WEIGHTED ESTIMATES FOR THE VARIATION-NORM CARLESON OPERATOR VIA SPARSE FORM DOMINATION

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Abstract. We prove a sparse form domination for the bilinear form associated with the variation-norm Carleson operator and deduce weighted estimates for this operator, improving on a prior result of the second author and Lacey [4].

1. Introduction

Let $r > 2$. In this note we prove weighted estimates for

$$C_r f(x) = \sup_{K, N_0 < \cdots < N_K} \left| \sum_{j=1}^{K} \int_{N_{j-1}}^{N_j} e^{ix\xi} \hat{f}(\xi) d\xi \right|^{r/2}$$

(known as the variation-norm Carleson operator), improving on a prior result of the second author with Lacey [4]. Our method of proof is different from [4], here we will deduce the estimates from a new sparse form estimate for $\langle C_r f, g \rangle$.

(In particular we don’t need to use interpolation as in prior works [4, 12].) Note that the simpler case $r = \infty$ corresponds to the classical Carleson operator whose boundedness implies a.e. convergence of the Fourier series, see e.g. [7, 8, 3]. Our consideration uses the sparse domination approach of Lacey [9] and Lerner [10] (cf. [11]), more specifically we follow the adaptation in Culiuc-Di Plinio-Ou [1] where the authors proved sparse domination for the trilinear form associated with the bilinear Hilbert transform and obtain weighted estimates for this operator.

Our main theorem is the following

**Theorem 1.1.** Let $w \in A_q$ some $q \geq 1$, and let $r > \max(2, \frac{p}{p-q})$. Then $C_r f$ maps $L^p(w)$ into itself.

(Transference of this estimate to Fourier series is discussed in [3].) Theorem 1.1 improves the lower bound assumed on $r$ in a previous result of the second author and Lacey [4], where $r > \max(2q, \frac{pq}{p-q})$. Theorem 1.1 is a consequence of the following sparse domination result. To state this result we first fix several notations.

Let $0 < \alpha < 1$. By a $\alpha$-sparse collection $S$ we mean a collection of (shifted) dyadic intervals $I$ such that there exists $E_I$ such that $(E_I)_{I \in S}$ are disjoint and $|E_I| \geq \alpha |I|$. For any $f$ let $(f)_I$ denote the average $\frac{1}{|I|} \int_I |f|$ and more generally $(f)_{I,p} = \left( \int_I |f|^p \right)^{1/p}$. Below is the sparse domination for $\langle C_r f, g \rangle$:

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Theorem 1.2. For any \( \alpha \in (0, 1) \) and any Schwartz functions \( f \) and \( g \) there is an \( \alpha \)-sparse collection \( S \) such that for any \( q > r' \)
\[
\langle C_{r'} f, g \rangle \leq \Lambda_S(f, g) := \sum_{I \in S} |I| (f)_{3L,q}(g)_{3L,1}
\]

1.1. Deducing weighted estimates from sparse domination. To demonstrate the idea, we first deduce the (unweighted) \( L^p \) estimates for \( C_{r'} \) in \( L^p \) in the range \( p > r' \) (as proved by Oberlin et al. \cite{12}) from sparse domination and the Hardy-Littlewood maximal theorem:
\[
\langle C_{r'} f, g \rangle \leq \sum_{I \in S} |E_I| \inf_{x \in I} M_{r' + \epsilon} f(x) M g(x)
\]
then, by choosing \( \epsilon > 0 \) such that \( p > r' + \epsilon \).

We now deduce Theorem 1.1 from the above sparse domination estimate. Recall \( w \in \mathcal{A}_q \), we let \( t = p/q \). Then \( p > t > r' \) thanks to the given hypothesis on \( r \). Let \( \sigma = w^{1-p'} \in \mathcal{A}_{p'} \). Then using sparse domination as above we get
\[
\langle C_{r'} (f \sigma), gw \rangle \leq \| M_t (f \sigma) M(gw) \|_1
\]
by choosing \( \epsilon > 0 \) such that \( p > r' + \epsilon \).

