3.1 Auslander–Reiten Quivers of Type $A_n$

In this section, let $Q$ be a quiver of type $A_n$, that is, the underlying unoriented graph of $Q$ is the Dynkin diagram of type $A_n$:

$$1 \rightarrow 2 \rightarrow 3 \rightarrow \ldots \rightarrow (n-1) \rightarrow n.$$  

We will see several ways to construct the Auslander–Reiten quiver of $Q$.

3.1.1 The Knitting Algorithm

The knitting algorithm owes its name to the fact that it recursively constructs one mesh after the other, from left to right. In order to get started one has to compute the indecomposable projective representations which are the leftmost indecomposable representations in the Auslander–Reiten quiver.

1. Compute the indecomposable projective representations $P(1), P(2), \ldots, P(n)$.

2. Draw an arrow $P(i) \rightarrow P(j)$ whenever there exists an arrow $j \rightarrow i$ in $Q_1$, in such a way that each $P(i)$ sits at a different level.

3. (Knitting) There are three types of meshes. Complete each mesh as shown in Fig. 3.1 in such a way that

$$\text{dim } L + \text{dim } \tau^{-1}L = \sum_{i=1}^{2} \text{dim } M_i.$$  

4. Repeat step 3 until you get negative integers in the dimension vector.

Observe that, every time we perform the third step, the representations $L$ and $M_i$ have been computed earlier and only $\tau^{-1}L$ is unknown.

The isoclasses of indecomposable representations of quivers of type $A_n$ are determined by their dimension vectors as follows. The dimension vector is always of the form $(0, \ldots, 0, 1, \ldots, 1, 0, \ldots, 0)$, and the corresponding representation is $M = (M_1, \varphi_0)$ with $M_1 = k$ if the dimension at $i$ is one, and $M_i = 0$ otherwise; and $\varphi_0 = 1$ if the dimension at $s(\alpha)$ and at $t(\alpha)$ is one, and $\varphi_0 = 0$ otherwise.

Example 3.1. Let $Q$ be the quiver

$$1 \leftarrow 2 \leftarrow 3 \rightarrow 4 \leftarrow 5.$$  

Then

$$P(1) = 1 \quad P(2) = 2 \quad P(3) = 2 \quad P(4) = 4 \quad P(5) = 5$$

$$P(3) = 2 \quad P(4) = 4 \quad P(5) = 5$$
and the Auslander-Reiten quiver is

![Diagram of the Auslander-Reiten quiver](image)

### 3.1.2 $\tau$-Orbits

The map $\tau$ is the Auslander-Reiten translation. In the Auslander-Reiten quiver, it is the translation that sends the rightmost point of a mesh to the lefmost point of the same mesh. The $\tau$-orbit of an indecomposable representation is the set of all representations that can be obtained by applying $\tau$ or $\tau^{-1}$ repeatedly to the representation. Thus the $\tau$-orbits in the Auslander-Reiten quivers of type $A_n$ consist of the representations that sit on the same level in the quiver.

Each $\tau$-orbit in the Auslander-Reiten quiver of type $A_n$ contains exactly one projective representation, so starting from the projectives, we can compute the whole quiver by computing the $\tau$-orbits.

There are several methods to compute $\tau$-orbits.

### 3.1.2.1 First Method: Auslander-Reiten Translation

Let $M$ be an indecomposable representation that is not injective. We want to compute the translation to the right $\tau^{-1}M$ of $M$. Start with an injective resolution

$$0 \rightarrow M \rightarrow I_0 \rightarrow I_1 \rightarrow 0,$$

and apply the inverse Nakayama functor $\nu^{-1}$. This functor maps the indecomposable injective representation $I(j)$ to the corresponding indecomposable projective representation $P(j)$; see Proposition 2.29 of Chap. 2. Then $\tau^{-1}M$ is given by the projective resolution:

$$0 \rightarrow \nu^{-1}I_0 \rightarrow \nu^{-1}I_1 \rightarrow \tau^{-1}M \rightarrow 0.$$

Let us compute $\tau^{-1}M$ for the module $M = 4$ in Example 3.1. The upper line in the following diagram shows an injective resolution of $M$, and the lower line shows the corresponding projective resolution of $\tau^{-1}M$ obtained by applying $\nu^{-1}$:

$$0 \rightarrow 4 \rightarrow 35 \rightarrow 3 \oplus 5 \rightarrow 0$$

Thus $\tau^{-1}M = 24$ which verifies the result of Example 3.1.

