Chapter 12

Fredholm determinant

12.1 Motivation

Fredholm determinant started in Fredholm’s investigation of the integral equation

$$(1 + K)u = f$$

where $K$ is the integral operator $Kf(y) = \int_Y K(x, y)f(y)dy$ mapping functions on $Y$ to functions on $X$. (Here $X, Y$ are compact metric spaces.) We know that $K$ is compact from $L^2(Y)$ to $L^2(X)$ if the kernel $K$ is square integrable in $X \times Y$. The setting originally considered is for continuous functions, for which we have the following results:

(i) If $K$ is a continuous function of $x$ and $y$ then $K$ is compact from $L^1(Y)$ to $C(X)$.

(ii) If $K$ is a continuous function of $x$ in the $L^1$ norm wrt to $y$, i.e. $\|K(x, \cdot)\|_{L^1_y}$ is continuous wrt $x$, then $K$ is compact from $C(Y)$ to $C(X)$.

Proof of these facts uses the Arzela Ascoli criteria: $S$ haudorff compact, functions equicontinuous and pointwise uniformly bounded, then the family is precompact in the sup norm on $S$.

For simplicity, assume $K : [0, 1]^2 \rightarrow \mathbb{C}$ is continuous and $f \in C[0, 1]$. Note that this is not a Hilbert space. By Riesz’s theorem the integral equation

$$u(x) + \int_0^1 K(x, y)u(y)dy = f(x)$$

is solvable on $C[0, 1]$ (i.e. given $f \in C[0, 1]$ one could find $u \in C[0, 1]$) iff the integral operator $K$ is injective on this space, or equivalently its range is the whole space. It can be shown that a vector is in the range of $K$ iff it is orthogonal to the null space of $K^*$ which has kernel $K^*(x, y) = K(y, x)$.

Fredholm investigated the above equation by discretizing the equation and appeal to linear algebra, and then taking a limit at the end. This give rises to a number called the Fredholm determinant of $1 + K$ (we simply say the Fredholm determinant for $K$), which determines whether the given integral equation is solvable or not. The determinant concept
has been extended to other settings, most commonly for $K$ being a trace class operator on some separable Hilbert space.

We first discuss Fredholm’s original approach for $C[0,1]$ (modulo some simplifications), and then we’ll discuss the Hilbert space theory later.

## 12.2 Fredholm’s approach for integral operators

We now investigate Fredholm’s approach towards solving the above equation. Fredholm’s idea is to use linear algebra, approximating the integral using discretized sums and taking an appropriate limit at the end.

$$u_i + h \sum K_{ij} u_j = f_i, \quad i = 1, \ldots, n,$$

where $h = 1/n$, $f_i = f(ih)$ and $K_{ij} = K(ih, jh)$ and $u_i = u(ih)$. The determinant of the matrix acting on the vector $(u_1, \ldots, u_n)$ is denoted by $D(h)$

$$D(h) = \det(I + h(K_{ij}))$$

which is clearly a polynomial in $h$ of degree $n$

$$D(h) = \sum_{m=0}^{n} c_m h^m$$

$$c_m = \frac{1}{m!} \left( \frac{d}{dh} \right)^m D(h)|_{h=0}$$

We then use the product rule, which say that if $C = \det(C_1, \ldots, C_n)$ is the determinant of a matrix with columns $C_1, \ldots, C_n$ then by multilinearity $\frac{d}{dh}C = \sum_k \det(C_1, \ldots, \frac{d}{dh}C_k, \ldots, C_n)$. In our case each column is linear in $h$, and $C_j(0) = (0, \ldots, 1, \ldots, 0)$ the unit is in the $j$th position. Therefore

$$D(h) = 1 + h \sum_j K_{jj} + \frac{h^2}{2} \sum_{i,j} \det \left( \begin{array}{cc} K_{ii} & K_{ij} \\ K_{ji} & K_{jj} \end{array} \right) + \ldots$$

For convenience let $K \left( \begin{array}{c} x_1, \ldots, x_k \\ y_1, \ldots, y_k \end{array} \right)$ denote the determinant of the matrix $K(x_i, y_j)$, then letting $h = 1/n$ and send $n \to \infty$ we obtain

$$D = \sum_{k=0}^{\infty} \frac{1}{k!} \int \ldots \int K \left( \begin{array}{c} x_1, \ldots, x_k \\ x_1, \ldots, x_k \end{array} \right) dx_1 \ldots dx_k$$

