The Tutte polynomial relations for planar and surface graphs

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The *chromatic polynomial* $\chi_\Gamma(Q)$ of a graph Γ, for $Q \in \mathbb{Z}_+$, is the number of colorings of the vertices of Γ with the colors $1, \ldots, Q$ where no two adjacent vertices have the same color.

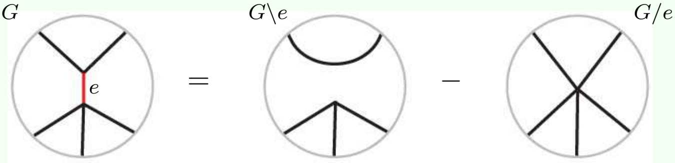
- $$\chi_\Gamma(Q) = \sum_{S \subset \{\text{edges of } \Gamma\}} (-1)^{|S|} \, Q^{k(S)}$$

where $k(S)$ is the number of connected components of the graph which has the same vertices as Γ and whose edge set is given by S.

- The contraction-deletion rule: given any edge e of Γ which is not a loop,

$$\chi_\Gamma(Q) = \chi_{\Gamma\setminus e}(Q) - \chi_{\Gamma/e}(Q)$$

If Γ contains a loop then $\chi_\Gamma \equiv 0$.
If Γ has no edges and V vertices, then $\chi_\Gamma(Q) = Q^V$.



W.T. Tutte (1969):

the "golden identity": for a planar triangulation T,

$$\chi_T(\phi+2) = (\phi+2)\,\phi^{3\,V(T)-10}\,(\chi_T(\phi+1))^2,$$

where $V(T)$ is the number of vertices of the triangulation.

ϕ denotes the golden ratio, $\phi = \frac{1+\sqrt{5}}{2}$.

Another Tutte's relation:

$$\chi_{Z_1}(\phi+1)+\chi_{Z_2}(\phi+1)=\phi^{-3}[\chi_{Y_1}(\phi+1)+\chi_{Y_2}(\phi+1)],$$

where Y_i, Z_i are planar graphs which are locally related as follows:

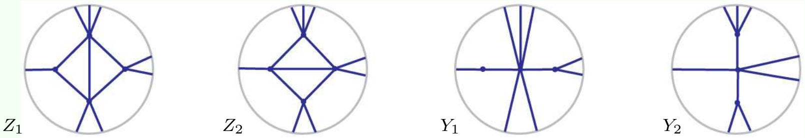


Figure:

Reference: P. Fendley and V. Krushkal, *Tutte chromatic identities from the Temperley-Lieb algebra*, Geometry and Topolology 13(2009), 709-741 [arXiv:0711.0016]

Outline:

Define the chromatic algebra $\mathcal{C}_n^Q$.

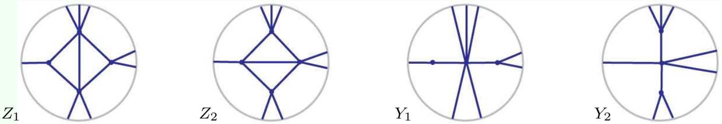
Basic idea: consider the contraction-deletion rule as a linear relation in the vector space spanned by graphs, rather than just a relation defining the chromatic polynomial.

The Markov trace of a graph is the chromatic polynomial of its dual.

Identities such as Tutte's can then be understood as finding elements of the *trace radical*: elements of the chromatic algebra which, multiplied by any other element of the algebra, are in the kernel of the Markov trace.

Tutte's polynomial relation:

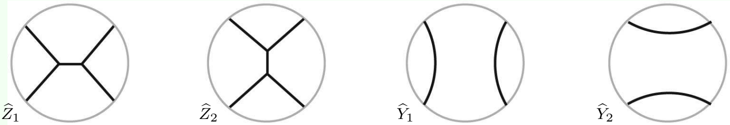
$$\chi_{Z_1}(\phi+1) + \chi_{Z_2}(\phi+1) = \phi^{-3}[\chi_{Y_1}(\phi+1) + \chi_{Y_2}(\phi+1)]$$



Prove that the relation

$$\widehat{Z}_1 + \widehat{Z}_2 = \phi^{-3}[\widehat{Y}_1 + \widehat{Y}_2]$$

holds in the chromatic algebra $\mathcal{C}_2^{\phi+1}$:



Outline:

Construct a map: chromatic algebra $\longrightarrow$ Temperley-Lieb algebra:

$$\mathcal{C}_n^Q \longrightarrow TL_{2n}^d, \quad Q = d^2$$

The trace radical in the Temerley-Lieb algebra is well-understood: the Jones-Wenzl projectors at special values of d.

