Chapter 6

Bounded and continuous functions on a locally compact Hausdorff space and dual spaces

Recall that the dual space of a normed linear space is a Banach space, and the dual space of $L^p$ is $L^q$ where $1/p + 1/q = 1$ if $1 < p < \infty$. If the underlying measure space is $\sigma$-finite then the dual of $L^1$ is $L^\infty$. What about the dual of $L^\infty$? Or its subspace $C_c(X)$ and $C_0(X)$ where say $X$ is locally compact Hausdorff. In this chapter we discuss several representation theorems related to these themes.

6.1 Riesz representation theorems

6.1.1 The dual space of $L^\infty$

The dual space of $L^\infty(X, \Sigma, \mu)$ consists of bounded (i.e. the total measure of $X$ is finite) and finitely additive signed/complex measures on $\Sigma$ that is absolutely continuous with respect to $\mu$.

Clearly if $\sigma$ is a such a measure we may define a linear functional $\ell_\sigma(f) := \int f \, d\sigma$, it is not hard to see that this is well defined as a bounded linear functional on $L^\infty$. (The fact that $\sigma \ll \mu$ ensures that if $f$ and $f'$ equals almost everywhere with respect to $\mu$, i.e. they represent the same $L^\infty$ function, then the two integrals agree.) Note that it is possible to define integration with respect to a finitely additive measure: start with positive function and define the integral as the supremum over integration of simple functions dominated by the given positive function. Then define integral of signed functions and complex valued functions etc.

Conversely, given a bounded linear functional $\ell$, after several reductions we may assume that $\ell$ is nonnegative, namely if $f \geq 0$ then $\ell(f) \geq 0$. Then we may define $\sigma(E) = \ell(1_E)$, it is not hard to see that $\sigma$ is a finitely additive bounded measure, and if $\mu(E) = 0$ then $1_E \equiv 0$ in $L^\infty(\mu)$ thus $\sigma(E) = \ell(1_E) = \ell(0) = 0$. Boundedness of the measures follows from choosing
the right input.

6.1.2 Measures on a locally compact Hausdorff spaces

There are several related and sometimes equivalent notions of measures on a given locally compact Hausdorff space. Here we list and compare some of them.

Regular Borel measures and Radon measures

Recall that a Borel set is a set in the $\sigma$-algebra generated by the open sets.

A regular Borel measure is a Borel measure (i.e. all Borel sets on $X$ are measurable) such that for any Borel set $E$ the following holds.

$$\mu(E) = \inf \{\mu(U) : U \text{ open}, E \subset U\}$$

$$\mu(E) = \sup \{\mu(K) : K \text{ compact}, K \subset E\}$$

A Radon measure is a locally finite regular Borel measure, i.e. in addition to being regular Borel it also assignes a finite value to all compact subsets of $X$.

Note that some textbooks define Radon measure as locally finite inner regular Borel measures, thus requiring only that measures of Borel sets could be approximated from inside using measures of compact subsets.

Baire sets and Baire measures

A Baire set is an element of the sigma algebra generated by all compact subsets of $X$ that are at the same time countable intersections of open sets.

A Baire measure is a measure on the Baire sigma algebra.

One could check that a Baire measure is also regular with respect to the Baire sigma algebra.

The Baire sigma algebra is the smallest sigma algebra such that all continuous functions are measurable.

It can be shown that any Baire measure extends uniquely to a Radon measure and vice versa.

6.1.3 The dual space of $C_0(X)$ and $C_c(X)$

Let $X$ be LCH. Since the closure of $C_c(X)$ under the uniform norm is $C_0(X)$, it suffices to consider the dual of $C_0(X)$. Note that if $X$ is not compact then $C(X)$ is not a normed linear space with the sup norm, and we have to look at its dual as a topological dual. More on this in later chapters.

We say that a signed measure is a signed Radon measure if it could be written as the difference of two Radon measures. We say that a complex valued measure is a complex valued Radon measure if its real and imaginary parts are signed Radon measures.
Theorem 27 (Riesz). Let $X$ be a LCH. Then the dual of $C_o(X)$ is $M(X)$ the space of complex valued Radon measures (locally finite regular Borel measures) on $X$. Namely the following map is an isometric isomorphism between this dual and $M(X)$

$$
\mu \mapsto \ell_\mu(f) = \int f d\mu
$$

and $\|\ell_\mu\| = \|\mu\|$ the total variation of $\mu$ defined by

$$
\sup \sum_{j=1}^m |\mu(A_j)| \text{ supremum over all collection } A_1, \ldots, A_m \text{ of disjoint subsets of } X.
$$

**Step 1:** We first show that if $\ell$ is a bounded positive linear functionals on $C_o,\mathbb{R}(X)$, the space of real valued continuous functions on $X$ that vanish at $\infty$, then there is a Radon measure $\mu$ such that for any $f \in C_o,\mathbb{R}(X)$ it holds that

$$
\ell(f) = \int f d\mu
$$

To see this, we would like to define $\mu$ by $\mu(E) = \ell(1_E)$, however $1_E$ is not continuous therefore one can not apply $\ell$ to this function. To get around this we may define for each open set $U \subset X$

**Definition:** $\mu(U)$ is defined to be the supremum over $\ell(f)$ where $f \in C_o(\mathbb{R})$ and the support of $f$ is inside $U$ and $0 \leq f \leq 1$ pointwise.

