The Tutte polynomial relations for planar and surface graphs

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The chromatic polynomial $\chi_{\Gamma}(Q)$ of a graph $\Gamma$, for $Q \in \mathbb{Z}_{+}$, is the number of colorings of the vertices of $\Gamma$ with the colors $1, \ldots, Q$ where no two adjacent vertices have the same color.

$$\chi_{\Gamma}(Q) = \sum_{S \subseteq \{\text{edges of } \Gamma\}} (-1)^{|S|} Q^{k(S)}$$

where $k(S)$ is the number of connected components of the graph which has the same vertices as $\Gamma$ and whose edge set is given by $S$. 
• The contraction-deletion rule: given any edge $e$ of $\Gamma$ which is not a loop,

$$
\chi_\Gamma(Q) = \chi_{\Gamma\setminus e}(Q) - \chi_{\Gamma/e}(Q)
$$

If $\Gamma$ contains a loop then $\chi_\Gamma \equiv 0$.

If $\Gamma$ has no edges and $V$ vertices, then $\chi_\Gamma(Q) = Q^V$. 

\begin{align*}
G & = G\setminus e & \quad & - & \quad G/e
\end{align*}
W.T. Tutte (1969):

the “golden identity”: for a planar triangulation \( T \),

\[
\chi_T(\phi + 2) = (\phi + 2) \phi^3 V(T) - 10 \chi_T(\phi + 1)^2,
\]

where \( V(T) \) is the number of vertices of the triangulation.

\( \phi \) denotes the golden ratio, \( \phi = \frac{1+\sqrt{5}}{2} \).
Another Tutte’s relation:

\[ \chi_{Z_1}(\phi + 1) + \chi_{Z_2}(\phi + 1) = \phi^{-3}[\chi_{Y_1}(\phi + 1) + \chi_{Y_2}(\phi + 1)], \]

where \( Y_i, Z_i \) are planar graphs which are locally related as follows:

Figure:

Outline:

Define the chromatic algebra $C_n^Q$.

Basic idea: consider the contraction-deletion rule as a linear relation in the vector space spanned by graphs, rather than just a relation defining the chromatic polynomial.

The Markov trace of a graph is the chromatic polynomial of its dual.

Identities such as Tutte’s can then be understood as finding elements of the trace radical: elements of the chromatic algebra which, multiplied by any other element of the algebra, are in the kernel of the Markov trace.
Tutte’s polynomial relation:

\[ \chi_{Z_1}(\phi + 1) + \chi_{Z_2}(\phi + 1) = \phi^{-3}[\chi_{Y_1}(\phi + 1) + \chi_{Y_2}(\phi + 1)] \]

Prove that the relation

\[ \widehat{Z}_1 + \widehat{Z}_2 = \phi^{-3} [\widehat{Y}_1 + \widehat{Y}_2] \]

holds in the chromatic algebra \( C_2^{\phi+1} \):
Outline:

Construct a map: \textbf{chromatic algebra} \longrightarrow \textbf{Temperley-Lieb algebra}:

\[ C_Q^Q \longrightarrow TL^d_{2n}, \quad Q = d^2 \]

The trace radical in the Temperley-Lieb algebra is well-understood: the \textit{Jones-Wenzl projectors} at special values of \( d \).

Pull them back to get elements in the trace radical of the chromatic algebra.
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Result: A generalization of Tutte’s relation for the chromatic polynomial at \( Q = 2 + 2 \cos \left( \frac{2\pi j}{n+1} \right) \). Recursive formula.
These values of $Q$: $Q = 2 + 2 \cos \left( \frac{2\pi j}{n+1} \right)$ are generalizations of Beraha numbers: $B_n = 2 + 2 \cos \left( \frac{2\pi}{n+1} \right)$ ($B_5 = \phi + 1$).

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Beraha experimentally observed (in the 1970s) that the zeros of the chromatic polynomial of large planar triangulations seem to accumulate near these numbers ($B_n$).