This is called the Fredholm determinant of the integral operator $K$. Note that this is a complex number.
12.2. FREDHOLM’S APPROACH FOR INTEGRAL OPERATORS

12.2.1 Convergence and Continuity of the determinant

We’ll show that the sum defining $D$ converges. We’ll use Hadamard’s inequality: for column vectors $v_1, \ldots, v_n \in \mathbb{R}^n$, it holds that

$$|\det(v_1|v_2| \ldots |v_n)| \leq n^{n/2} \prod_{i=1}^{n} |v_j|_\infty$$

This follows from the fact that the volume of the parallelopipde is at most the product of the side lengths.

Now, the given assumption on $K$ implies that $\sup_{x,y} |K(x, y)| \leq M$ for some finite $M$. So by Hadamard inequality we have

$$K \left( \begin{array}{c} x_1, \ldots, x_k \\ y_1, \ldots, y_k \end{array} \right) \leq (Mk^{1/2})^k$$

which implies the desired convergence.

Let $\|\|$ denote the sup norm on $[0,1]^2$ below, we obtain

**Lemma 9.** Let $F$ and $G$ be two functions on $[0,1]^2$. Then

$$|\det(F(x_i, x_j)) - \det(G(x_i, x_j))| \leq n^{1+\frac{n}{2}} \|F - G\|_{\sup} \max(\|F\|, \|G\|)^{n-1}$$

here all matrices are $n \times n$.

**Proof.** For convenience, let $F$ be the $n \times n$ matrix with entries $F(x_i, x_j)$ and $G$ be the $n \times n$ matrix with entries $G(x_i, x_j)$. It is clear that

$$\det(F) - \det(G) = \det M_1 + \det M_2 + \cdots + \det M_n$$

where $M_k$ is the matrix whose first $k-1$ rows are the same as $G$, and the $k$th row is the same as $F-G$, and the last $n-k$ rows are the same as $F$.

Apply Hadamard inequality we obtain the first desired estimate. □

We now discuss how to solve the integral equation using Fredholm’s determinant. The idea is to solve it at the discrete level and then send $h \to 0$ via $n \to \infty$. Via this heuristic we obtain

$$(I + K)(I + L) = (I + L)(I + K) = I$$

where

$$L f = \int L(x, y) f(y) dy$$

$$L(x, y) = -D^{-1} \sum_{k \geq 0} \frac{1}{k!} \int \cdots \int K \left( \begin{array}{c} x, x_1, \ldots, x_k \\ y, y_1, \ldots, y_k \end{array} \right) d\xi_1 \cdots d\xi_k$$

We now ready to prove
Theorem 40. Let $K$ acts on $C[0,1]$. Let $K$ be a continuous kernel.

(i) If $D = 0$ then the operator $I + K$ has a nontrivial null space and therefore is not invertible.

(ii) Conversely if $D \neq 0$ then the operator $I + K$ is invertible, furthermore its inverse is given by $I + L$ where $L$ is defined above.

Proof: We first note that if $K_1$ and $K_2$ are two integral operators with kernel $K_1(x,y)$ and $K_2(x,y)$ respectively then $(1 + K_1)(1 + K_2) = 1 + K_3$ where $K_3$ is another integral operator whose kernel is given by

$$K_3(x,y) = K_1(x,y) + K_2(x,y) + \int K_1(x,z)K_2(z,y)dz$$

(i) Now, assume that $D \neq 0$. We need to show

$$K(x,y) + L(x,y) + \int K(x,z)L(z,y)dz = 0$$

$$L(x,y) + K(x,y) + \int L(x,z)K(z,y)dz = 0$$

We will show the first equality. For convenience of notation let

$$R(x,y) := \sum_{k \geq 0} \frac{1}{k!} \int \cdots \int K \left( \begin{array}{c} x, x_1, \ldots, x_k \\ y, y_1, \ldots, y_k \end{array} \right) d\xi_1 \cdots d\xi_k$$

Then $L = -\frac{1}{D}R$.