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2. Outline of the Proof

For simplicity of notation we will show Theorem 1.2 for \( \alpha = 1/2 \), the general case is entirely similar. Also since \( r' < 2 \) and using monotonicity of \( L^q \) averages we may assume without loss of generality that \( q < 2 \).

To show Theorem 1.2 we discretize \( C_{r'} f \) using the argument of Oberlin et. al., and \( \langle C_{r'} f, g \rangle \) becomes a finite linear combination of discrete Carleson operators 
\[
\sum_{P \in \mathcal{P}} |I| \langle f, \phi_p \rangle \langle \tilde{\phi}_p a_p, g \rangle 
\]
where \( \langle \phi_p \rangle \) is a collection of Fourier wave packets that are \( L^1 \) normalized (see later sections for background), and \( \tilde{\phi}_p \) is a frequency truncated version of \( \phi_p \) that arises from the \( r \)-variation norm, which we will also define later.

Let \( r > 2 \). Then it is proved in \cite{5} that the projection \( f \mapsto \langle f, \tilde{\phi}_p \rangle \) maps \( L^p(\mathbb{R}, w) \) into \( \mathcal{L}^p(S, \mu) \) for \( p > r' \) where \( S \) and \( \mu \) are defined in Section 4 and weak-type embedding holds at the endpoint \( p = r' \). A local version of this estimate is implicit in \cite{5}, we formulate this result below. An continuous analogue of this result is a special case of G. Uraltsev’s Ph.D. thesis (personal communication).
Theorem 2.1. Given \( \lambda > 0 \) let \( I \) be the collection of \( P \) such that \( I_P \subset \{ Mf > \lambda \} \) here \( M \) is the dyadic maximal function. Then the following holds for \( q > r' \):
\[
\| \langle f, \tilde{\phi}_p \rangle 1_I \|_{L^q(P,S,\mu)} \lesssim \lambda^{1/q'} \| f \|^{1/q}_{L^1}.
\]

The sparse estimate (Theorem 1.2) will be deduced from this estimates together with the local embedding theorem of Di Plinio-Ou \cite{2} using arguments of Lacey–Lerner et al., details are discussed in later sections. We first recall some background for wavelets.

3. Wave packets

We first recall the notion of a classical wave packet associated with a Heisenberg rectangles in the phase plane, i.e. \( p = I_p \times \omega_p \subset \mathbb{R}^2 \) a rectangle of unit area (commonly refered to as a tile). (The precise technical definition for wave packets below could be relaxed or adapted to more general settings.) For convenient for each interval \( I \) we let \( c(I) \) denote its center. Typically we will assume that \( I_p \) belongs to a given dyadic grid on \( \mathbb{R} \) and similarly \( \omega_p \) belongs to another given dyadic grid. For simplicity we will assume that \( I_p \) and \( \omega_p \) are standard dyadic intervals \( [2^n k, 2^n (k+1)] \) and let \( P \) be the set of these tiles.

Let \( C_3 > 0 \) be fixed. A function \( \phi_p \) is a classical \( L^p \)-Fourier wave packet adapted to \( p \) if the Fourier transform of \( \phi_p \) is supported in \( C_3 \omega_p \) and for \( M \) sufficiently large the following holds for all \( 0 \leq n, N \leq M \):
\[
\frac{d^n}{dx^n} \left[ e^{-i2\pi x c(\omega_p)} \phi_p(x) \right] \lesssim_{N,n} \frac{1}{|I_p|^{n+1/p}} \left( 1 + \frac{|x - c(I_p)|}{|I_p|} \right)^{-N}
\]

In this paper we will assume that all unmodified wave packets are \( L^2 \) packets.

We will also consider the modified version of this wave packet which arises from a maximal or variation-norm operator.