(Note that we may not be able to do this if $U$ wasn’t open since such a continuous function may not exists; for open $U$ the existence of $f$ is due to Urysohn’s lemma).

Now we obtain a premeasure on open sets which is a ring of sets, thus by Caratheodory we may extend $\mu$ to the sigma algebra generated by these sets, which are exactly the Borel sets.

We want to show that $\ell(f) = \int f d\mu$ for every $f \in C_o(X)$. Without loss of generality assume $0 \leq f \leq 1$. Since $f$ vanishes at infinity the sets $A_k = \{ f \geq k/n \}$ are compact, and we may find continuous functions $f_k$ such that

$$
1_{A_k} \leq n f_k \leq 1_{A_{k-1}}
$$

and $f = \sum_{k=1}^n f_k$. We will use the following lemma

**Lemma 4.** If $E \subset F$ are compact subsets and $f$ is continuous on $X$ such that $1_E \leq f \leq 1_F$ then $\mu(E) \leq \ell(f) \leq \mu(F)$.

Using the lemma and positivity of $\ell$ it follows that

$$
\frac{1}{n} \sum_{k=1}^n \mu(A_k) \leq \ell(f) \leq \frac{1}{n} \sum_{k=1}^n \mu(A_{k-1})
$$

consequently

$$
|\ell(f) - \int f d\mu| \leq \frac{1}{n} \mu(A_0) \leq \frac{1}{n} \mu(\text{supp}(f)) = O(\frac{1}{n})
$$
by sending \( n \to \infty \) we obtain the desired claim.

We now show the lemma.

For the second estimate, recall that by definition \( \mu \) is outer regular, thus \( \mu(F) = \inf\{\mu(U) : F \subset U, U \text{ open}\} \). Now for every \( U \) open containing \( F \) we have \( 0 \leq f \leq 1_U \) thus by definition of \( \mu(U) \) it follows that \( \ell(f) \leq \mu(U) \). Consequently \( \mu(F) \geq \ell(f) \).

For the first estimate, it suffices to show that for every \( \epsilon > 0 \) we have \( \mu(E) \leq (1 + \epsilon)\ell(f) \).

Let \( U_\epsilon = \{ f > \frac{1}{1+\epsilon} \} \) which is an open set that contains \( E \). Thus

\[
\mu(E) \leq \mu(U_\epsilon) = \sup\{\ell(g) : 0 \leq g \leq 1, \supp(g) \subset U_\epsilon\}
\]

\[
\leq \sup\{\ell(g) : 0 \leq g \leq (1 + \epsilon)f\}
\]

\[
\leq (1 + \epsilon)\ell(f)
\]

We now show regularity properties for \( \mu \). (It is clear that \( \mu \) is finite on compact set using Urysohn’s lemma.) Outer regularity of \( \mu \) follows from construction using premeasure, and it remains to show inner regularity. Let \( E \) be Borel, it suffices to show that

\[
\mu(E) \leq \sup\{\mu(K) : K \subset E \text{ compact}\}
\]

now using outer regularity it suffices to do this for \( E \) open. Let \( U \) be an open set, we have

\[
\mu(E) = \sup\{\ell(f) : 0 \leq f \text{ continuous} \leq 1, \sup(f) \subset U\}
\]

\[
\leq \sup\{\mu((\sup(f)) \ldots\} = \sup\{\mu(K) : K \subset E \text{ compact}\}
\]

**Step 2:** We now reduce the desired result to the positive setting. This is done via writing bounded linear functionals on \( C_0(X) \) as a linear combination of positive linear functionals. To see this note that without loss of generality it suffices to consider bounded linear functional on \( C_{0,R}(X) \) real valued members of \( C_0(X) \). Let \( \ell_+ \) be defined by

\[
\ell_+(f) = \sup\{\ell(g) : 0 \leq g \leq f, g \in C_{0,R}(X)\}
\]

and let \( \ell_- = \ell_+ - \ell \), it is not hard to check that both \( \ell_+ \) and \( \ell_- \) are positive linear functionals on \( C_{0,R}(X) \).

### 6.2 The Stone-Weierstrass approximation theorem

Let \( X \) be compact Hausdorff. Let \( P \) be a subspace of \( C_R(X) \) such that \( P \) is also an algebra inside \( C_R(X) \), i.e. \( p_1 p_2 \in P \) if both \( p_1, p_2 \in P \).