Pull them back to get elements in the trace radical of the chromatic algebra.

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Result: A generalization of Tutte's relation for the chromatic polynomial at $Q = 2 + 2\cos\left(\frac{2\pi j}{n+1}\right)$. Recursive formula.

These values of Q: $Q = 2 + 2\cos\left(\frac{2\pi j}{n+1}\right)$ are generalizations of

Beraha numbers: $B_n = 2 + 2\cos\left(\frac{2\pi}{n+1}\right) \quad (B_5 = \phi + 1)$.

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Beraha experimentally observed (in the 1970s) that the zeros of the chromatic polynomial of large planar triangulations seem to accumulate near these numbers (B_n).

Another Tutte's result:

$$|\chi_T(\phi + 1)| \leq \phi^{5-k}$$

where T is a planar triangulation and k is the number of its vertices.

Analogue for other Beraha numbers??

The Temperley-Lieb algebra in degree n, TL_n, is an algebra over $\mathbb{C}[d]$ generated by $1, E_1, \ldots, E_{n-1}$ with the relations

$$E_i^2 = E_i, \qquad E_i E_{i\pm1} E_i = \frac{1}{d^2}\, E_i, \qquad E_i E_j = E_j E_i \text{ for } |i-j| > 1.$$

Define $TL = \cup_n TL_n$. The indeterminate d may be specialized to a complex number, and then it is denoted TL_n^d.

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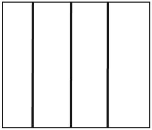
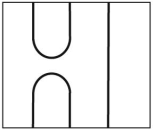
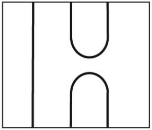
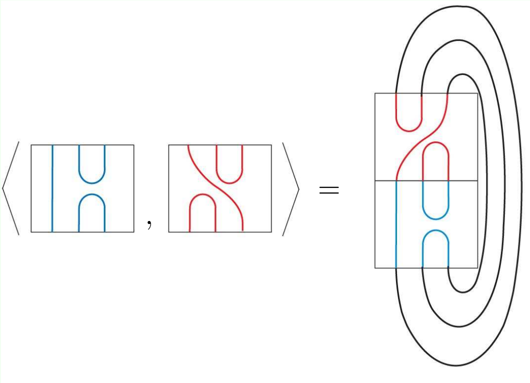
Pictorially, an element of TL_n is a linear combination of $1-$dimensional submanifolds in a rectangle R. Each submanifold meets both the top and the bottom of the rectangle in exactly n points. The multiplication corresponds to vertical stacking of rectangles. Generators:

$1 =$  $E_1 = \frac{1}{d}$  $E_2 = \frac{1}{d}$ 

Relation: Any circle in a picture may be erased and then the element in the algebra is multiplied by d

The trace tr_d: $TL_n^d \longrightarrow \mathbb{C}$ is defined on rectangular pictures by connecting the top and bottom endpoints by disjoint arcs and evaluating $d^{\#circles}$.

The scalar product on TL_n is defined by $\langle a, b \rangle = tr(a\,\bar{b})$.



$= d^2$

The chromatic algebra is defined as isotopy classes of graphs in a rectangle modulo local relations:

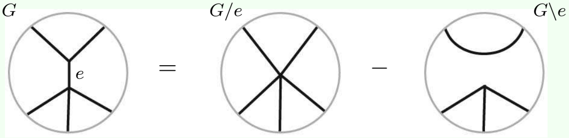


Figure: Relation (1) in the chromatic algebra

Figure: Relations (2), (3) in the chromatic algebra

Consider the set $\mathcal{G}_n$ of the isotopy classes of planar graphs G embedded in the rectangle R with n endpoints at the top and n endpoints at the bottom of the rectangle.