Let \( a_1, \ldots, \) be a sequence of measurable functions on \( \mathbb{R} \) that is eventually zero pointwise (however the number of nonzero terms may depend on \( x \)) and let \( r \) be such that \( |a_1|^{r'} + \cdots + |a_n|^{r'} + \cdots \leq 1 \) pointwise.

Let \( N_1 < M_1 \leq N_2 < M_2 \leq \ldots \) be a (pointwise increasing) sequence of measurable functions that could take value in \( [-\infty, \infty] \).

By a scale-truncated wave packet we mean \( \tilde{\phi}_p(x) = \phi_p(x) a_j(x) \) if \( N_j(x) \leq |I_p| < M_j \) (such \( j \) would be unique) and \( \tilde{\phi}_p(x) = 0 \) if no such \( j \) exists. For simplicity we’ll assume that the unmodified \( \phi_p \) is frequency supported in \( \omega_p \) in this case.

By a frequency-truncated wave packet we mean \( \tilde{\phi}_p(x) = \phi_p(x) a_j(x) \) if \( N_j \in \omega_{p,left} \) and \( N_{j+1} \in \omega_{p,right} \). Here \( \omega_{p,left} \) and \( \omega_{p,right} \) are two intervals that stay strictly on the left and the right of \( \omega_p \) and at most one of them is infinite (i.e. a half line). For simplicity in this paper we’ll assume that \( \omega_{p,left} \) is a half line and \( \omega_{p,right} \) is a translation of \( \omega_p \) to the right by \( n|\omega_p| \) where \( n \geq 2 \) is a sufficiently large fixed integer such that \( C_2 \omega_p \) is disjoint from \( C_2 \omega_{p,right} \) and \( \omega_{p,left} \) for some finite fixed \( C_2 > C_3 \).
By a classical wavelet projection we mean the maps from $f$ to the collection of numbers $\langle f, \phi_p \rangle$, $p \in \mathcal{P}$. Similarly we consider the modified projection $\langle f, \tilde{\phi}_p \rangle$ for the truncated wave packets $\tilde{\phi}_p$.

4. Basic facts about outer measure spaces

We first recall several notions from [6] regarding an outer measure space $(X, S, \mu)$, which consists of the following ingredients:

(i) A countable set $X$. In the setting of this paper $X$ will be discrete, in fact we will often assume $X$ is finite (the underlying estimates are uniform over $|X|$).

(ii) An outer measure $\mu$ generated from a pre-measure on $E$ a covering of $X$ using non-empty subsets.

(iii) A size $S$ taking values in $[0, \infty]$ to each pair $(f, E)$ where $E \in \mathcal{E}$ and $f : X \to \mathbb{C}$ measurable, that is monotonic, homogeneous and quasi subadditive with respect to $f$ i.e. for any fixed $E \in \mathcal{E}$ it holds that $S(f + g)(E) \leq S(f)(E) + S(g)(E)$ and $S(\lambda f)(E) = |\lambda| S(f)(E)$ and if $|f| \leq |g|$ pointwise then $S(f)(E) \leq S(g)(E)$.

We may then the $\mathcal{L}^\infty$ norm by $\text{sup}_{E \in \mathcal{E}} S(f)(E)$, and for $\mathcal{L}^p$ we use the Lorentz norm approach:

$$\|f\|_{\mathcal{L}^p(X, S, \mu)} = \sup_{\lambda > 0} \lambda \mu(S(f) > \lambda)^{1/p},$$

$$\|f\|_{\mathcal{L}^r(X, S, \mu)} = \left( \int_0^\infty \lambda^{p-1} \mu(S(f) > \lambda) d\lambda \right)^{1/p}.$$

Here the distribution function is defined as follows: for Borel measurable $F \subset X$ we define $\text{outsup}_F S(f) := \|f 1_F\|_{\mathcal{L}^\infty(X, S, \mu)}$. Then for any $\lambda \in \mathbb{R}$ let $\mu(S(f) > \lambda) := \inf\{\mu(F) : \text{outsup}_{X \setminus F} S(f) \leq \lambda\}$.