**Theorem 28.** Assume that \( P \) separates point, i.e. if \( x_1 \neq x_2 \) are elements of \( X \) then there exists one element \( p \in P \) such that \( p(x_1) \neq p(x_2) \). Then one of the following holds: \( \overline{P} = C_R(X) \) or there is some \( x_0 \in X \) such that \( \overline{P} = \{ f \in C_R(X) : f(x_0) = 0 \} \).
Note that there are also versions for complex valued functions, in which case the same conclusion holds if we assume further that \( P \) is closed under conjugation \( p \in P \) then \( \overline{p} \in P \).

There are also versions for \( C_0(X) \) and \( C_{0, \mathbb{R}}(X) \) where \( X \) is locally compact Hausdorff.

The original Weierstrass approximation theorem is for \( X \) being compact intervals, which implies that polynomials are dense inside continuous functions. Applications in probability (moment problems etc.)

Proof: Since \( P \) is also an algebra that separate points, without loss of generality we may assume that \( P \) is closed. We divide the proof into two steps.

Step 1: We will show that \( P \) is a lattice, namely if \( f, g \in P \) then so are \( \max(f, g) \) and \( \min(f, g) \).

Step 2: Using the lattice property of \( P \), we will show that if \( f \in C_\mathbb{R}(X) \) be such that for every \( x \neq y \) in \( X \) there exists \( h \in P \) such that \( h(x) = f(x) \) and \( h(y) = f(y) \), then \( f \in P \).

Combining these two steps, we prove the theorem as follows. Given any \( x \neq y \) consider the algebra \((h(x), h(y)), h \in P \). Clearly this is a subalgebra subspace of \( \mathbb{R}^2 \) and it can’t be \( \{(0, 0)\} \) since \( P \) separate points. Thus it could be either \( 0 \times \mathbb{R} \) or \( \mathbb{R} \times 0 \) or \( \mathbb{R}^2 \). Now in the last case by step 2 any \( f \in C_\mathbb{R}(X) \) must be in \( P \) as desired. In the first case for instance, \( h(x) = 0 \) for all \( h \in P \) and so we must have \( \{h(y), h \in P\} = \mathbb{R} \) for any other \( y \). So by Step 2 again it follows that any function \( f \) such that \( f(x) = 0 \) must be in \( P \) as desired.

Proof of step 1: since \( \max(f, g) \) and \( \min(f, g) \) could be written as linear combination of \( |f + g| \) and \( |f - g| \) and \( f \) and \( g \), therefore it suffices to show that if \( f \in P \) then so is \( |f| \). The idea is to approximate \( |f| \) with a polynomial of \( f \) uniformly. Indeed, clearly \( f \) is bounded therefore without loss of generality assume \( |f| \leq 1 \), it then suffices to show that \( |x| \) could be approximated uniformly by polynomials on \([-1, 1]\). To see this use the Taylor expansion of \( \sqrt{1 - t} = 1 - \frac{1}{2}t + \ldots \) which converges uniformly on \( 0 \leq t \leq 1 \), then write

\[
|x| = \sqrt{x^2} = \sqrt{1 - (1 - x^2)} = 1 - \frac{1}{2}(1 - x^2) + \ldots
\]

Proof of step 2: Let \( \epsilon > 0 \). It suffices to show that there exists \( g \in P \) such that \( |g(x) - f(x)| \leq \epsilon \) uniformly over \( x \in X \). Assume that for every \( x \in X \) there exists \( g_x \in P \) such that \( |g_x(x) - f(x)| < \epsilon \) and \( g_x(y) \leq f(y) + \epsilon \) for every \( y \in X \). Then the desired \( g \) could be constructed as follows. By continuity for each \( x \in X \) there exists \( U_x \subset X \) neighborhood of \( x \) such that \( \sup_{U_x} |g_x - f| < \epsilon \). By compactness of \( X \) one could refine the collection \( U_x, x \in X \) to a finite subcollection \( U_{x_1}, \ldots, U_{x_m} \), and we simply let

\[
g = \max(g_{x_1}, \ldots, g_{x_m})
\]

which is in \( P \) by the lattice property and clearly \( \sup_X |g - f| \leq \epsilon \).

Now to show the existence of such a \( g_x \) we fix \( x \) and notice that, using the given hypothesis, for each \( y \in Y \) we may find \( h_y \in P \) and two open neighborhoods of \( x \) and \( y \) respectively, denoted respectively by \( U_y \) and \( V_y \) (note that \( x \) is fixed so we ignore the dependence on \( x \)) such that

\[
\sup_{U_y} |h_y - f| < \epsilon, \quad \sup_{V_y} |h_y - f| < \epsilon
\]
By compactness of \( X \) again we may refine the covering \( V_y, y \in Y \) to a finite subcovering \( V_{y_1}, \ldots, V_{y_n} \), and we simply let \( g_x = \min(h_{y_1}, \ldots, h_{y_n}) \) which has the desired property with \( U_x := U_{y_1} \cap \cdots \cap U_{y_n} \).