Now, computing the determinant $K \left( \begin{array}{c} x, x_1, \ldots, x_k \\ y, y_1, \ldots, y_k \end{array} \right)$ using its first row, we obtain

$$K \left( \begin{array}{c} x, x_1, \ldots, x_k \\ y, y_1, \ldots, y_k \end{array} \right) = K(x,y)K \left( \begin{array}{c} x_1, x_2, \ldots, x_k \\ y_1, y_2, \ldots, y_k \end{array} \right) +$$

$$+ \sum_{j=1}^{k} (-1)^j K(x,y_j) \left( \begin{array}{c} x_1, x_2, \ldots, x_{j-1}, x_{j+1}, \ldots, x_k \\ y_1, y_2, \ldots, y_{j-1}, y_{j+1}, \ldots, y_k \end{array} \right)$$

here $(y_j)$ means one omit $y_j$.

Now let $y_1 = x_1$, ... $y_k = x_k$ and integrate the above over $x_1, \ldots, x_k \in [0,1]$. Then the integrals of the last $k$ terms in the above expansion are actually the same, one could prove this by simple change of variable. It follows that

$$\int \cdots \int K \left( \begin{array}{c} x, x_1, \ldots, x_k \\ y_1, x_2, \ldots, x_k \end{array} \right) dx_1 \cdots dx_k$$

$$= K(x,y) \int \cdots \int K \left( \begin{array}{c} x_1, x_2, \ldots, x_k \\ x_1, x_2, \ldots, x_k \end{array} \right) dx_1 \cdots dx_k +$$
2. FREDHOLM’S APPROACH FOR INTEGRAL OPERATORS

+ kK(x, x_1) \left( \begin{array}{c} x_1, x_2, \ldots, x_k \\ y, x_2, \ldots, x_k \end{array} \right) dx_1 \ldots dx_k

Dividing by k! and summing over k \geq 0, we obtain

R(x, y) = K(x, y)D - \int K(x, x_1)R(x_1, y)dx_1

which implies the desired equality K(x, y) + L(x, y) + \int K(x, z)L(z, y)dz = 0.

For the second equality, one argues similarly, computing the determinant using the first column.

(ii) Since D = 0, by part (i) we have

R(x, y) + \int K(x, z)R(z, y)dz = 0

for every x, y. If R \neq 0 then one could find one y such that g(.) = R(.y) is not the zero function (note that it is continuous), and it satisfies g + Kg = 0, therefore 1 + K is not injective.

It is however possible that R \equiv 0. If so, consider the following functions

D(z) = \sum_{k \geq 0} \frac{z^k}{k!} \int \ldots \int K \left( \begin{array}{c} x_1, \ldots, x_k \\ x_1, \ldots, x_k \end{array} \right) dx_1 \ldots dx_k

R(x, y, z) := \sum_{k \geq 0} \frac{z^{k+1}}{k!} \int \ldots \int K \left( \begin{array}{c} x, x_1, \ldots, x_k \\ y, y_1, \ldots, y_k \end{array} \right) d\xi_1 \ldots d\xi_k

One could think of D(z) and R(x, y, z) as the version of D and R with zK instead of K. It is clear that D and R are entire functions. Since D(1) = 0 the entire function D has a zero of finite order n \geq 1 at z = 1. By algebraic manipulation we have

\int R(x, x, z)dx = zD'(z)

therefore there is some (x, y) such that R(x, y, z) can not vanish at z = 1 with order more than n - 1. Then for some 1 \leq \ell < n it holds that

R(x, y, z) = (z - 1)^\ell g(x, y) + O((z - 1)^{\ell+1})

where g(x, y) \neq 0, and g is continuous in x, y (to see this note that g is the uniform limit of a sequence of continuous functions on [0, 1]^2). We recall that R(x, y, z) = zK(x, y)D(z) - z \int K(x, x_1)R(x_1, y, z)dx_1 thus by dividing everything by (z - 1)\ell and then send z \to 1 we obtain

\int K(x, x_1)g(x_1, y)dx_1

and since g \neq 0 continuous it follows that 1 + K is not injective. □
Theorem 41. Assume that $K$ is Holder $c$-continuous where $c > 1/2$. Then the nonzero eigenvalues of $K$ on $C[0,1]$ (only countability many of them since $K$ is compact) satisfies $\sum_j |\lambda_j| < \infty$ and for every $z \in \mathbb{C}$ we have

$$D(z) = \prod_j (1 + z\lambda_j)$$

and we also have the trace formula

$$\int K(x,x)dx = \sum_j \lambda_j$$

Proof. We recall Hadamard’s factorization theorem (or may be a consequence of this theorem): Let $f$ be an entire function such that for some finite positive $C_1, C_2$ and $\rho \in [0,1)$ it holds for every complex number $z$ that