Many standard properties of classical $L^p$ spaces can be proved for outer $L^p$ spaces, see [6] for details.

5. Outer measure spaces

Let $C_2$ be any finite constant in $(C_3, \infty)$. Let $n_0 \geq 2$ be a fixed integer and let $\omega_{p, \text{right}} = \omega_p + n_0|\omega_p|$, assume $n_0$ is large enough such that $C_2\omega_p$ and $C_2\omega_{p, \text{right}}$ are disjoint. For each $p \in \mathcal{P}$ let $\tilde{\omega}_p$ be the following enlargement of $\omega_p$:

$$\tilde{\omega}_p = \text{convex-hull}(C_2\omega_p \cap C_2\omega_{p, \text{right}})$$

We now describe the covering set $E$. A subset $T$ of $\mathcal{P}$ is in $E$ if there exists a dyadic interval $I_T$ and a real number $\xi_T$ such that for all $p \in T$ it holds that

$$I_p \subset I_T \quad \text{and} \quad \omega_T := [\xi_T - \frac{1}{2|I_T|}, \xi_T + \frac{1}{2|I_T|}] \subset \tilde{\omega}_p$$

We say that $T$ is overlapping if $\xi_T \in C_2\omega_p'$ for all $p \in T$ and $T$ is lacunary if $\xi_T \notin C_2\omega_p'$ for all $p \in T$; clearly any $T \in \mathcal{E}$ can be splitted into two elements of $\mathcal{E}$ one of each type.
For each interval $I$ let $w(I)$ be $\int_I w$, and we consider the following size:

$$S_{t,\text{overlap}}(F)(T) = \sup_{E \text{ is overlapping}} \frac{1}{E \cap T} \left| w(I_E) \right|^{1/t} \left| \left\{ P \mid \sum_{p \in E} \left| F(p) \right|^{t} w(I_p) \right\} \right|^{1/t}$$

and define $S_{t,\text{lacunary}}$ similarly.

Let $S = S_{t,\text{overlap}} + S_{t,\text{lacunary}}$. We generate the outer measure $\mu_w$ on $\mathbb{P}$ by defining for each $A \subset \mathbb{P}$

$$\mu_w(A) = \inf\{ \sum_T w(I_T) \}$$

the inf is over countable covering of $A$ using elements of $E$.

For convenience of notation let $F(p) = \langle f, \phi_p \rangle$ and $G(p) = \langle f, \tilde{\phi}_p \rangle$.

6. Local Embedding for Frequency-Truncated Wavelet Projections

We will use the same set up as in the last section and let $\tilde{S}_w = S_{t,\text{lacunary}} + S_{t,\text{overlap}}$. Let density denote the following quantity: for each $A \subset \mathbb{P}$ let

$$\text{density}(g, A) = \sup_{T \in E, T \subset A} \left( \frac{1}{|I_T|} \int_{X_{I_T}} |g|^{r'} \sum_{j \in N_j \subset T} |d_j|^{r'} \right)^{1/r'}$$

We first prove an estimate

**Lemma 6.1.** For any $A \subset \mathbb{P}$ and $0 \leq \delta \leq 1$ it holds that

$$S(\langle f, \tilde{\phi}_p \rangle, A) \leq (\sup_{p \in A} \frac{1}{|I_p|} \int |f|^{r(1-\delta)} \tilde{\chi}_{I_p}^{N_p})^{1/r'} \text{density}(|f|^{1-\delta}, A)$$

**Proof.** Let $M(A)$ be the right hand side. It suffices to show that if $T$ is lacunary then for any sequence $a_p$ we have

$$\sum_{p \in T} |I_P| a_p \langle f, \tilde{\phi}_p \rangle \lesssim |I_T| S_{t,\text{lacunary}}(A, T) M_A$$

then taking $a_p = \langle f, \tilde{\phi}_p \rangle$ we obtain the desired estimate. By breaking $T$ into smaller subsets that are spatially disjoint (if necessary) we may assume that there is some $p \in T$ such that $I_P = I_T$.