### 3.1.2.2 Second Method: Coxeter Functor

Choose a sequence of vertices $(i_1, i_2, \ldots, i_n)$, with $i_j \neq i_\ell$ if $i \neq \ell$, as follows:

- $i_1$ is a sink of $Q$;
- $i_2$ is a sink of the quiver $s_{i_1}Q$ obtained from $Q$ by reversing all arrows that are incident to the vertex $i_1$;
- $i_t$ is a sink of $s_{i_{t-1}} \cdots s_{i_2}s_{i_1}Q$, for $t = 2, 3, \ldots, n$.

Thus in Example 3.1 such a sequence would be $(1, 4, 2, 3, 5)$.

Next, we need the notion of reflections $s_i : \mathbb{R}^n \rightarrow \mathbb{R}^n$ defined by $s_i(x) = x - 2B(x, e_i)e_i$, where $\{e_1, \ldots, e_n\}$ is a basis of $\mathbb{R}^n$ and $B$ is a symmetric bilinear form defined by

$$B(e_i, e_j) = \begin{cases} 1 & \text{if } i = j \\ -1/2 & \text{if } i \text{ is adjacent to } j \text{ in } Q \\ 0 & \text{otherwise}. \end{cases}$$

In other words, $s_i(\sum j a_j e_j) = \sum j a'_j e_j$, where $a'_j = a_j$ if $j \neq i$ and $a'_j = -a_i + \sum_{j \neq i} a_j$, where the sum is over all vertices $j$ that are adjacent to $i$ in $Q$.

Finally, we define a so-called Coxeter element $c = s_{i_1}s_{i_2}\ldots s_{i_n}$ as a product of reflections using the sequence of vertices defined above. Thus in Example 3.1 such a Coxeter element would be $c = s_1s_3s_5s_3s_5s_3s_5$. 
3.1 Auslander–Reiten Quivers of Type \( \Lambda_n \)

Now we define yet another matrix, the **Coxeter matrix** \( \Phi \), as \( \Phi = -C^T (C^{-1}) \), and its inverse is \( \Phi^{-1} = -C(C^{-1})^T \), the superscript \( T \) here denotes the transpose of a matrix. Then

\[
\Phi \text{ dim } M = \text{ dim } \tau M, \text{ if } M \text{ is not projective and } \Phi \text{ dim } P(j) = -\text{dim } I(j),
\]

whereas

\[
\Phi^{-1} \text{ dim } M = \text{ dim } \tau^{-1} M, \text{ if } M \text{ is not injective and } \Phi^{-1} \text{ dim } I(j) = -\text{dim } P(j).
\]

In our Example 3.1, we have

\[
C = \begin{bmatrix}
1 & 1 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix},
(C^{-1}) = \begin{bmatrix}
1 & -1 & 0 & 0 & 0 \\
0 & 1 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & -1 & 1 & -1 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

\[
\Phi = \begin{bmatrix}
-1 & 1 & 0 & 0 & 0 \\
-1 & 0 & 1 & 0 & 0 \\
-1 & 0 & 1 & -1 & 1 \\
0 & 0 & 1 & -1 & 1 \\
0 & 0 & 1 & -1 & 0
\end{bmatrix}, \Phi^{-1} = \begin{bmatrix}
0 & 0 & -1 & 1 & 0 \\
1 & -1 & 0 & 0 & 1 & 0 \\
1 & -1 & 1 & 0 & 1 & -1 \\
1 & -1 & 1 & -1 & 1 & 0 \\
1 & -1 & 1 & -1 & -1 & 1
\end{bmatrix}
\]

so that the dimension of \( \tau^{-1} \) can be computed by \( \Phi^{-1} (0, 0, 0, 1, 0)^T \) which is equal to \((1, 1, 1, 1, 1)^T \). On the other hand, \( \Phi \text{ dim } P(4) = \Phi (0, 0, 0, 1, 0)^T = (0, 0, -1, -1, -1)^T = -\text{dim } I(4) \).

3.1.3 **Diagonals of a Polygon with \( n + 3 \) Vertices**

In this section, we give a geometric way to construct the Auslander–Reiten quiver of a quiver \( Q \) of type \( \Lambda_n \) from a triangulation of a polygon. This method works only for quivers of type \( \Lambda_n \).

