Chapter 10

Spectral theorems for bounded self-adjoint operators on a Hilbert space

Let $H$ be a Hilbert space. For a bounded operator $A : H \to H$ its Hilbert space adjoint is an operator $A^* : H \to H$ such that $\langle Ax, y \rangle = \langle x, A^* y \rangle$ for all $x, y \in H$. We say that $A$ is bounded self adjoint if $A = A^*$.

In this chapter we discussed several results about the spectrum of a bounded self adjoint operator on a Hilbert space. We emphasize that in this chapter $A$ is bounded, there is also a notion of unbounded self adjoint operator which we will discuss in subsequent chapters.

10.1 Diagonalization form

The first result says that $A$ could be diagonalized using some change of basis.

**Theorem 36.** Let $A : H \to H$ be a bounded self-adjoint operator on a Hilbert space $H$. Then there exists some $L^2(X, \mu)$ and $U : L^2(X, \mu) \to H$ isometric isomorphism such that for some bounded $M$ on $(X, \mu)$ and every $f \in L^2(X, \mu)$ it holds that

$$(U^{-1}AU)f(x) = M(x)f(x), \quad x \in X$$

**Proof.** We note that if $H$ is the direct sum of subspaces $H_1$ and $H_2$ such that $H_1$ and $H_2$ are invariant under $A$ then it suffices to prove the theorem for the restriction of $A$ to each subspace. This applies to direct sums indexed by larger index sets.

Now it is not hard to see that $H$ could be written as an orthogonal direct sum of subspaces of the form $\text{span}(A^n \xi, n \geq 0)$ where $\xi \in H$. (The proof uses Zorn’s lemma.) Note that these subspaces are invariant under $A$, therefore it suffices to show the theorem when $H = \text{span}(A^n \xi, n \geq 0)$ for some fixed $\xi$.

Now, recall from spectral calculus for bounded operators on a Banach space that if $f$ is analytic on a domain containing $\sigma(A)$ then we could define $f(A)$ and furthermore $\sigma(f(A)) =$
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\( f(\sigma(A)) \). For polynomials we could do this directly using factorization into linear factors for polynomials.

In the case of bounded self adjoint operator we will show below that \( \sigma(A) \subset \mathbb{R} \). (Note that this fact also holds for unbounded case, but we will not discuss that in this section.)

**Lemma 8.** Let \( A \) be bounded self adjoint on a complex Hilbert space \( H \). Then \( \sigma(A) \subset \mathbb{R} \).

To see this lemma, we will show that if \( \lambda \in \mathbb{C} \) has nonzero imaginary part then \( \lambda \in \rho(A) \). To do this, we will show that

\[
| \langle x, (\lambda - A)x \rangle | \geq c \|x\|^2 > 0
\]

for some \( c \) depending on \( \lambda \). This would imply that \( \lambda - A \) is invertible using an application of the Lax-Milgram theorem: consider the bilinear form \( B(x, y) = \langle x, (\lambda - A)y \rangle \), which is nondegenerate once we proved the above estimate, thus given any \( z \) we could find \( y \) such that \( \langle x, z \rangle = B(x, y) \) for all \( x \in H \), which implies that \( z = (\lambda - A)y \), thus \( \lambda - A \) is bijective on \( H \) and so is boundedly invertible and so \( \lambda \in \rho(A) \) as desired.

To show the above estimate, simply write \( \lambda = a + ib \) where \( a, b \in \mathbb{R} \) and \( b \neq 0 \), then using the self-adjoint property of \( A \) it follows that \( \langle x, (\lambda - A)x \rangle \) is a real number, therefore

\[
| \langle x, (a + ib - A)x \rangle | = | \langle x, (a - A)x \rangle + ib\|x\|^2 | \geq b\|x\|^2
\]

as desired. This completes the proof of the above lemma.