$$|f(z)| \leq C_1 \exp(C_1|z|^\rho)$$

Assume that $f(0) \neq 0$. Then $f$ has at most a countable number of zeros, furthermore

$$\sum_{z: f(z)=0} \frac{1}{|z|} < \infty$$

and for every complex number $\lambda$ it holds that

$$f(\lambda) = f(0) \prod_{z: f(z)=0} \left(1 - \frac{\lambda}{z}\right)$$

We plan to use this theorem to show the first part of the theorem. We note that the second part of the theorem, i.e. the trace formula would then follows from

$$\int K(x,x)dx = D'(z)|_{z=0}$$

(which is part of the definiton of $D$) and the absolute convergence of the product $\prod_j (1 + z\lambda_j)$ (viewed as an infinite power series for $z$).

First, we will show that for $z \neq 0$ it holds that: $D(z) = 0$ if and only if $-1/z$ is an eigenvalue of $K$. This is simply a consequence of our last theorem applied to $D = \det(1+zK)$.

Now, we will show that if $K$ is Holder $c$-continuous then

$$|D(z)| \lesssim \exp(O(|z|^\frac{2}{1+2c}))$$

To see this, using Stirling’s formula it suffices to show that, for some $C$ finite,

$$|D(z)| \lesssim \sum_{n \geq 0} C^n |z|^{2n/(1+2c)}$$
which is equivalent to showing that, for some $C$ finite,

$$|D(z)| \lesssim \sum_{k \geq 0} \frac{C^k |z|^k}{k^{(1+c)k}}$$

Using the definition of $D$ it suffices to show that given any $x_1, \ldots, x_k, y_1, \ldots, y_k$ it holds that for some $C > 0$ finite

$$|K\begin{pmatrix} x_1, & x_2, & \ldots, & x_k \\ y_1, & y_2, & \ldots, & y_k \end{pmatrix}| \leq (Ck^{1/2})^k k^{-c}$$

Note that incomparision with (12.1) we gain a factor of $k^{-c}$. To see this it suffices to show that

$$|K\begin{pmatrix} x_1, & x_2, & \ldots, & x_k \\ y_1, & y_2, & \ldots, & y_k \end{pmatrix}| \leq (Ck^{1/2})^k \prod_{j=1}^{k-1} |y_{j+1} - y_j|^c$$

(note that the constant $C$ may be different in different display). Indeed, wlog we may assume $y_1 \leq y_2 \cdots \leq y_k$, then by the geometric arithmetical mean inequality we have $\prod_{j=1}^{k-1} |y_{j+1} - y_j| \leq (\frac{1}{k-1})^{k-1}$, as desired.

Now, note that if $c_1, \ldots, c_k$ are $k$ column vectors ($k \times 1$) then

$$|\det(c_1 | \ldots | c_k)| = |\det(c_1, c_2 - c_1, \ldots, c_k - c_{k-1})|$$

therefore using Hadamard’s inequality we obtain

$$|K\begin{pmatrix} x_1, & x_2, & \ldots, & x_k \\ y_1, & y_2, & \ldots, & y_k \end{pmatrix}| \leq \prod_j (\sum_{n=1}^{k} |K(x_n, y_{j+1}) - K(x_n, y_j)|^2)^{1/2}$$

which implies the desired estimate using Holder continuity of $K$. □

### 12.3 Fredholm determinant for operators on Hilbert spaces

Let $H$ be a separable Hilbert space over $\mathbb{C}$. The Fredholm determinant $\det(1 + K)$ could be defined for trace class operators $K$ on a Hilbert space $H$. Note that this is not the same as the setting considered in the last section since $C[0,1]$ with the sup norm is not a Hilbert space.

We first define the singular values of a compact operator $T$ on $H$. Let $T^*$ be its adjoint, clearly $T^*T$ is a nonnegative self adjoint operator on $H$, so one could define its square root $A = \sqrt{T^*T}$ using functional calculus. Note that $T^*T$ and $A$ are both compact operators with nonnegative eigenvalues.

The singular values of $T$ are defined to be the positive eigenvalues of $A$, counting with multiplicity.
**Definition:** We say that $T$ is a trace class operator if the sum of its singular values is finite (counting with multiplicity). In that case the trace norm of $T$ is defined to be this sum, denoted by $\|T\|_{\text{tr}}$.