Start with a regular polygon with \( n + 3 \) vertices. A **diagonal** in the polygon is a straight line segment that joins two of the vertices and goes through the interior of the polygon, and a **triangulation** of the polygon is a maximal set of non-crossing diagonals. Such a triangulation cuts the polygon into triangles, hence the name. Given a triangle with sides \( a, b, c \), we say that the side \( a \) is clockwise of the side \( b \) if going along the boundary of the triangle in the clockwise direction corresponds to the sequence \( a, b, c, a, b, c \ldots \).
3.1 Auslander–Reiten Quivers of Type $A_n$

We will associate a triangulation $T_Q$ to our type $A_n$ quiver $Q$ as follows: Let 1 be a vertex in the quiver that has only one neighbor. Draw a diagonal that cuts off a triangle $\Delta_0$ and label that diagonal 1. If $1 \leftarrow 2$ is an arrow in $Q$, then draw the unique diagonal 2 such that 1, 2 and one boundary segment of the polygon form a triangle $\Delta_1$ in such a way that diagonal 2 is clockwise of diagonal 1 in the triangle $\Delta_1$. If, on the other hand, $1 \rightarrow 2$ is an arrow in $Q$, draw the unique diagonal 2 such that diagonal 2 is counterclockwise of diagonal 1 in the triangle $\Delta_1$; see Fig. 3.2. Continue this procedure up to diagonal $n$.

In this way the quiver

$$1 \leftarrow 2 \leftarrow 3 \rightarrow 4 \leftarrow 5$$

of Example 3.1 gives rise to the triangulation

Since $T_Q$ is a triangulation of the polygon, any other diagonal $\gamma$ which is not already in $T_Q$ will cut through a certain number of diagonals in $T_Q$; in fact, any such diagonal $\gamma$ is uniquely determined by the set of diagonals in $T_Q$ that $\gamma$ crosses. To such a diagonal $\gamma$, we associate a representation $M_\gamma = (M_i, \varphi_\gamma)$ of $Q$ by letting

$$M_i = \begin{cases} k & \text{if the diagonal } \gamma \text{ crosses the diagonal } i; \\ 0 & \text{otherwise}; \end{cases}$$

and setting $\varphi_\gamma = 1$ whenever $M_{i(a)} = M_{i(a)} = k$, and $\varphi_\gamma = 0$ otherwise. In the example, the diagonal $\gamma$ crosses the diagonals 1, 2, and 3, and the corresponding representation is

$$k \leftarrow 1 \leftarrow k \leftarrow 1 \leftarrow 0 \rightarrow 0 \leftarrow 0 \leftarrow 0.$$

The map $\gamma \mapsto M_\gamma$ is a bijection from the set of diagonals that are not in $T_Q$ and the set of isoclasses of indecomposable representations of $Q$.

The Auslander–Reiten translation $\tau$ is given by an elementary clockwise rotation of the polygon, so in our example $\tau$ of $\gamma$ is the diagonal that cuts through the diagonals 4 and 5.

The projective representation $P(i)$ is given by $\tau^{-1}$ of the diagonal $i$, and the injective representation $I(i)$ is given by $\tau$ of the diagonal $i$. In our example $P(1)$ is the diagonal that cuts through the diagonal 1 only and $I(1)$ is the diagonal $\gamma$.

The complete Auslander–Reiten quiver can be easily constructed now starting with the projectives and applying the elementary rotation to compute the $\tau$-orbits until we reach the injective in each $\tau$-orbit, and the Auslander–Reiten quiver is

![Diagram](image)

Note that any arrow in the Auslander–Reiten quiver acts on the diagonal by pivoting one of the endpoints of the diagonal to its counterclockwise neighbor.
1.4 Computing Hom Dimensions, Ext Dimensions, and Short Exact Sequences

Given two indecomposable representations $M$ and $N$, we want to have information about the space of morphisms $\text{Hom}(M, N)$. The Auslander–Reiten quiver allows us to compute the dimension of this space easily, at least if $M$ and $N$ lie in the same connected component.

1.4.1 Dimension of $\text{Hom}(M, N)$

Let $Q$ be a type A quiver and let $M$, $N$ be two indecomposable representations of $Q$. We can compute the dimension of the vector space $\text{Hom}(M, N)$ using the relative position of $M$ and $N$ in the Auslander–Reiten quiver. For this we need to introduce some terminology:

A path $M_0 \to M_1 \to \cdots \to M_s$ in the Auslander–Reiten quiver is called a sectional path if $\tau M_{i+1} \neq M_{i-1}$ for all $i = 1, \ldots, s - 1$. Let $\Sigma_+(M)$ be the set of all indecomposable representations that can be reached from $M$ by a sectional path, and let $\Sigma_-(M)$ be the set of all indecomposable representations from which one can reach $M$ by a sectional path.