Another Tutte’s result:

$$|\chi_T(\phi + 1)| \leq \phi^{5-k}$$

where $T$ is a planar triangulation and $k$ is the number of its vertices.

An analogue for other Beraha numbers??
The **Temperley-Lieb algebra** in degree $n$, $TL_n$, is an algebra over $\mathbb{C}[d]$ generated by $1, E_1, \ldots, E_{n-1}$ with the relations

\[
E_i^2 = E_i, \quad E_i E_{i\pm1} E_i = \frac{1}{d^2} E_i, \quad E_i E_j = E_j E_i \text{ for } |i-j| > 1.
\]

Define $TL = \bigcup_n TL_n$. The indeterminate $d$ may be specialized to a complex number, and then it is denoted $TL_n^d$. 
The Temperley-Lieb algebra in degree \( n \), \( T\!L_n \), is an algebra over \( \mathbb{C}[d] \) generated by \( 1, E_1, \ldots, E_{n-1} \) with the relations

\[
E_i^2 = E_i, \quad E_iE_{i\pm 1}E_i = \frac{1}{d^2} E_i, \quad E_iE_j = E_jE_i \quad \text{for} \quad |i-j| > 1.
\]

Define \( T\!L = \bigcup_n T\!L_n \). The indeterminate \( d \) may be specialized to a complex number, and then it is denoted \( T\!L_d^n \).

Pictorially, an element of \( T\!L_n \) is a linear combination of \( 1 \)–dimensional submanifolds in a rectangle \( R \). Each submanifold meets both the top and the bottom of the rectangle in exactly \( n \) points. The multiplication corresponds to vertical stacking of rectangles. Generators:

\[
1 = \quad E_1 = \frac{1}{d} \quad E_2 = \frac{1}{d}
\]
Relation: Any circle in a picture may be erased and then the element in the algebra is multiplied by $d$

The trace $\text{tr}_d: TL_n^d \longrightarrow \mathbb{C}$ is defined on rectangular pictures by connecting the top and bottom endpoints by disjoint arcs and evaluating $d\#\text{circles}$.

The scalar product on $TL_n$ is defined by $\langle a, b \rangle = \text{tr}(a \bar{b})$. 
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\[ = d^2 \]
The chromatic algebra is defined as isotopy classes of graphs in a rectangle modulo local relations:

\[
G = G/e - G/e\]

\[
G/e = (Q - 1) \cdot (Q - 1)
\]

\[
Q = 0.
\]

**Figure:** Relation (1) in the chromatic algebra

**Figure:** Relations (2), (3) in the chromatic algebra
Consider the set $G_n$ of the isotopy classes of planar graphs $G$ embedded in the rectangle $R$ with $n$ endpoints at the top and $n$ endpoints at the bottom of the rectangle.

Let $F_n$ denote the free algebra over $\mathbb{C}[Q]$ with free additive generators given by the elements of $G_n$. The multiplication is given by vertical stacking. Define $F = \bigcup_n F_n$.

The *chromatic algebra* in degree $n$, $\overline{C}_n$, is an algebra over $\mathbb{C}[Q]$ which is defined as the quotient of the free algebra $F_n$ by the ideal $I_n$ generated by the relations (1), (2), (3).

1. If $e$ is an inner edge of a graph $G$ which is not a loop, then $G = G/e - G\setminus e$.
2. If $G$ contains an inner edge $e$ which is a loop, then $G = (Q - 1) G\setminus e$.
3. If $G$ contains a 1-valent vertex (in the interior of the rectangle), then $G = 0$. 
The trace, \( \text{tr}_\chi: \bar{C}^Q \rightarrow \mathbb{C} \) is defined on the additive generators (graphs) \( G \) by connecting the endpoints of \( G \) by arcs in the plane (denote the result by \( \bar{G} \)) and evaluating

\[
Q^{-1} \cdot \chi_{\bar{G}}(Q).
\]