Let $\mathcal{F}_n$ denote the free algebra over $\mathbb{C}[Q]$ with free additive generators given by the elements of $\mathcal{G}_n$. The multiplication is given by vertical stacking. Define $\mathcal{F} = \cup_n \mathcal{F}_n$.

The *chromatic algebra* in degree n, $\overline{\mathcal{C}}_n$, is an algebra over $\mathbb{C}[Q]$ which is defined as the quotient of the free algebra $\mathcal{F}_n$ by the ideal I_n generated by the relations (1), (2), (3).

(1) If e is an inner edge of a graph G which is not a loop, then $G = G/e - G\backslash e$.
(2) If G contains an inner edge e which is a loop, then $G = (Q-1)\, G\backslash e$.
(3) If G contains a $1-$valent vertex (in the interior of the rectangle), then $G = 0$.

The trace, $tr_\chi \colon \bar{\mathcal{C}}^Q \longrightarrow \mathbb{C}$ is defined on the additive generators (graphs) G by connecting the endpoints of G by arcs in the plane (denote the result by $\overline{G}$) and evaluating

$$Q^{-1} \cdot \chi_{\widehat{\overline{G}}}(Q).$$

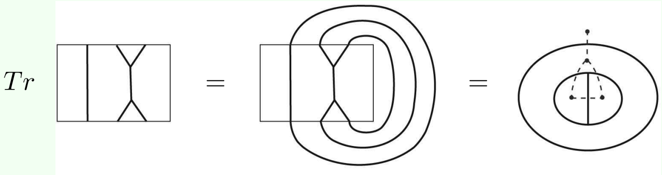


Figure: An example of the evaluation of the trace: The trace $= (Q-1)^2(Q-2)$.

There is a presentation of the chromatic algebra in terms of trivalent graphs:

$\mathcal{C}_n^Q$ is isomorphic to the algebra generated by trivalent graph in a rectangle, modulo local relations:

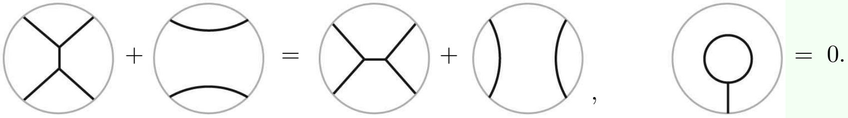


Figure: Relations in the trivalent presentation of the chromatic algebra.

Consider the algebra homomorphism $\Phi\colon \mathcal{C}_n^{d^2} \longrightarrow TL_{2n}^d$:

The factor in the definition of Φ corresponding to a $k-$valent vertex is $d^{(k-2)/2}$. The overall factor for a graph G is the product of the factors $d^{(k(V)-2)/2}$ over all vertices V of G.

Let G be a planar graph. Then

$$Q^{-1}\,\chi_Q(\widehat{G}) \,=\, \Phi(G)$$

Here $Q = d^2$. Therefore, the following diagram commutes:

$$\begin{array}{ccc} \mathcal{C}_n^Q & \xrightarrow{\ \Phi\ } & TL_{2n}^d \\ \Big\downarrow {\scriptstyle tr_\chi} & & \Big\downarrow {\scriptstyle tr_d} \\ \mathbb{C} & \xrightarrow{\ =\ } & \mathbb{C} \end{array}$$

For example, for the theta-graph G,

$$Q^{-1}\chi_Q(\widehat{G}) \,=\, (Q-1)(Q-2) \,=\, d^4 - 3d^2 - 4 \,=\, \Phi(G).$$

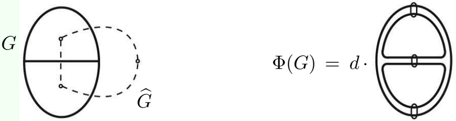
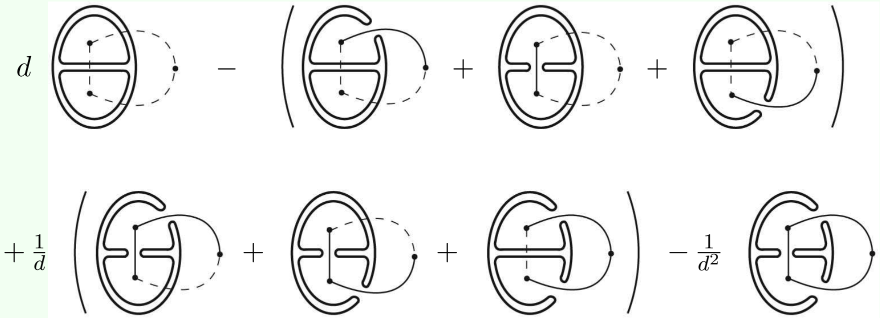