Now let $\mathcal{R}_+(M)$ be the set of all indecomposable representations whose position in the Auslander–Reiten quiver is in the slanted rectangular region whose left boundary is $\Sigma_-(M)$. We call $\mathcal{R}_+(M)$ the maximal slanted rectangle in the Auslander–Reiten quiver whose leftmost point is $M$. Then $\dim \text{Hom}(M, N)$ is either 1 or 0, and it is 1 if and only if $N$ lies in $\mathcal{R}_+(M)$.

3.1 Auslander–Reiten Quivers of Type $A_n$

![Fig. 3.3](image)

Fig. 3.3 Dimension of $\text{Hom}(M, -)$ for $M = P(4)$ on the left and $M = S(2)$ on the right. The position of the representation $M$ is at the leftmost 1 in each case; the numbers 0, 1 indicate the dimension of $\text{Hom}(M, N)$ for each indecomposable representation $N$.

We illustrate this concept in Fig. 3.3 for the Auslander–Reiten quiver of Example 3.1. On the left side of Fig. 3.3, the module $M$ is the indecomposable projective $P(4)$. Its position in the Auslander–Reiten quiver is the leftmost 1 in the figure, so this 1 indicates that $\dim \text{Hom}(M, M) = 1$. A basis for this vector space is the identity morphism $1_M$. Each indecomposable representation $N$ is located at a specific point in the Auslander–Reiten quiver; the number 0 or 1 at that point indicates the dimension of $\text{Hom}(M, N)$ for each $N$.

In the Auslander–Reiten quiver on the right-hand side of Fig. 3.3, the module $M$ is the simple module $S(2)$. Again its position is the leftmost 1 in that figure. The rectangle on which $\text{Hom}(M, -)$ is nonzero reduces in this case to a single line.

Symmetrically, we denote by $\mathcal{R}_-(N)$ the maximal slanted rectangle in the Auslander–Reiten quiver whose rightmost point is $N$. We can compute the dimension of $\text{Hom}(-, N)$ using $\mathcal{R}_-(N)$. Thus the data in the left picture in Fig. 3.3 also computes the $\dim \text{Hom}(-, N)$ for $N = I(4)$.

Note that if $M = P(i)$ is an indecomposable projective, then it follows from Theorem 2.11 that the representations in $\mathcal{R}_-(P(i))$ are precisely the indecomposable representations $N$ such that $N_i \neq 0$. It then follows from Exercise 2.7 of Chap. 2 that there is a unique rightmost point in $\mathcal{R}_-(P(i))$ which must be the position of the indecomposable injective representation $I(i)$. In particular, $\mathcal{R}_-(P(i)) = \mathcal{R}_+(I(i))$.

Figure 3.4 shows an example where the right end of the $\mathcal{R}_+(M)$ does not really have the shape of a rectangle, because the Auslander–Reiten quiver ends before the rectangle is completed. This happens exactly when $M$ is not projective.

3.1.4.2 Dimension of $\text{Ext}^1(M, N)$

Next we compute the dimensions of the vector spaces $\text{Ext}^1(M, N)$ for indecomposable representations $M, N$ of type $A$. If $M$ is projective, then this space is zero, by Exercise 2.11 of Chap. 2, so let us assume that $M$ is not projective. Thus $\tau M$ is a
3 Examples of Auslander–Reiten Quivers

Fig. 3.4 Dimension of $\text{Hom}(M, -)$ where $M$ is the representation whose dimension vector is $(0, 1, 1, 1, 1)$

Fig. 3.5 Dimension of $\text{Ext}^1(M, -)$ for $M = I(3)$

3.1 Auslander–Reiten Quivers of Type $A_n$

isomorphism, there is exactly one other possibility for $E$. For representations of type $A$, we can compute $E$ simply from the relative positions of $M$ and $N$ in the Auslander–Reiten quiver:

Fig. 3.6 Computing short exact sequences

Let $M, N$ be indecomposable representations of a quiver of type $A$ such that $\text{Ext}^1(M, N) \neq 0$. Then $N$ must lie in $\mathcal{R}_-(\tau M)$ and this implies that $\Sigma_+(N)$ and $\Sigma_-(M)$ have either 1 or 2 points in common, and these points correspond to the indecomposable summands of $E$.