We now discuss functional calculus for bounded self adjoint operators. Note that since \( \sigma(A) \) is bounded and closed it will follow that \( \sigma(A) \) is a compact subset of \( \mathbb{R} \), and thus we could define \( f(A) \) even if \( f \) is merely continuous (which would be weaker than the analytic assumption required by the complex method generally applied to all bounded operators). The idea is to use the Weierstrass theorem and define \( f(A) \) to be the limit in operator norm of \( p_n(A) \) where \( (p_n) \) is a sequence of polynomials that approximates \( f \). To see that this could be done, it suffices to show that if \( g \) is a polynomial then

\[
\|g(A)\| = \sup_{x \in \sigma(A)} |g(x)|
\]

To see this last claim, we first show it for real polynomial. In fact we will consider \( g(x) = x \). Then as we proved before

\[
\sup_{x \in \sigma(A)} |g(x)| = \lim_{n} \|A^n\|^{1/n} \leq \|A\|
\]

while \( \|A\| \leq \sup_{x \in \sigma(A)} |g(x)| \) using either the spectral theorem, or by elementary methods.\[1\]

The real polynomial case then follows from the spectral mapping theorem \( \sigma(g(A)) = g(\sigma(A)) \)

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\[1\] Here we could easily see that \( \|Ax\|^2 = \langle A^2x, x \rangle \leq \|A^2\|\|x\|^2 \) thus by repeating we obtain \( \|A^n\|^2 - n \leq \|A\| \) as desired.
and the fact that \( g(A) \) which is self adjoint when \( g \) is real polynomial. To allow for complex polynomial \( p \), simply write
\[
\|p(A)\|^2 = \|p(A)^* p(A)\| = \|p(p^*) (A)\| = \sup_{\lambda \in \sigma(A)} (p^*) (A) = \sup_{\lambda \in \sigma(A)} |p(\lambda)|^2
\]

We note that as a consequence of the definition we also have

**Corollary 4.** For all continuous \( g \) on \( \sigma(A) \) it holds that \( \|g(A)\| = \sup_{x \in \sigma(A)} |g(x)| \).

Now, we may define a linear functional on the space of polynomials on \( \sigma(A) \) as follows: given such a polynomial \( p \), let \( L(p) = \langle p(A) \xi, \xi \rangle \), it is clear that \( \|L\| \leq \sup_{x \in \sigma(A)} |p(x)| \). Using Weierstrass’s theorem it follows that we could extend \( L \) to the space of continuous functions on \( \sigma(A) \). Using self-adjointness of \( A \) it is clear that \( L \) is positive: if \( f \geq 0 \) then write \( f = g^2 \) and \( g = \lim p_n \) limit of polynomials then \( L(f) = L(g^2) = \lim \langle p_n(A)^2 \xi, \xi \rangle = \lim \|p_n(A)\xi\|^2 \geq 0 \). Thus by the Riesz representation theorem we may write \( L(f) = \int f d\mu \) where \( \mu \) is some finite Borel measures on \( \sigma(A) \).

We now construct the operator \( U \), initially from the space of continuous functions on \( \sigma(A) \) to \( H \). If \( q \) is a polynomial then let \( Uq = q(A)\xi \), clearly
\[
\int_{\sigma(A)} |q(x)|^2 d\mu = L(qq^*) = \langle (qq^*) (A) \xi, \xi \rangle = \|q(A)\xi\|^2 = \|Uq\|^2
\]
thus the restriction of \( U \) to the polynomials is an isometry and the image of the polynomials under \( U \) is clearly dense inside \( H \) by the given assumption. Thus \( U \) extends to an isometric isomorphism from \( L^2(\sigma(A), d\mu) \) to \( H \).