Figure: An example of the evaluation of the trace: The trace

\[
= (Q - 1)^2(Q - 2).
\]
There is a presentation of the chromatic algebra in terms of \textit{trivalent graphs}:

\[ \mathcal{C}_n^Q \text{ is isomorphic to the algebra generated by trivalent graph in a rectangle, modulo local relations:} \]

\[ + = + = 0. \]

\textbf{Figure:} Relations in the trivalent presentation of the chromatic algebra.
Consider the algebra homomorphism \( \Phi : C_n^{d^2} \rightarrow TL_{2n}^d \):

\[
\Phi = -\frac{1}{d}.
\]

The factor in the definition of \( \Phi \) corresponding to a \( k \)-valent vertex is \( d^{(k-2)/2} \). The overall factor for a graph \( G \) is the product of the factors \( d^{(k(V)-2)/2} \) over all vertices \( V \) of \( G \).
Let $G$ be a planar graph. Then

$$Q^{-1} \chi_Q(\hat{G}) = \Phi(G)$$

Here $Q = d^2$. Therefore, the following diagram commutes:

For example, for the theta-graph $G$,

$$Q^{-1} \chi_Q(\hat{G}) = (Q - 1)(Q - 2) = d^4 - 3d^2 - 4 = \Phi(G).$$
The expansions of $Q^{-1} \chi_Q(\hat{G})$, $\Phi(G)$ where $G$ is the theta graph.
The trace radical of an algebra $A$ consists of elements $a$ such that $\langle a, b \rangle = 0$ for all $b$ in $A$.

**Corollary**: The pullback of the trace radical in $TL^d_{2n}$ to $C^{d^2}_n$ is in the trace radical of the chromatic algebra.

The trace radical of the Temperley-Lieb algebra is well-understood (Jones, Wenzl, Goodman):

It is non-trivial precisely for $d = 2\cos(\pi j/n)$, and for these values it is generated by the Jones-Wenzl projector.
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Chromatic algebra

\[ \hat{Z}_1 + \hat{Z}_2 = \phi^{-3} [\hat{Y}_1 + \hat{Y}_2] \]

\(\Phi\) maps the dual of Tutte’s relation to the 4-th Jones-Wenzl relation (at \(d = \phi\)):

\[
P^{(4)} = \left| \begin{array}{|c|c|c|c|}
\hline
& \frac{d}{d^2 - 2} & \frac{1}{d^2 - 2} \\
\hline
\end{array} \right|
- \frac{d^2 + 1}{d^3 - 2d} \left| \begin{array}{|c|c|c|c|}
\hline
& & \ \ \\
\hline
\end{array} \right|
- \frac{d}{d^4 - 3d^2 + 2} \left| \begin{array}{|c|c|c|c|}
\hline
& & \ \ \\
\hline
\end{array} \right|
+ \frac{1}{d^4 - 3d^2 + 2}
\]
Figure: A recursive formula for the pull-back \( P^{(2m)} \) of the Jones-Wenzl projector \( P^{(2m)} \) in the chromatic algebra.
Theorem

For a planar triangulation \( \hat{G} \),

\[
\chi_{\hat{G}}(\phi + 2) = (\phi + 2) \phi^{3V(\hat{G})-10} (\chi_{\hat{G}}(\phi + 1))^2
\]  

(1)

where \( V(\hat{G}) \) is the number of vertices of \( \hat{G} \).
Theorem

For a planar triangulation $\hat{G}$,

$$\chi_{\hat{G}}(\phi + 2) = (\phi + 2) \phi^3 V(\hat{G})^{-10} (\chi_{\hat{G}}(\phi + 1))^2$$  \hspace{1cm} (2)

where $V(\hat{G})$ is the number of vertices of $\hat{G}$.

Idea of the proof: Construct a map

$$C^{\phi+2} \rightarrow C^{\phi+1} \times C^{\phi+1}$$

and apply the trace:

$$C^{\phi+2} \rightarrow (C^{\phi+1}/R) \times (C^{\phi+1}/R)$$

$$\downarrow \hspace{1cm} = \hspace{1cm} \downarrow$$

$$C \hspace{1cm} = \hspace{1cm} C$$
More conceptually, Tutte’s golden identity is a consequence of level-rank duality for $SO(N)$ topological quantum field theories.