We illustrate this situation in Fig. 3.6; the representation $N$ is marked by $\Theta$ and the representation $M$ by $\Theta$. The representations in $\Sigma_+(N)$ are marked by $-$ or $\Theta$ (for $N$) and those in $\Sigma_-(M)$ by $+$ or $\Theta$ (for $M$). The points of intersection are marked $\pm$. The example of the left-hand side of Fig. 3.6 corresponds to the short exact sequence:

$$0 \rightarrow 5 \rightarrow 3 \rightarrow 0$$

and the example on the right-hand side corresponds to the short exact sequence:

$$0 \rightarrow 3 \rightarrow 2 \rightarrow 3 \rightarrow 0.$$
3.3 Auslander-Reiten Quivers of Type $D_n$

In this section, let $Q$ be a quiver of type $D_n$, that is, the underlying unoriented graph $Q$ is the Dynkin diagram of type $D_n$.

We will use the different techniques from Sect. 3.1 to construct the Auslander-Reiten quiver of $Q$.

3.1 The Knitting Algorithm

We can use this algorithm in almost the same way as for type $A_n$, with the difference that now, there is a fourth type of mesh:

$$
\begin{array}{c}
M_1 \\
L \rightarrow M_2 \\
M_3
\end{array}
\begin{array}{c}
M_1 \\
L \rightarrow M_2 \rightarrow \tau^{-1}L \\
M_3
\end{array}
\begin{array}{c}
M_2 \\
M_3
\end{array}
$$

The isoclasses of indecomposable representations of quivers of type $D_n$ are determined by their dimension vectors $d = (d_1, \ldots, d_n)$ as follows. The entries of the dimension vector are either 0, 1 or 2, and if we have $d_i = 2$, then $i$ is one of the vertices $2, 3, \ldots, n - 2$.

For all vertices $j$ with $i \leq j \leq n - 2$ we have $d_j = 2$.

$d_{i-1} \geq 1$ and $d_{n-1} = d_n = 1$.

Thus the vertices $i$ with $d_i = 2$ form a subgraph of type $A$ that contains the vertex $n - 2$.

The vertices $i \neq n - 1, n$ with $d_i = 1$ also form a subgraph of type $A$, and if $i \neq 2$ for all $j$, then all the vertices $i$ with $d_i = 1$ form a subgraph of type $A$ or a subgraph of type $D$.

Graphically, we can represent some of these configurations as follows:

$$
\begin{array}{c}
1 \\
0 - \cdots - 0 - 1 - \cdots - 2 \\
1
\end{array}
\begin{array}{c}
4
\end{array}
\begin{array}{c}
3
\end{array}
\begin{array}{c}
1
\end{array}
\begin{array}{c}
0 - \cdots - 0 - 1 - \cdots - 1 - \cdots - 1
\end{array}
\begin{array}{c}
5
\end{array}
$$

The corresponding representation is $M = (M_i, \varphi_i)$ with $M_i = k^{d_i}$; and $\varphi_i = 1$ if $d_{\alpha(i)} = d_{\alpha(i+1)}$, $\varphi_i = 0$ if one of $d_{\alpha(i)}$, $d_{\alpha(i+1)}$ is zero. If one of the $d_i$ is 2, then there are exactly three arrows that connect a vertex with dimension 1 to a vertex with dimension 2; two of these arrows, let us call them $\beta_1$, $\beta_2$, connect the vertex $n - 2$ with the vertices $n - 1$ and $n$, the vector space of dimension two being at $n - 2$, while the third arrow $\alpha_i$ connects two vertices $i$ and $i + 1$, the vector space of dimension two being at vertex $i + 1$. Consider the one-dimensional subspace of $M_{i+1}$ given by

\[\{ \text{im } \varphi_{i+1} \text{ if } \alpha_i \text{ points to } i + 1, \]
\[\text{ker } \varphi_{i+1} \text{ otherwise.} \]

