1. Durfee Squares

Recall a classical binomial identity

\[
\binom{n+m}{n} = \sum_{j=0}^{n} \binom{n}{j} \binom{m}{j}.
\]

This can be quickly shown by choosing \( n \) representatives from a team of \( n + m \) people with \( n \) girls and \( m \) boys and noting \( \binom{n}{j} = \binom{n}{n-j} \). We have the following quantum version of this equation.

**Proposition 1.** We have

\[
\sum_{j=0}^{n} q^{j^2} \left[ \begin{array}{c} n \\ j \end{array} \right] \left[ \begin{array}{c} m \\ j \end{array} \right] = \left[ \begin{array}{c} n + m \\ n \end{array} \right].
\]

We will prove this using weighted partitions. But first, consider as an example the partition \( \lambda = (5, 4, 3, 3, 1, 1, 1) \), with Young diagram

```
  1 1 1
  1 1
  1 1 1
  1 1
  1
```

How large a square can we fit inside this partition? If the square is anchored against the upper-left corner, then the answer is clearly a square with side length 3. This square is called the *Durfee square* of the partition. We call the side-length of the Durfee square the partition’s *Durfee length*. If we pull the partition’s Durfee square away from the rest of the partition, we leave behind partitions to the right and below the square, as seen here.
Note that the partition to the right of the Durfee square has at most three parts, and the partition below the square has parts at most three. We now prove Proposition 1.

**Proof.** Recall the theorem from some previous lecture

\[
\sum_{\lambda \subseteq R_{n,m}} q^{\lambda} = \left[ n + m \right]_n
\]

and notice that the right-hand side is equal to the right-hand side of our proposition. We look for a formula for the weight of a partition \( \lambda \in R_{n,m} \) in terms of its Durfee square. Decompose \( \lambda \) into the triple \((j, \mu, \nu)\), where \( j \) is the Durfee length, \( \mu \) is the partition remaining to the right of the square, and \( \nu \) is the partition remaining below the square. Then we can rewrite the weight of \( \lambda \) as

\[
|\lambda| = j^2 + |\mu| + |\nu|.
\]

Since the Durfee square has \( j \) parts, \( \mu \) can have at most \( j \) parts, and \( \nu \) can have at most \( n - j \) parts. Since the Durfee square’s parts are all \( j \), the partition \( \mu \)'s parts are at most \( m - j \), and \( \nu \)'s parts are at most \( j \). That is, we have a bijection

\[
R_{n,m} \overset{1-1}{\leftrightarrow} \{(j, \mu, \nu) \mid 0 \leq j \leq n, \mu \in R_{n,m-j}, \nu \in R_{n-j,j}\}.
\]

Now compute

\[
\sum_{\lambda \subseteq R_{n,m}} q^{\lambda} = \sum_{j=0}^{n} q^{j^2} \sum_{\mu \in R_{n,m-j}} q^{\mu} \sum_{\nu \in R_{n-j,j}} q^{\nu} = \sum_{j=0}^{n} q^{j^2} \left[ m \right] \left[ n \right],
\]

where the last line follows from Equation (1.1) and simplification. \( \square \)

If we take the limits \( n \to \infty \) and \( m \to \infty \) in Proposition 1, we obtain the identity

\[
\sum_{j=0}^{\infty} \frac{q^{j^2}}{(1 - q)(1 - q^2)\cdots(1 - q^j)^2} = \prod_{n=1}^{\infty} \frac{1}{1 - q^n}.
\]

Interpreting this in terms of generating functions, we see that the right-hand side is the generating function for all partitions, while the left-hand side is the generating function for partitions in terms of their Durfee square decomposition. So the identity (1.2) follows directly from such a combinatorial argument (similar to the proof of Proposition 1) as well.
2. Frobenius Symbols

Now we consider an alternate way to decompose a partition according to its Durfee square. This time, we cut the diagram down the diagonal of the Durfee square, and throw away the half boxes. As an example, consider the cut on the previous example partition.

Note that although the boxes remaining after the cut do not form Young diagrams, they can easily be realigned so that they do. We take the conjugate of the partition \((3, 2, 1, 1, 1, 1)\) so that it has at most three parts; this allows us to write

\[(5, 4, 3, 1, 1, 1) \leftrightarrow (\begin{array}{c} 4 \\ 2 \\ 0 \\ 6 \\ 2 \\ 1 \end{array})\,.

The two-rowed array on the right is called a Frobenius symbol.

