1}}{Q_{H,t}} \quad D_{F,t+1} = (1 - \theta) \frac{P_{F,t+1} A_{F,t+1}}{Q_{F,t}} \]

First-order components are

\[ d_{H,t+1} (1) = a_{H,t+1} (1) - q_{H,t} (1) \quad d_{F,t+1} (1) = a_{F,t+1} (1) + p_{F,t+1} (1) - q_{F,t} (1) \]

and the return differentials are

\[ d_{H,t+1} (1) - d_{F,t+1} (1) = a_{H,t+1} (1) - a_{F,t+1} (1) - p_{F,t+1} (1) - (q_{H,t} (1) - q_{F,t} (1)) \]

\[ r_{H,t+1} (1) - r_{F,t+1} (1) = \frac{1 - \psi}{1 - \psi \theta} [(q_{H,t+1} (1) - q_{F,t+1} (1)) - (q_{H,t} (1) - q_{F,t} (1))] \]

\[ + \frac{\psi (1 - \theta)}{1 - \psi \theta} (d_{H,t+1} (1) - d_{F,t+1} (1)) \]

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Using (128), (129) and (130) the dynamics of the Home country’s net foreign assets are

\[ \text{nfa}_t(1) = \text{nfa}_{t+1}(1) + \text{tb}_{t+1}(1) + \text{nd}_{t+1}(1) + \text{nk}_{t+1}(1) \]

\[ = \text{nfa}_t(1) + \text{tb}_{t+1}(1) + \psi (1 - \theta) \frac{1}{1 - \psi} \text{nfa}_t(1) \]

\[ - (1 - \theta) \frac{1}{\psi} (r_{H,t+1}(1) - r_{F,t+1}(1)) \]

where we used the fact that

\[ \text{nd}_{t+1}(1) = \frac{\psi (1 - \theta)}{1 - \psi} \text{nfa}_t(1) - (1 - \theta) (1 - k(0)) (d_{H,t+1}(1) - d_{F,t+1}(1)) \]

We rewrite the dynamics of the net foreign asset position as

\[ \text{nfa}_{t+1}(1) = \text{tb}_{t+1}(1) + R(0) \cdot \text{nfa}_t(1) \]

\[ - GA(0) \cdot R(0) (r_{H,t+1}(1) - r_{F,t+1}(1)) \]

where \( GA(0) = \frac{1 - \psi}{\psi} (1 - k(0)) \) is the zero-order component of the gross asset position. Iterating forward we get

\[ \text{nfa}_t(1) = - \sum_{s=1}^{\infty} \left( \frac{1}{R(0)} \right)^s \text{tb}_{t+s}(1) + GA(0) \sum_{s=1}^{\infty} \left( \frac{1}{R(0)} \right)^{s-1} (r_{H,t+s}(1) - r_{F,t+s}(1)) \]

(139) shows that a net debt \( \text{nfa}_t(1) < 0 \) has to be offset by future trade surpluses \( \text{tb}_{t+s}(1) > 0 \) or a higher return on Foreign equity than on Home equity, that is a higher return on home assets than liabilities \( r_{H,t+s}(1) - r_{F,t+s}(1) < 0 \).

Future returns can in turn be split between capital gains and dividend yields:

\[ \text{nfa}_t(1) = - \sum_{s=1}^{\infty} \left( \frac{1}{R(0)} \right)^s \text{tb}_{t+s}(1) - \sum_{s=1}^{\infty} \left( \frac{1}{R(0)} \right)^s [\text{ndy}_{t+s}(1) + \text{nk}_{t+s}(1)] \]

where \( \text{nk}_{t+s} \) is given by (130) and:

\[ \text{ndy}_{t+1}(1) = - (1 - \theta) (1 - k(0)) (d_{H,t+1}(1) - d_{F,t+1}(1)) \]