Under the composition of the identity maps $\varphi_{n-3} \cdots \varphi_{i+1}$, this one-dimensional subspace is sent to a one-dimensional subspace $\ell_1$ of $M_{n-2}$. Consider also the following two one-dimensional subspaces $\ell_2$ and $\ell_3$ of $M_{n-2}$:

\[\ell_2 = \{ \text{im } \varphi_{i+1} \text{ if } \beta_1 \text{ points to } n - 2, \]
\[\text{ker } \varphi_{i+1} \text{ otherwise;} \]

and

\[\ell_3 = \{ \text{im } \varphi_{i+1} \text{ if } \beta_2 \text{ points to } n - 2, \]
\[\text{ker } \varphi_{i+1} \text{ otherwise.} \]

Then the condition on the three maps $\varphi_{n-3}, \varphi_{i+1}$ and $\varphi_{i+1}$ is that the three one-dimensional subspaces are pairwise distinct. This corresponds to the "generic" situation as opposed to the special case where two (or more) of these subspaces are equal.

Example 3.2. Let $Q$ be the quiver

Then

\[P(1) = \frac{1}{2}, \quad P(2) = 2, \quad P(3) = \frac{3}{2}, \quad P(4) = \frac{4}{2}, \quad P(5) = 5.\]
3.3 Auslander–Reiten Quivers of Type $D_n$

3.3.2.2 Second Method: Coxeter Functor

As in Sect. 3.1.2.2, we define a sequence of vertices $(i_1, i_2, \ldots, i_n)$, with $i_j \neq i_k$, if $j \neq k$, as follows.

- $i_1$ is a sink of $Q$.
- $i_2$ is a sink of the quiver $s_{i_1} Q$ obtained from $Q$ by reversing all arrows that are incident to the vertex $i_1$.
- $i_k$ is a sink of $s_{i_{k-1}} \cdots s_{i_2} s_{i_1} Q$, for $k = 2, 3, \ldots, n$.

Then we define the Coxeter element $c = s_{i_1} s_{i_2} \cdots s_{i_n}$ as a product of reflections using this sequence of vertices.

Thus in Example 3.2, we can take the sequence $(2, 5, 1, 3, 4)$, and its Coxeter element is $c = s_2 s_5 s_1 s_3 s_4$.

Let us use this Coxeter element to compute the dimension vector of $\tau^{-1} M$. In Example 3.2. We have $\dim M = (1, 1, 1, 0, 1)$. Thus $\dim \tau^{-1} M$ is equal to

$$s_2 s_5 s_3 s_4 (e_1 + e_2 + e_3 + e_4 + e_5) = s_2 s_5 s_1 s_3 (e_1 + e_2 + 2e_3 + e_4 + e_5) = s_2 s_5 (e_2 + 2e_3 + e_4 + e_5) = e_2 + e_3 + e_4 + e_5$$

which again confirms the result obtained in Example 3.2.

As in type $A$, we can also use the Cartan matrix $C$ and the Coxeter matrix $\Phi = -C^t C^{-1}$ in order to compute the action of the Coxeter element. In our example, we have

$$C = \begin{bmatrix}
1 & 0 & 0 & 0

1 & 1 & 1 & 1

0 & 0 & 1 & 1

0 & 0 & 0 & 1

0 & 0 & 1 & 1
\end{bmatrix},

(C^{-1}) = \begin{bmatrix}
1 & 0 & 0 & 0

-1 & -1 & 0 & 0

0 & 0 & 1 & -1

0 & 0 & 0 & 1

0 & 0 & 1 & 0
\end{bmatrix}$$

$$\Phi = \begin{bmatrix}
0 & -1 & 1 & 0 & 0

1 & -1 & 0 & 0

1 & -1 & 1 & 1 & -1

1 & -1 & 1 & 0 & -1

0 & 0 & 1 & 0 & -1
\end{bmatrix},

\Phi^{-1} = \begin{bmatrix}
-1 & 1 & 0 & 0

-1 & 1 & 0 & 0

0 & 0 & 1 & -1

0 & 0 & 1 & -1

0 & 0 & 1 & 0
\end{bmatrix}.$$

Thus for the representation $M$ above, we can compute the dimension vector of $\tau^{-1} M$ as $\Phi^{-1} \dim M = \Phi^{-1}(1, 1, 1, 0, 1)^t = (0, 1, 2, 1, 1)^t$.

On the other hand, $\tau M$ has dimension vector $\Phi(1, 1, 1, 0, 0)^t = (0, 1, 0, 0, 0)^t$. 

3.2 $\tau$-Orbits

As in type $A$, there are several ways to compute the $\tau$-orbits.