Now factorize $f = f_0 f_1$ where $|f_0| = |f|^{1-\delta}$ and $|f_1| = |f|^{\delta}$. Now using estimates from Oberlin et. al. [12], Proposition 5.1], the LHS is controlled by

$$\int_{I_T} |\sum_P |I_P| a(P) \tilde{\phi}_p f| + \sum_{k \geq 1} \int_{2^{k} I_T \cap 2^{k-1} I_T} \sum_P |I_P| a(P) \tilde{\phi}_p f|$$

$$\lesssim \left( \int_{I_T} \sum_P |I_P| a(P) \tilde{\phi}_p f_0|^{r'} \right)^{1/r'} \left( \int_{I_T} |f_1|^{r'} \right)^{1/r} + \sum_{k \geq 1} \left( \int_{2^{k} I_T \cap 2^{k-1} I_T} \sum_P |I_P| a(P) \tilde{\phi}_p f_0|^{r'} \right)^{1/r'} \left( \int_{2^{k} I_T \cap 2^{k-1} I_T} |f_1|^{r'} \right)^{1/r}$$

$$\lesssim \sum_{k \geq 0} 2^{-Mk} |I_T|^{1/r'} S(a, T) \text{density}(f_0, T) \left( \int_{2^{k} I_T \cap 2^{k-1} I_T} |f_1|^{r'} \right)^{1/r}$$
and again we let \( \delta \) making \( r\delta \)

We now choose \( M \) such that \( \sup \) and \( \liminf \) the argument of Culiuc-Di Plinio-Ou. Without loss of generality assume that \( f \) and \( g \) are compactly supported \( C^\infty \). For each interval \( J \) let \( I_j \) be maximal elements of \( I_{f,J,p} \cup I_{g,J} \) for \( r'' < p < 2 \).

Then the intervals in \( I_J \) are disjoint and their union has total measure < \( |J|/2 \).

We start with \( (J_j) \) a nested sequence of intervals covering \( \mathbb{R} \) such that \( supp(f) \) and \( supp(g) \) are contained in \( 3J_j \) any \( j \). Then for each member \( J \) of the sequence
we build $S_0(J) = \{J\}$ and $S_{k+1}(J) = \bigcup_{I \in S_k} I_I$, and let $S(J) = \bigcup_{k\geq 0} S_k(J)$. Eventually we obtain the 1/2-sparse grid
\[ S = \bigcup_j S(J_j) \]

Let $P_c(J)$ denote the subset of $P$ containing $P$ such that $I_P \subset J$, clearly
\[ P = \bigcup_j P_c(J_j) \]

It suffices to show that if $Q$ is $P_c(J_j)$ any $j$ then
\[ \sum_{P \in Q} |I_P| \langle g, \phi_P \rangle \langle \bar{\phi}_P, f \rangle \lesssim \sum_{I \in S(J_j)} |I|(f)_{3t, p}(g)_{3t} \]

Let $\Lambda_Q(f, g)$ denote the bilinear form on the left hand side. To see the above estimate we show that there is a constant $C$ such that for any $P$
\[ \Lambda_P(f_{13I}, g_{13I}) \leq C |J|(f)_{3I, p}(g)_{3I} + \sum_{I \in I_J} \Lambda_{P_c(I)}(f_{13I}, g_{13I}) \]

Indeed, decompose the left hand side into $\Lambda_{I_J}(f_{13I}, g_{13I}) + \Lambda_{I_I}(f_{13I}, g_{13I}).$ Note that the given assumption implies $p' < r$, so let $r > t > p'$. Then $t > p'$ and $t' > r'$, so the first term is controlled by
\[ \| (f_{13I}, \phi_P) 1_{I_I} \|_{L'(P, S_{1, \mu})} \times \| (g, \bar{\phi}_P) 1_{I'_I} \|_{L'(P, S_{2, \mu})} \]
\[ \lesssim |J|^{1/t}(f)_{3I, p}|J|^{1/r'}(g)_{3I} \]
\[ = |J|(f)_{3I, p}(g)_{3I} \]