**Properties:** $T$ and $T^*$ has the same trace norm, and if $T$ is trace class then so is $TB$ and $BT$ where $B$ is any bounded operator on $H$, and $\|\cdot\|_{\text{tr}}$ satisfies the triangle inequality.

$$\|T_1 + T_2\|_{\text{tr}} \leq \|T_1\|_{\text{tr}} + \|T_2\|_{\text{tr}}$$

This inequality is an immediate consequence of the following equivalent characterization of the trace norm:

**Lemma 10.** For any trace class operator $T$

$$\|T\|_{\text{tr}} = \sup_{f_n,e_n} \sum_n |\langle Tf_n,e_n\rangle|$$

where the sup is taken over all $(f_n)$ and $(e_n)$ orthonormal bases of $H$.

To show this characterization we will first derive the polar factorization of $T$: namely there is a partial unitary operator $U$ such that $T = UA$, here unitary of $U$ simply means that $U^*U$ when acting on the range of $A$ is the identity operator. This should be thought of as the operator analogue of the usual polar factorization of a complex number.

To define $U$, note that $\|Au\| = \|Tu\|$ for all $u \in H$, therefore we may define an isometry $U$ from $\text{range}(A)$ to $\text{range}(T)$ by mapping $Au$ to $Tu$.

One then extends $U$ to $H$ by letting $U$ to be zero on the orthogonal complement of $\text{range}(A)$. It follows immediately that $U^*H \subset \overline{\text{range}(A)}$: one simply notice that for every $z$ in the orthogonal complement of $\text{range}(A)$ and $z' \in H$ it holds that $\langle z, U^*z' \rangle = \langle Uz, z' \rangle = 0$.

Now let $z$ in the closure of the range of $A$. We want to show that $(U^*U - 1)z = 0$, which is the desired local unitary property of $U$. Using the isometric property of $U$ on $\overline{\text{range}(A)}$, for every $z'$ in the closure of the range of $A$ we have

$$\langle z', z \rangle = \langle Uz', Uz \rangle = \langle z', U^*Uz \rangle$$

therefore $(U^*U - 1)z$ belongs to the orthogonal complement of $\overline{\text{range}(A)}$. But $U^*U - 1$ leaves $\text{range}(A)$ invariant since the range of $U^*$ is inside $\text{range}(A)$. This contradiction completes the proof of the local unitary property of $U$.

We are now back to proving the above equivalent characterization of the trace class norm of $T$. Let $(F_n)$ be a complete set of normalized eigenvectors of $A = \sqrt{T^*T}$. (Note that since $A^* = A$ we could find such a set.) Let $G_n = UF_n$. (Note that $\|F_n\| = \|G_n\| = 1$.) Then

$$\sum_n |\langle TF_n,G_n\rangle| = \sum_n |\langle UAF_n,UF_n\rangle| = \sum_n |\langle AF_n,F_n\rangle| = \|T\|_{\text{tr}}$$

therefore it remains to show that

$$\sum_n |\langle Tf_n,e_n\rangle| \leq \|T\|_{\text{tr}}$$
for any pair of orthogonal bases \((f_n)\) and \((e_n)\). Let \(s_n\) be the singular values of \(T\), namely \(AF_n = s_n F_n\). Then expand \(f_n\) into \((F_n)\) we have

\[ f_n = \sum_k \langle f_n, F_k \rangle F_k \]

and by an application of Fubini’s theorem

\[ \sum_n |\langle T f_n, e_n \rangle| \leq \sum_k |s_k| \sum_n |\langle f_n, F_k \rangle \langle G_k, e_n \rangle| \]

(it will be clear from the proof that the double sum is absolutely summable)

\[ \leq \sum_k |s_k| (\sum_n |\langle f_n, F_k \rangle|^2)^{1/2} (\sum_n |\langle G_k, e_n \rangle|^2)^{1/2} \]

\[ \leq \sum_k |s_k| \|F_k\| \|G_k\| = \sum_k |s_k| = \|T\|_{tr} \]

This completes the proof of the characterization. \(\square\)