In particular, the level-rank duality implies that the $SO(3)_4$ and $SO(4)_3$ theories are isomorphic, and the latter splits into a product of two copies of $SO(3)_{3/2}$,

$$SO(3)_4 \rightarrow SO(3)_{3/2} \otimes SO(3)_{3/2}$$

The partition function of an $SO(3)$ theory is given in terms of the chromatic polynomial, specifically $\chi(\phi + 2)$ for $SO(3)_4$ and $\chi(\phi + 1)$ for $SO(3)_{3/2}$. 
The Tutte polynomial of a graph $G$:

$$T_G(X, Y) = \sum_{H \subseteq G} X^{c(H)-c(G)} Y^{n(H)}.$$ 

The summation is taken over all spanning subgraphs $H$ of $G$.

$c(H)$ denotes the number of connected components of the graph $H$, and $n(H)$ is the nullity of $H$, defined as the rank of the first homology group $H_1(H)$.

($n(H)$ may also be computed as $c(H) + e(H) - v(H)$, where $e$ and $v$ denote the number of edges and vertices of $H$, respectively.)
Basic properties of the Tutte polynomial: the contraction-deletion rule, and the duality

$$T_G(X, Y) = T_{G^*}(Y, X)$$

where $G$ is a planar graph, and $G^*$ is its dual.

(The vertices of $G^*$ correspond to the connected regions in the complement of $G$ in the plane, and two vertices are connected by an edge in $G^*$ whenever the two corresponding regions are adjacent.)
Now suppose $G$ is a ribbon graph (a graph embedded in a surface $\Sigma$). Consider the polynomial

$$P_{G,\Sigma}(X, Y, A, B) = \sum_{H \subset G} X^{c(H) - c(G)} Y^{k(H)} A^{s(H)/2} B^{s^\perp(H)/2}$$

Here $s(H)$ dentes the genus of the surface obtained as a regular neighborhood of the graph $H$ in $\Sigma$, and $s^\perp(H)$ is the genus of the surface obtained by removing a regular neighborhood of $H$ from $\Sigma$. Denote by $i$ the embedding $G \longrightarrow \Sigma$, and define

$$k(H) := \dim (\ker (i_* : H_1(H; \mathbb{R}) \longrightarrow H_1(\Sigma; \mathbb{R}))).$$
The polynomial $P$ satisfies the contraction-deletion rule,

$$P_G = P_{G\setminus e} + P_{G/e},$$

and it satisfies a duality relation, analogous to the duality of the Tutte polynomial on planar graphs:

$$P_G (X, Y, A, B) = P_{G^*} (Y, X, B, A).$$

Reference: V. Krushkal, Graphs, links, and duality on surfaces, arXiv:0903.5312
The well-known **Bollobás-Riordan polynomial** of ribbon graphs is defined by

\[
BR_{G,S}(X, Y, Z) = \sum_{H \subset G} (X - 1)^{r(G) - r(H)} Y^{n(H)} Z^{c(H) - bc(H) + n(H)}.
\]

Let \( v(H) \), \( e(H) \) denote the number of vertices, respectively edges, of \( H \), and let \( c(H) \) be the number of connected components. Then \( r(H) = v(G) - c(H) \), \( n(H) = e(H) - r(H) \), and \( bc(H) \) is the number of boundary components of the surface \( S \).

The polynomial \( BR_{G,S} \) is a **universal** polynomial of ribbon graphs, satisfying the contraction-deletion rule.

The Bollobás-Riordan polynomial of a ribbon graph may be obtained as a specialization of the polynomial \( P_G \):

\[
BR_{G,S}(X, Y, Z) = Y^g P_G,\Sigma(X - 1, Y, YZ^2, Y^{-1}).
\]