Because the parts of a partition are weakly decreasing, by performing this cut the parts of the two new partitions will be strictly decreasing. In general, given a partition \(\lambda\) of Durfee length \(s\), we can write

\[\lambda \leftrightarrow \left(\begin{array}{c} a_1 \\ b_1 \\ a_2 \\ b_2 \\ \cdots \\ a_s \\ b_s \end{array}\right)\]

where the integer sequences \((a_i)\) and \((b_i)\) are strictly decreasing and non-negative. We also have the identity

\[(2.1) \quad |\lambda| = s + \sum a_i + \sum b_i.\]

Define the polynomials

\[J_+(z) = (1 + (zq^0)q^0)(1 + (zq^1)q^1)(1 + (zq^2)q^2)\ldots,\]

\[J_-(z) = (1 + z^{-1}q^0)(1 + z^{-1}q^1)(1 + z^{-1}q^2)\ldots\]

and set

\[(2.2) \quad J(z) = J_+(z) \cdot J_-(z).\]

We use the Durfee square decomposition described above to find a formula for the constant term of \(J(z)\). We denote this constant term by \([z^0]J(z)\) (and the coefficient of \(z^k\) by \([z^k]J(z)\), for general \(k\)).

**Proposition 2.** We have

\[[z^0]J(z) = \prod_{n=1}^{\infty} \frac{1}{1 - q^n}.\]
Proof. Since $J_+$ contributes positive powers of $z$ and $J_-$ contributes negative powers, the expressions that contribute to the constant term are products of an equal number of terms, say $s$, from $J_+$ and $J_-$. Pick $s$ terms from $J_+$ of the form $(zq)^{a_i}$, so that $a_i$ is strictly decreasing. Similarly, pick terms from $J_-$ of the form $z^{-1}q^{b_i}$, again with $b_i$ decreasing. Then $J_+$ contributes the constant $q^{s}q^{a_1+a_2+...+a_s}$ and $J_-$ contributes $q^{b_1+b_2+...+b_s}$. Using Equation (2.1), this gives us

$$[z^0]J(z) = \sum_{s=0}^{\infty} \sum_{a_1 > ... > a_s \geq 0 \atop b_1 > ... > b_s \geq 0} q^{s+a_1+...+a_s+b_1+...+b_s} = \sum_{\lambda} q^{\lambda} = \prod_{n=1}^{\infty} \frac{1}{1-q^n}. \quad \square$$

3. Jacobi’s Triple Product Identity

We have a number of “sum = product” identities. Another such identity, given by Jacobi, is

$$(3.1) \sum_{n \in \mathbb{Z}} z^n q^{\frac{n(n+1)}{2}} = \prod_{n=1}^{\infty} (1 - q^n)(1 + zq^n)(1 + z^{-1}q^{n-1}).$$

This equation is known as Jacobi’s triple product identity. To prove it, we start with the following lemma.

Lemma 3. We have $J(qz) = q^{-1}z^{-1}J(z)$.

Proof. This follows from the following computation:

$$J(qz) = \prod_{n=1}^{\infty} (1 + zq^{n+1})(1 + z^{-1}q^{n-2})$$

$$= \frac{1}{1 + zq} \prod_{n=0}^{\infty} (1 + zq^{n+1}) \cdot (1 + z^{-1}q^{n-1}) \prod_{n=2}^{\infty} (1 + z^{-1}q^{n-2})$$

$$= q^{-1}z^{-1}J_+(z)J_-(z) = q^{-1}z^{-1}J(z). \quad \square$$

We now prove the following reformulation of the Jacobi triple product identity (3.1).

Proposition 4. We have

$$\left( \sum_{n \in \mathbb{Z}} z^n q^{\frac{n(n+1)}{2}} \right) \left( \prod_{n=1}^{\infty} \frac{1}{1-q^n} \right) = \prod_{n=1}^{\infty} (1 + zq^n)(1 + z^{-1}q^{n-1}).$$

Proof. Notice that the right-hand side of the identity is equal to $J(z)$ given in (2.2). For $n \geq \mathbb{Z}$, we define

$$A_n(q) = [z^n]J(z).$$

That is, we have

$$J(z) = \sum_{n \in \mathbb{Z}} A_n(q)z^n.$$
Then by expanding both sides of Lemma 3 in a Laurent series, we find
\[ \sum_{n \in \mathbb{Z}} A_n(q) q^n z^n = q^{-1} \sum_{n \in \mathbb{Z}} A_n(q) z^{n-1}. \]
For these Laurent polynomials to be equal, they must be equal term-by-term. That is,
\[ q^n A_n(q) = q^{-1} A_{n+1}(q). \]
Then by induction on \( n \), we have that, for \( n \in \mathbb{Z} \),
\[ A_n(q) = q^{\frac{n(n+1)}{2}} A_0(q). \]
Since Proposition 2 gives us a formula for \( A_0(q) \), we have that
\[ J(z) = \sum_{n \in \mathbb{Z}} z^n q^{\frac{n(n+1)}{2}} A_0(q) = \left( \sum_{n \in \mathbb{Z}} z^n q^{\frac{n(n+1)}{2}} \right) \left( \prod_{n=1}^{\infty} \frac{1}{1 - q^n} \right). \]
\[ \square \]
Note that if we specialize this identity to one variable by making the transformations
\[ q \mapsto q^3 \quad z \mapsto -q^{-1} \]
we recover Euler’s pentagonal number identity.