This is a closed book, closed notes exam, to be taken in a single 75 minute period. There are two problems which will be weighted equally. For full credit, be sure to show and explain all your work. You are welcome to use additional pages for your work; attach any that you wish to be graded.

Name: __________________________

Signature: ________________________
1. An equilibrium gas of photons confined to volume $V$ at temperature $T$ has energy

$$U = aVT^4,$$

where $a$ can be related to fundamental constants. Note that the number of photons $N$ is not conserved in this system, so the chemical potential $\mu$ can be taken as zero and $N$ is not a useful thermodynamic variable.

(a) Using the above and the laws of thermodynamics, derive the entropy of the gas to be

$$S = \frac{4}{3}aVT^3 + \text{const}.\]

(b) Derive the equation of state $P = P(V, T)$, where $P$ is the pressure of the gas.

(c) The universe contains a gas of photons. Cosmological evidence suggests that when the universe was $t_1 = 400$ kyears old the photon gas had a temperature of about 2700 K, and that the universe is now $t_2 = 13.7$ Gyears old with a temperature of 2.7 K. Given the speed at which photons travel, it is plausible to assume that the expansion between these times was reversible. If we take the universe to be spherical with radius $r$, by what factor $r_2/r_1$ has its size increased between these two times? Neglect any coupling between the photons and other particles or fields in the universe.
2. Suppose that the Gibbs free energy of a magnetic material can be expressed

\[
\frac{G}{V} = \alpha T \left(1 - \log \frac{T}{\tau}\right) - \mathcal{H} \mathcal{M} + (aT - b)\mathcal{M}^2 + \beta \mathcal{M}^4
\]

where \( V \) is the (fixed) volume of the system, \( T \) is the temperature, \( \mathcal{H} \) is the magnetic field, and \( \mathcal{M} \) is the magnetization. The other parameters \( \alpha, \beta, \tau, a \) and \( b \) are positive constants. For simplicity, treat \( \mathcal{H} \) and \( \mathcal{M} \) as scalars here. Note that this material is not an ideal Curie magnet of the type we have considered previously.

(a) Explain why \( \frac{\partial G}{\partial \mathcal{M}} |_{\mathcal{H}, T} = 0 \).

(b) Suppose first that \( \mathcal{H} \) and \( \mathcal{M} \) are very small. Determine the equation of state \( \mathcal{M} = \mathcal{M}(\mathcal{H}, T) \). Is your result consistent with the assumption of small \( \mathcal{M} \)?

(c) Assume now that \( \mathcal{H} = 0 \), but allow \( \mathcal{M} \) to be large. Calculate \( \mathcal{M}(T) \); if you find multiple solutions, determine which one(s) are correct.

(d) Calculate and sketch the heat capacity \( C_{\mathcal{H}}(T) \) of the material for \( \mathcal{H} = 0 \).

(e) Does this system exhibit a phase transition? If so, is it first or second order?