**Trace:** Given a trace class operator one may define a linear functional, namely the trace of \(T\)

\[ \text{Trace}(T) = \sum_n \langle Tf_n, f_n \rangle \]

where \((f_n)\) is any orthonormal basis of \(H\). This definition is independent of the choice of the basis. Let \((g_n)\) be another orthonormal basis, then by expanding \(f_n\) into this new basis we have

\[ \sum_n \langle Tf_n, f_n \rangle = \sum_n \sum_k \langle f_n, g_k \rangle \langle Tg_k, f_n \rangle \]

it is not hard to see that this double sum is abs convergence (using Cauchy Schwartz). Then by Fubini

\[ \sum_n \langle Tf_n, f_n \rangle = \sum_k (\sum_n \langle f_n, g_k \rangle \langle Tg_k, f_n \rangle) \]

using orthogonality of \(f_n\) and normalization of \(f_n\) we obtain

\[ = \sum_k (\sum_n \sum_j \langle f_n, g_k \rangle \langle f_j, f_n \rangle \langle Tg_k, f_j \rangle) \]

\[ = \sum_k \langle Tg_k, g_k \rangle \]

By definition it is clear that \(|\text{Trace}(T)| \leq \|T\|_{tr}\) and the sum defining the trace is absolutely convergent.

Now, Lidskii’s trace formula says that
Theorem 42 (Lidskii). If \( T \) is trace class on a separable Hilbert space \( H \) then

\[
\text{Trace}(T) = \sum_j \lambda_j
\]

where \( \lambda_j \) are the eigenvalues of \( T \).

Note that \( T \) is compact so it has a countable set of nonzero eigenvalues. It is clear that the sum of the eigenvalues is bounded above by the trace norm.

Proof. The proof of this formula could be divided into two steps: first one consider the cases when \( T \) does not have any nonzero eigenvalues, and then in the second step we reduce the general case to this setting.

Let’s assume for now that \( T \) does not have any nonzero eigenvalues, then we want to show that \( \text{Trace}(T) = 0 \). Let \( s_1, s_2, \ldots \) be the singular values of \( T \). Then we’ll show that for every \( \lambda > 0 \)

\[
e^{\lambda \text{Trace}(T)} \leq O(1 + |\lambda|^M)e^{\lambda \sum_{j>M} s_j}
\]

(12.2)

(the implicit constant is independent of \( \lambda \)), from here by sending \( \lambda \rightarrow \infty \) we obtain \( |\text{Trace}(T)| \leq \sum_{j>M} s_j \); and the desired estimate now follows from the fact that \( \sum_j s_j = \|T\|_\text{tr} < \infty \). To show the above claim, we approximate \( T \) by finite rank operators, say \( T_n = P_nTP_n \rightarrow T \) where \( P_n \) is the projection into the first \( n \) basis vectors of \( H \), here one may fix any orthonormal basis. Let \( D_n(\lambda) = \det(1 + \lambda T_n) \), here the determinant is defined using linear algebra, in other words if \( \Lambda_n \) is the set of eigenvalues of \( T_n \), then \( D_n(\lambda) = \prod_{\alpha \in \Lambda_n}(1 + \lambda \alpha) \). We will show that uniformly on any compact subsets of the complex plane it holds that

\[
e^{\lambda \text{Trace}(T)} = \lim_{n \rightarrow \infty} D_n(\lambda)
\]

Note that by definition we have \( \|T_n - T\| \rightarrow 0 \) (in operator norm) and \( \text{Trace}(T_n) \rightarrow \text{Trace}(T) \). Since the spectral radius of \( T \) is 0 it is clear that the spectral radius \( \sigma(T_n) \) of \( T_n \) converges to 0 too. In particular given any bounded set of the complex plane we may choose \( n \) large such that this bounded set is contained inside the ball of radius \( 1/\sigma(T_n) \). Furthermore one could also show that \( s_j(T_n) \leq s_j(T) \) if the singular values of \( T \) and \( T_n \) are ordered in decreasing order.

Now, it is not hard to see that

\[
\frac{D'_n(\lambda)}{D_n(\lambda)} = \sum_{\alpha \in \Lambda_n} \frac{\alpha}{1 + \lambda \alpha}
\]

\[
= \text{Trace}(T_n) + O(\sum_{k \geq 2} (|\lambda|\sigma(T_n))^{k-1}||T_n||_\text{tr})
\]

therefore uniformly over \( \lambda \) in a bounded subset of \( \mathbb{C} \) it holds that

\[
\lim_{n \rightarrow \infty} \frac{D'_n(\lambda)}{D_n(\lambda)} - \text{Trace}(T) = 0
\]
12.3. Fredholm Determinant for Operators on Hilbert Spaces

which implies the desired limiting equality \([12.3]\).