Finally we will show that \( U^{-1} A Uf(x) = xf(x) \) for all \( f \in L^2 \), note that this equality is understood in the almost everywhere since with respect to \( \mu \), furthermore \( m(x) := x \) is bounded in \( \sigma(A) \) thanks to compactness of \( \sigma(A) \). Thanks to Weierstrass’s theorem again it suffices to show this equality for \( f \) being polynomials. In that case let \( g(x) = xf(x) \) also a polynomial, we then have
\[
U^{-1} A Uf = U^{-1} (Af(A)\xi) = U^{-1} (g(A)\xi) = g(x) = xf(x)
\]
\( \square \)

### 10.2 Projection-valued measure and spectral projection

Recall that \( P : H \to H \) is a projection is \( P^2 = P \). We say that it is an orthogonal projection if \( \ker(P) \) and \( \text{range}(P) \) are orthogonal subspaces, which would be the case if and only if \( P = P^* \) (the underlying assumption is that \( P \) is bounded). Recall a basic fact:

**Theorem 37.** Given any closed subspace \( K \) of \( H \) there is an orthogonal projection \( P_K \) onto \( K \): \( \text{range}(P) \) is \( K \) and \( \ker(P) \) is exactly \( K^\perp \).
In this section we take a closer look at the spectral representation of $A$.

We say that a family $P_\Omega$ indexed by Borel subsets $\Omega \subset \mathbb{R}$ if a (compactly supported) projection-valued measure if

(i) $P_\Omega$ is an orthogonal projection on $H$

(ii) $P = 0$ (and $P_{[-M,M]} = I$ for some $M$ sufficiently large).

(iii) If $\Omega = \bigcup_{n \geq 1} \Omega_n$ disjoint union then $P_\Omega \xi = \sum P_{\Omega_n} \xi$ convergence in the norm.

(iv) If $\Omega_1$ and $\Omega_2$ are Borel sets then $P_{\Omega_1} P_{\Omega_2} = P_{\Omega_1 \cap \Omega_2}$.

We note that property (iv) is a corollary of the first three properties.

We first note that we could construct a measure out of $(P_\Omega)$ when testing the projections on some vector $\phi \in H$. Namely let $\mu_{\phi_1,\phi_2}(\Omega) = \langle P_\Omega \phi_1, \phi_2 \rangle$ then this defines a compactly supported complex Borel measure on $\mathbb{R}$ with finite total mass, in fact it is not hard to see that

$$\|\mu_{\phi_1,\phi_2}\| \leq \|P_{\mathbb{R}}\| = 1$$

therefore we could integrate any bounded Borel measurable function $f$ and obtained a bounded bilinear form

$$T_f(\phi_1, \phi_2) = \int f(\lambda) d\mu_{\phi_1,\phi_2}(\lambda)$$

Then by Riesz representation theorem we may write $T_f(\phi_1, \phi_2) = \langle B \phi_1, \phi_2 \rangle$ where $B$ is a bounded linear operator on $H$. If we approximate $f$ using simple functions, it can be seen that $B$ is the limit in the weak operator topology of the corresponding linear combinations of $P_\Omega$ (note that this implies $B$ is the limit in the strong operator topology too). We will let $\int f(\lambda) dP_\lambda$ denote this operator $B$ and we think of this as the integration of $f$ over the measure induced by the family $(P_\Omega)$. In particular, $\int \lambda dP_\lambda$ is a bounded self adjoint operator on $H$.

It turns out that the converse of this is also true:

**Theorem 38.** Given a bounded self adjoint operator $A$ on a complex Hilbert space $H$ let $P_\Omega = 1_\Omega(A)$ then $(P_\Omega)$ is a compactly supported projection valued measure, and $\int f(\lambda) dP_\lambda$ converges to $f(A)$ (as defined by spectral caculus) in the strong operator norm ($T_j \to T$ means $\|T_j x - Tx\| \to 0$ for all $x$). Furthermore this is the unique projection valued operator with this property.

To check that $\int f(\lambda) dP_\lambda$ converges to $f(A)$ in strong operator norm topology, it suffices to show convergence in the weak operator norm, then it suffices to check that if $f = 1_\Omega$ then $\langle \int 1_\Omega dP_\lambda \phi_1, \phi_2 \rangle = \langle 1_\Omega(A) \phi_1, \phi_2 \rangle$ for all $\phi_1, \phi_2 \in H$. By definition the left hand side the same as $\int 1_\Omega dP_{\phi_1,\phi_2} = \langle P_\Omega \phi_1, \phi_2 \rangle$ which is the same as the right hand side.