Figure:



The expansions of $Q^{-1}\,\chi_Q(\widehat{G})$, $\Phi(G)$ where G is the theta graph.

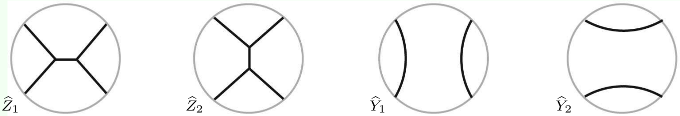
The trace radical of an algebra A consists of elements a such that $\langle a, b \rangle = 0$ for all b in A.

Corollary: The pullback of the trace radical in TL^{d}_{2n} to $\mathcal{C}^{d^2}_{n}$ is in the trace radical of the chromatic algebra.

The trace radical of the Temperley-Lieb algebra is well-understood (Jones, Wenzl, Goodman):

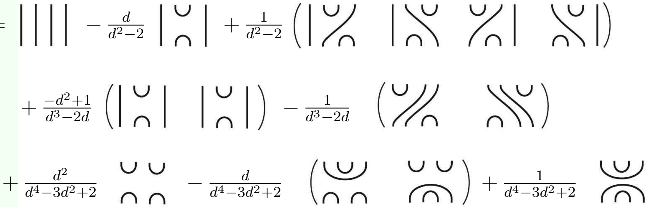
It is non-trivial precisely for $d = 2\,cos(\pi j/n)$, and for these values it is generated by the Jones-Wenzl projector.

$$\widehat{Z}_1 + \widehat{Z}_2 = \phi^{-3}\,[\,\widehat{Y}_1 + \widehat{Y}_2\,]$$



Φ maps the dual of Tutte's relation to the 4-th Jones-Wenzl relation (at $d = \phi$):

$P^{(4)} =$



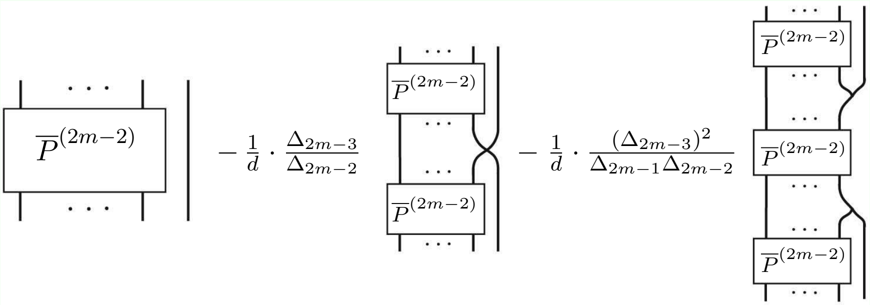


Figure: A recursive formula for the pull-back $\overline{P}^{(2m)}$ of the Jones-Wenzl projector $P^{(2m)}$ in the chromatic algebra.

Theorem

For a planar triangulation $\widehat{G}$*,*

$$\chi_{\widehat{G}}(\phi+2) = (\phi+2)\,\phi^{3\,V(\widehat{G})-10}\,(\chi_{\widehat{G}}(\phi+1))^2 \qquad (1)$$

where $V(\widehat{G})$ *is the number of vertices of* $\widehat{G}$*.*

Theorem

For a planar triangulation $\widehat{G}$,

$$\chi_{\widehat{G}}(\phi+2) = (\phi+2)\,\phi^{3\,V(\widehat{G})-10}\,(\chi_{\widehat{G}}(\phi+1))^2 \qquad (2)$$

where $V(\widehat{G})$ is the number of vertices of $\widehat{G}$.