For the second term we decompose $f_{13I} = f_{3I} + f_{13I}$ and similarly for $g$ for each $I \in I_I$. It remains to show that
\[ \sum_{I \in I_I} \Lambda_{P_c(I)}(f_{13I}, g_{13I}) \leq |J|(f)_{3I, p}(g)_{3I} \]
and similarly $\sum_{I \in I_J} \Lambda_{P_c(I)}(f_{13I}, g_{13I}) \leq |J|(f)_{3I, p}(g)_{3I}.$

Without loss of generality assume that $\text{supp}(f) \subset 3J$ and $\text{supp}(g) \subset 3J$. From the definition of $I_I$ it suffices to show that
\[ \Lambda_{P_c(I)}(f_{13I}, g_{13I}) \lesssim |I| \inf_{x \in I} M_p(f)(x)M(g)(x) \]
\[ \Lambda_{P_c(I)}(f_{13I}, g_{13I}) \lesssim |I| \inf_{x \in I} M_p(f)(x)M(g)(x) \]

We will first prove the second estimate, the argument is similar for the first estimate and will present later.

We devide $P_c(I)$ into subsets $P_{c,k}(I)$ where $2^k I_P \subset I$ but $2^{k+1} I_P \notin I$ and it suffices to show the analogous estimate with exponential decay in $k$. Let $L_k$ be the collection of intervals $L$ such that for some $P \in P_{c,k}(I)$ we have $I_P = L$. If $k = 0$ it is clear that one of the endpoints of $L$ must be an endpoint of $I$ therefore it is clear that $\sum |L| \lesssim |I|$, on the other hand if $k > 0$ then $L$ have finite overlap. Thus it suffices to show for each $L \in L_k$ and any $p > 1$
\[ \Lambda_{P_{c,L}}(f_{13I}, g_{13I}) \lesssim 2^{-Nk}|L| \inf_{x \in L} M_p(f)(x)M(g)(x) \]
(Without loss of generality we may assume $p \leq 2$.) Given any interval $L$ let $\chi_L(x) = (1 + \frac{(x-c(L))^2}{|L|^2})$. For any $p \leq 2$ we have, recall that $\phi_P$ is $L^1$ normalized, therefore

$$\sup_{P: I_P = L} | \langle f 1_{3I}, \phi_P \rangle | \leq |L|^{-1} \| f 1_{3I} \chi_L^{-N} \|_1$$

On the other hand

$$\sum_{P: I_P = L} | \langle g 1_\{3I\}^*, \tilde{\phi}_P \rangle | \leq |L|^{-1} \| g 1_\{3I\}^* \chi_L^{-N} \|_1$$

To see the first estimate, we note that the left hand side is bounded above by

$$\| g 1_\{3I\}^* \chi_L^{-N} \|_1 \sum_{P: I_P = L} | \tilde{\phi}_P(x) | \chi_L^N(x) \|_\infty .$$

By the variational truncation condition, for each $x$ in the sum inside the second norm there is only one term, which is bounded above by $O(|L|^{-1})$.

It follows that, via classical $L^1 L^\infty$ Holder the desired estimate follows

$$\Lambda_{P_\infty(L)}(f 1_{3I}, g 1_\{3I\}^*) \leq 2^{-NK} |L| \inf_{x \in J} M f(x) \inf_{x \in J} M g(x) .$$

Now for the first estimate we proceed similarly, and in the final step we estimate

$$\Lambda_{P_\infty(L)}(f 1_{3I}^*, g 1_{3I}) \leq |L| \sup_{P: I_P = L} | \langle f 1_{3I}^*, \phi_P \rangle | (\sum_{P: I_P = L} | \langle f 1_{3I}, \phi_P \rangle |)$$

$$\leq 2^{-NK} |L| \inf_{x \in J} M f(x) \inf_{x \in J} M g(x)$$

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