Now using \([12.3]\), for any \(\lambda > 0\) we have

\[
e^{-|\text{Trace}(T)|} \leq \liminf_{n \to \infty} \prod_{\alpha \in \Lambda_n} (1 + \lambda |\alpha|)
\]

We will show that the last limit is bounded above by \(\prod_{j=1}^{\infty}(1 + \lambda s_j(T))\) where \(s_j(T)\) are the singular values of \(T\), which easily implies the desired estimate \([12.2]\). To show this, using \(s_j(T_n) \leq s_j(T)\) it suffices to show that

\[
\sum_{\alpha \in \Lambda_n} \log(1 + \lambda |\alpha|) \leq \sum_{j=1}^{n} \log(1 + \lambda s_j(T_n))
\]

Thanks to convexity of \(\log\), one could show that this inequality is a consequence of the fact that

\[
\prod_{\alpha \in \Lambda_n} |\alpha| \leq \prod_{j=1}^{N} s_j(T_n)
\]

which in turn could be easily verified (equality holds if \(T_n\) is nonsingular).

Now to reduce the general case to the above setting, we consider the subspace \(K_1\) of \(H\) spanned by the eigenfunctions and generalized eigenfunctions of \(T\). Let \(K_2\) be the orthogonal complement of \(K_1\). Using linear algebra it is clear that the trace of the restriction of \(T\) to \(K_1\) is exactly the sum of the eigenvalues of \(T\). More precisely if \((g_n)\) is a basis for \(K_2\) and \((f_m)\) is a basis for \(K_1\) then we may take the union of the two as a basis for \(H\) and

\[
\text{Trace}(T) = \sum_m \langle Tf_m, f_m \rangle + \sum_n \langle Tg_n, g_n \rangle
\]

thus using the previous argument it suffices to show that \(T^*\) leaves \(K_2\) invariant and the only eigenvalue of \(T^*\) on \(K_2\) is 0 (it is clear that \(T_2\) is also compact and trace class). These properties could be easily checked.

\(\square\)

**Determinant** We now discuss \(\det(1 + T)\) for trace class operators \(T\) on a separable Hilbert space \(H\).

Define an inner product on \(H^k\) by

\[
\langle (w_1, \ldots, w_k), (v_1, \ldots, v_k) \rangle = \det(\langle w_i, v_j \rangle)_{i,j}
\]

Then \(T\) extends to a trace class operator \(T_k\) on \(H^k\), defined by \(T_k(w_1, \ldots, w_k) = (Tw_1, \ldots, Tw_k)\), with \(\|T_k\|_{tr} \leq \|T\|_{tr}^k\). Then define

\[
\det(1 + T) = \sum_{k \geq 0} \text{Trace}(T_k)
\]
One could check that if $T$ is finite rank then this determinant is the same as what we would obtain from linear algebra, and $\det(1 + .)$ is locally Lipschitz wrt to the trace class norm. Approximating $T$ by a sequence of finite rank operators (for which $\det(1 + T)$ could be defined using standard linear algebra), we could show that $\det(1 + T)$ is the limit of the corresponding determinants, and thus this limit is independent of the choice of the sequence. Now, one could use the polar decomposition $T = UA$ and approximate $A$ by projections into finite dimensional subspaces of $H$ spanned by eigenfunctions of $A$. By this approximation scheme it can be shown that
\[
\det(1 + T) = \prod (1 + \lambda_j)
\]
where $\lambda_j$ are the eigenvalues of $T$ and
\[
\det[(1 + T_1)(1 + T_2)] = \det(1 + T_1) \det(1 + T_2)
\]
for any two trace class operators $T_1$ and $T_2$.

Using $\exp(T) = 1 + T + \cdots + \frac{T^n}{n!} + \cdots$ we could also define $\det(\exp T)$ too, and in fact $\det(\exp T) = e^{\Trace(T)}$.

As an example, if $T$ is an integral operator on some $L^2(X)$ with kernel $K(x, y)$ with mild assumptions on $K$ and $X$, then it could be shown that
\[
\Trace(T_k) = \frac{1}{k!} \int \cdots \int K \left( \begin{array}{c} x_1, \ x_2, \ \cdots \ \cdots \ x_k \\ x_1, \ x_2, \ \cdots \ x_k \end{array} \right) dx_1 \cdots dx_k
\]
and the definition of the determinant coincides with the previous section.