Idea of the proof: Construct a map

$$\mathcal{C}^{\phi+2} \longrightarrow \mathcal{C}^{\phi+1} \times \mathcal{C}^{\phi+1}$$

and apply the trace:

$$\begin{array}{ccc} \mathcal{C}^{\phi+2} & \longrightarrow & (\mathcal{C}^{\phi+1}/R) \times (\mathcal{C}^{\phi+1}/R) \\ \downarrow & & \downarrow \\ \mathbb{C} & \xrightarrow{=} & \mathbb{C} \end{array}$$

More conceptually, Tutte's golden identity is a consequence of level-rank duality for $SO(N)$ topological quantum field theories.

In particular, the level-rank duality implies that the $SO(3)_4$ and $SO(4)_3$ theories are isomorphic, and the latter splits into a product of two copies of $SO(3)_{3/2}$,

$$SO(3)_4 \longrightarrow SO(3)_{3/2} \otimes SO(3)_{3/2}$$

The partition function of an $SO(3)$ theory is given in terms of the chromatic polynomial, specifically $\chi(\phi+2)$ for $SO(3)_4$ and $\chi(\phi+1)$ for $SO(3)_{3/2}$.

The Tutte polynomial of a graph G:

$$T_G(X,Y) = \sum_{H \subset G} X^{c(H)-c(G)}\, Y^{n(H)}.$$

The summation is taken over all spanning subgraphs H of G.

$c(H)$ denotes the number of connected components of the graph H, and $n(H)$ is the *nullity* of H, defined as the rank of the first homology group $H_1(H)$.

($n(H)$ may also be computed as $c(H) + e(H) - v(H)$, where e and v denote the number of edges and vertices of H, respectively.)

Basic properties of the Tutte polynomial: the contraction-deletion rule, and the duality

$$T_G(X,Y) = T_{G^*}(Y,X)$$

where G is a planar graph, and G^* is its dual.

(The vertices of G^* correspond to the connected regions in the complement of G in the plane, and two vertices are connected by an edge in G^* whenever the two corresponding regions are adjacent.)

Now suppose G is a ribbon graph (a graph embedded in a surface Σ). Consider the polynomial

$$P_{G,\Sigma}(X,Y,A,B) \;=\; \sum_{H\subset G} X^{c(H)-c(G)}\; Y^{k(H)}\; A^{s(H)/2}\; B^{s^{\perp}(H)/2}$$

Here $s(H)$ dentes the genus of the surface obtained as a regular neighborhood of the graph H in Σ, and $s^{\perp}(H)$ is the genus of the surface obtained by removing a regular neighborhood of H from Σ. Denote by i the embedding $G \longrightarrow \Sigma$, and define

$$k(H) := \dim\left(\ker\left(i_*\colon\; H_1(H;\mathbb{R}) \longrightarrow H_1(\Sigma;\mathbb{R})\right)\right).$$

The polynomial P satisfies the contraction-deletion rule,

$$P_G = P_{G \smallsetminus e} + P_{G/e},$$

and it satisfies a duality relation, analogous to the duality of the Tutte polynomial os planar graphs:

$$P_G\,(X, Y, A, B) \; = \; P_{G^*}\,(Y, X, B, A).$$

Reference: V. Krushkal, *Graphs, links, and duality on surfaces*, arXiv:0903.5312

The well-known Bollobás-Riordan polynomial of ribbon graphs is defined by

$$BR_{G,S}(X,Y,Z) \;=\; \sum_{H\subset G} (X-1)^{r(G)-r(H)}\, y^{n(H)}\, Z^{c(H)-bc(H)+n(H)}.$$

Let $v(H), e(H)$ denote the number of vertices, respectively edges, of H, and let $c(H)$ be the number of connected components. Then $r(H) = v(G) - c(H)$, $n(H) = e(H) - r(H)$, and $bc(H)$ is the number of boundary components of the surface S.

The polynomial $BR_{G,S}$ is a universal polynomial of ribbon graphs, satisfying the contraction-deletion rule.

The Bollobás-Riordan polynomial of a ribbon graph may be obtained as a specialization of the polynomial P_G:

$$BR_{G,S}(X,Y,Z) \;=\; Y^g\, P_{G,\Sigma}(X-1, Y, YZ^2, Y^{-1}).$$