### 10.3 Spectral representation and decomposition

**Absolutely continuous, singular, and point spectral**

Recall that $H$ has the spectral representation $L^2(X, \mu)$ which comes from direct summing the spectral representations $L^2(\mu_j)$ of $K_j$. We may decompose $\mu_j$ into three parts (absolutely
continuous, point, and singular parts) using the Radon Nikodym theorem. This leads to decomposition of \( K \) and also decomposition of \( H \) into the absolutely continuous, singular, and point spectrum \( H^{(p)}, H^{(s)}, \) and \( H^{(c)} \). Note that these are orthogonal subspaces and the corresponding spectral measure of the cyclic subspace has the inherited properties. Say if \( x \in H^{(c)} \) then the spectral measure of \( A \) on \( \text{span}(A^n x) \) is absolutely continuous with respect to the Lebesgue measure.

**Uniqueness of the spectrum** Let \( A \) be bounded self adjoint on \( H \) and assume that \( H \) be separable. Recall from prior sections that there is a decomposition of \( H \) into orthogonal direct sum of \( K_1, K_2, \ldots \) where \( K_j \) are orthogonal closed subspaces of \( H \) and each of them is invariant under \( A \), furthermore there is a linear isometry \( U_j \) mapping some \( L^2(\sigma(A), \mu_j) \) into \( K_j \) such that \( U_j^{-2} AU_j \) acts on this \( L^2 \) space by multiplication, in fact \( U_j^{-1} f(A) U_j g(x) = f(x)g(x) \) for all bounded measureable \( f \) and \( g \in L^2(\mu_j) \). One could certainly modify \( d\mu_j \) multiplicatively using a bounded positive function, this would affect the isometry part of the map \( U \) but the \( L^2 \) space remains the same and the corresponding action of \( f(A) \) is still pointwise multiplication.

Now if \( H \) has another decomposition into an orthogonal direction sum of closed subspaces \( L_1, L_2, \ldots \) with spectral representations \( L^2(T_j, \nu_j) \) for \( L_j \), then upto a set of (spectral) measure 0 we have \( \bigcup S_j = \bigcup T_k \), namely we will show that for any \( k \) the set difference \( T_k \setminus \bigcup S_j \) has zero measure under \( \nu_k \). Note that if we assume furthermore that these measures are absolutely continuous with respect to the Lebesgue measure then \( \bigcup S_j = \bigcup T_k \) up to set of Lebesgue measure 0.

To see this take \( f \in L^2(T_k, \nu_k) \), and let \( F \) be any bounded Borel measurable function. Note that \( S_j \) and \( T_k \) are invariant under \( F(A) \), and \( f \) coressponds to some \( h_f \in H \). We now decompose \( h_f \) into \( h_j \) where \( h_j \in K_j \), and \( h_j \) in turn corresponds to \( f_j \in L^2(S_j, \mu_j) \). Then

\[
\int |F(x)f(x)|^2 d\mu_k = \|h_f\|^2 = \|h_1\|^2 + \|h_2\|^2 + \cdots = \int |F(x)|^2 |f_1|^2 d\mu_1 + \int |F(x)|^2 |f_2(x)|^2 d\mu_2 + \cdots
\]

Therefore if \( T_k \setminus \bigcup S_j \) has positive \( \mu_k \) measure we may choose \( F \) to be the characteristic function of this set and \( f \equiv 1 \), clearly the left hand side is now positive while the right hand side is 0, contradiction.

**Spectral multiplicity**

Consider the spectral representation of \( H \), assumed separable, using the orthogonal direct sum of \( K_1, \ldots \) with \( K_j = L^2(S_j, \mu_j) \) where \( S_j \) is the support of \( \mu_j \).

Given \( \lambda \in \mathbb{R} \) we define its spectral multiplicity with respect to this representation as the number of \( j \) such that \( \lambda \in S_j \).

Assume for simplicity that the spectrum of \( A \) is absolutely continuous with respect to the Lebesgue measure. Then it can be shown that the spectral multiplicity is independent of the spectral representation of \( H \); this is a theorem of Hellinger.