3.2.1 First Method: Auslander–Reiten Translation

Let us compute $\tau^{-1} M$ for the module $M = \frac{1}{2_5} 3_2$ in Example 3.2. The upper line in the following diagram shows an injective resolution of $M$, and the lower line shows the projective resolution of $\tau^{-1} M$ obtained by applying $\nu^{-1}$.

$$0 \longrightarrow \frac{1}{2_5} 3_2 \longrightarrow \frac{1}{3_2} 3_5 \longrightarrow \frac{4}{3} 4_4 \longrightarrow \frac{4}{3} 4_5 \longrightarrow 0$$

Thus $\tau^{-1} M = \frac{4}{2_5}$ which verifies the result of Example 3.2.
3.3.3 Arcs of a Punctured Polygon with n Vertices

In this section, we give a geometric construction of the Auslander–Reiten quiver of a quiver $Q$ of type $D_n$, similar to the construction in Sect. 3.1.3. Instead of a triangulated polygon, we work with a punctured punctured polygon. The diagonals in the polygon must be replaced by certain curves that are called arcs in the punctured polygon. If the boundary of the polygon has $n$ vertices, then we have exactly $n^2$ arcs given as follows:

For every vertex $a$ on the boundary of the polygon, we have the $n - 2$ arcs shown in the left picture of Fig. 3.8, and for the puncture, we have the $n$ arcs shown in the middle and the $n$ arcs shown in the right picture of Fig. 3.8. Note that for each boundary vertex $a$, there are two arcs from $a$ to the puncture, and we use a little tag on the arc to distinguish them. The arcs at the puncture that have a tag are called notched and the ones without a tag are called plain.

Also note that, given two boundary vertices $a \neq b$, there is exactly one arc connecting $a$ and $b$ if $a$ and $b$ are neighbors on the boundary and exactly two arcs if $a$ and $b$ are not neighbors, see Fig. 3.9.

Contrary to the case of the diagonals in the polygon, it is not so straightforward to say when two arcs $\gamma$ and $\gamma'$ in the punctured polygon cross.

We denote the number of crossings by $e(\gamma, \gamma')$. If one of the two arcs has both endpoints on the boundary of the polygon, the number of crossing should be intuitively clear, and we show several examples in Fig. 3.10. Note that in this case $e(\gamma, \gamma')$ can be 0, 1, or 2. For a rigorous definition of crossing numbers we would need the notion of homotopy, which would take us too far away from the subject of this book.

If both arcs $\gamma$ and $\gamma'$ are incident to the puncture and $a$ and $a'$ denote their respective endpoints on the boundary, we define

$$e(\gamma, \gamma') =
\begin{cases}
0 & \text{if } \gamma \text{ and } \gamma' \text{ are both plain}, \\
0 & \text{if } \gamma \text{ and } \gamma' \text{ are both notched}, \\
0 & \text{if } a = a', \\
1 & \text{if } \gamma, \gamma' \text{ have opposite tagging and } a \neq a'.
\end{cases}$$

We say that two arcs cross if their crossing number is at least 1, and a triangulation is a maximal set of non-crossing diagonals. A triangulation does not necessarily cut the polygon into triangles, even if one allows triangles to have curved edges. Some triangulations are shown in Fig. 3.11.

Now let $Q$ be a quiver of Dynkin type $D_n$. We associate a triangulation $T_{Q}$ to $Q$ as follows: Start with an arc $\gamma_1$ that cuts off a triangle $\Delta_0$. If 1 in $Q$, then let $\gamma_2$ be the unique arc that forms a triangle $\Delta_1$ together with $\gamma_1$ and a boundary segment in such a way that $\gamma_1$ is counterclockwise from $\gamma_2$ in $\Delta_1$. If on the other
3.3 Auslander–Reiten Quivers of Type $D_n$

of Example 3.2 gives rise to the triangulation

Since $T_Q$ is a triangulation of the punctured polygon, any arc $\gamma$ which is not already in $T_Q$ will cut through a certain number of diagonals in $T_Q$; in fact, any such arc $\gamma$ is uniquely determined by the set of diagonals in $T_Q$ that $\gamma$ crosses. To such a diagonal $\gamma$, we associate the indecomposable representation $M_\gamma = (M_\gamma, \varphi_\gamma)$ of $Q$ whose dimension at vertex $i$ is given by the number of crossings $e(\gamma, \gamma_i)$ between the arc $\gamma$ and the arc $\gamma_i$ of the triangulation that corresponds to the vertex $i$ of the quiver. In the example, the arc $\gamma$ crosses the arcs 1, 4, 5 once and 2, 3 twice, and the corresponding representation is isomorphic to

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

The map $\gamma \mapsto M_\gamma$ is a bijection from the set of arcs that are not in $T_Q$ and the set of isoclasses of indecomposable representations of $Q$.

The Auslander–Reiten translation $\tau$ is given by an elementary clockwise rotation of the punctured polygon with simultaneous change of the tags at the puncture. So in our example

$$\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$
3.3 Auslander–Reiten Quivers of Type $D_n$

3.3.4 Computing Hom Dimensions, Ext Dimensions, and Short Exact Sequences

As in type $A$, we can compute the dimensions of the Hom and Ext spaces easily from the Auslander–Reiten quiver in type $D$.

3.3.4.1 Dimension of $\text{Hom}(M, N)$

Let $Q$ be a type $D$ quiver and let $M, N$ be two indecomposable representations of $Q$. We can compute the dimension of the vector space $\text{Hom}(M, N)$ using the relative position of $M$ and $N$ in the Auslander–Reiten quiver. The maximal slanted rectangles of type $\sigma^t$ have to be replaced by maximal hammocks. It is a little harder to describe these hammocks than the rectangles. Several examples are illustrated in Fig. 3.14.

Recall that a path $M_0 \to M_1 \to \cdots \to M_s$ in the Auslander–Reiten quiver is called a sectional path if $\tau M_{i+1} \neq M_{i-1}$ for all $i = 1, \ldots, s - 1$. As in type $A$, we define $\Sigma_{\to}(M)$ to be the set of all indecomposable representations that can be reached from $M$ by a sectional path and $\Sigma_{\leftarrow}(M)$ to be the set of all indecomposable representations from which one can reach $M$ by a sectional path.

We can now construct the hammock by the following algorithm, refer to Fig. 3.14. Start by labeling each vertex in $\Sigma_{\leftarrow}(M)$ with the number 1. Then consider the almost split sequence $0 \to M \to E \to \tau^{-1} M \to 0$. Note that each summand of $E$ lies in $\Sigma_{\to}(M)$ and that $\tau^{-1} M$ does not. Label the vertex $\tau^{-1} M$ by the number of indecomposable summands of $E$ minus the label of $M$. Thus the label at $\tau^{-1} M$ is either 0, 1 or 2 depending on whether the mesh in the Auslander–Reiten quiver between $M$ and $\tau^{-1} M$ has 1, 2 or 3 middle vertices, respectively.

Fig. 3.13 Auslander–Reiten quiver of type $D_5$ in terms of arcs in a punctured polygon
3.3 Auslander–Reiten Quivers of Type $D_n$

is $\Sigma_n(f(i))$, and thus the hammock consists of all modules that are nonzero at the vertex $i$.

### 3.3.4.2 Ext$^1$ and Short Exact Sequences

We can compute Ext$^1$ as in type $A$ thanks to the formula

$$\dim \text{Ext}^1(M, N) = \dim \text{Hom}(N, \tau M).$$

Thus the dimension of $\text{Ext}^1(M, -)$ is determined by the maximal hammock ending at $\tau M$.

Since the dimension of $\text{Ext}^1(M, N)$ can be as large as 2, it is not so easy to find the short exact sequences that represent the elements of $\text{Ext}^1(M, N)$. We know that each element can be represented by short exact sequences of the form $0 \to N \to E \to M \to 0$, where $E$ is some representation of $Q$, but there might be several choices for $E$. In the example in Fig. 3.15, there are four non-split short exact sequences starting at $N$ and ending at $M$, namely

$$
0 \to N \to E_1 \oplus E_2 \oplus H_2 \to M \to 0 \\
0 \to N \to F_1 \oplus F_2 \oplus H_2 \to M \to 0 \\
0 \to N \to G_1 \oplus G_2 \to M \to 0 \\
0 \to N \to H_1 \oplus H_2 \to M \to 0.
$$

It is important to note that while there are four non-split short exact sequences, the dimension of $\text{Ext}^1(M, N)$ is only two. Thus any two of the above sequences span the vector space $\text{Ext}^1(M, N)$.

![Fig. 3.15 Computing short exact sequences in type D](image)