

# Picture TQFTs, categorification, and localization

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(joint work with Ben Cooper)

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## Outline:

- Construction of “Picture TQFTs”
  - The Jones-Wenzl projectors
  - Evaluation at a root unity
- Categorification
  - of the Jones-Wenzl projectors
- Towards categorification of TQFTs and 3-manifold invariants:
  - Evaluation at a root of unity as a localization

The Temperley-Lieb algebra  $TL_n$ : generators  $1$  and  $e_i$ ,  $0 < i < n$ , satisfying the relations:

- 1  $e_i e_j = e_j e_i$  if  $|i - j| \geq 2$ .
- 2  $e_i e_{i \pm 1} e_i = e_i$
- 3  $e_i^2 = [2] e_i$

where

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}} = q^{-(n-1)} + q^{-(n-3)} + \dots + q^{n-3} + q^{n-1}$$

Each generator  $e_i$  can be pictured as a diagram consisting of  $n$  chords between two collections of  $n$  points on two horizontal lines in the plane.

Example:  $n = 3$ :

$$1 = \begin{array}{|c|c|c|} \hline \\ \hline \end{array}, \quad e_1 = \begin{array}{|c|c|} \hline \cup \\ \cap \\ \hline \end{array} \quad \text{and} \quad e_2 = \begin{array}{|c|c|} \hline \cup \\ \cap \\ \hline \end{array}.$$

The multiplication is given by vertical composition of diagrams.

## Definition

The *Temperley-Lieb space*  $\mathrm{TL}(\Sigma)$  of a labelled surface  $\Sigma$  is the set of all  $\mathbb{Z}[q, q^{-1}]$  linear combinations of isotopy classes of 1-manifolds (“multi-curves”)  $F \subset \Sigma$  subject to the local relation: removing a simple closed curve bounding a disk in  $\Sigma$  from a multi-curve is equivalent to multiplying the resulting element by the quantum integer  $[2] = q + q^{-1}$ .

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$TL(\Sigma)$  is the 2-dimensional version of the Kauffman skein module of  $\Sigma \times [0, 1]$ .

The Temperley-Lieb algebra  $TL_n$  corresponds to  $\Sigma = D^2$

## Definition

Let  $Q$  be an element of the Temperley-Lieb algebra. The *ideal*  $\langle Q \rangle$  generated by  $Q$  in  $\text{TL}(\Sigma)$  is the smallest submodule containing all elements obtained by gluing  $Q$  to  $B$  where  $\Sigma = D^2 \cup (\Sigma \setminus D^2)$ , and  $B$  is any element of  $\text{TL}(\Sigma \setminus D^2)$ .

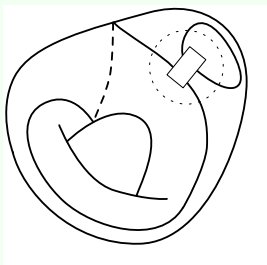


Figure: An element of the ideal  $\langle Q \rangle \subset \text{TL}(\Sigma)$

The notion of an ideal  $\langle Q \rangle$  allows one to consider an element  $Q$  of the Temperley-Lieb algebra as a local relation “ $Q = 0$ ” among multi-curves on a surface  $\Sigma$ .



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### **Theorem** (Goodman-Wenzl)

For  $q$  a primitive root of unity,  $q = e^{2\pi i/n}$ , there exists a unique local relation (ideal  $\langle Q \rangle$ ) such that the quotient of  $TL(\Sigma)/\langle Q \rangle$  is non-trivial.

In this case  $Q =$  **Jones-Wenzl projector**  $p_{n-1}$ , and  $TL(\Sigma)/\langle p_{n-1} \rangle$  is the Turaev-Viro theory associated to  $\Sigma$ .

For  $q$  not a root of unity,  $TL(\Sigma)$  has no non-trivial local relations.

Summary of the construction of the Turaev-Viro theory  $TV_n(\Sigma)$ :

- 1 Consider the  $\mathbb{Z}[q, q^{-1}]$ -module  $TL(\Sigma)$ ,
- 2 Specialize  $q = e^{2\pi i/n}$   
(Algebraically: quotient the coefficient ring by the cyclotomic polynomial  $\phi_n(q)$ )
- 3  $TV_n(\Sigma) = TL(\Sigma) / \langle p_{n-1} \rangle$ .

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An alternative construction: Skip step 2 and define

$$\overline{TV}_n(\Sigma) = TL(\Sigma)/\langle p_{n-1} \rangle$$

The result is closely related to  $TV_n(\Sigma)$  (the coefficient ring is divided by  $\phi_n(q^2) = \phi_n(q)\phi_n(-q)$  rather than  $\phi_n(q)$ )

The Jones-Wenzl projectors are elements of the Temperley-Lieb algebra that are uniquely characterized by two properties:

- 1  $p_n - 1$  belongs to the subalgebra generated by  $\{e_1, e_2, \dots, e_{n-1}\}$

In other words,  $p_n = 1 + \dots$

- 2  $e_i p_n = p_n e_i = 0$  for all  $i = 1, \dots, n - 1$  “killed by turnbacks”

## Origins and applications of the Jones-Wenzl projectors:

- Representation theory of  $U_q(\mathfrak{sl}_2)$
- A “nice basis” of the Kauffman skein module of  $\text{annulus} \times [0, 1]$
- Building blocks in the definition of quantum spin networks
- Used in constructions of the Reshetikhin-Turaev and Turaev-Viro  $SU(2)$  quantum invariants of 3-manifolds
- The colored Jones polynomial

## The Jones-Wenzl projectors

$$p_n = \text{[Diagram: A box labeled } n \text{ with four vertical lines (two on the left, two on the right) passing through it.]}$$

may be defined by the inductive formula:

$$\text{[Diagram: Box } n \text{ with 4 lines]} = \text{[Diagram: Box } n-1 \text{ with 4 lines and a vertical line to the right]} - \frac{[n-1]}{[n]} \text{[Diagram: Two boxes } n-1 \text{ stacked vertically with 4 lines and a loop on the right side]}$$

Example. The second projector  $p_2$ :

$$\boxed{\text{Diagram of } p_2} = \boxed{\text{Diagram of } 1} - \frac{1}{q + q^{-1}} \boxed{\text{Diagram of } \text{cup}}$$

Example: The second projector  $p_2$ :

$$\begin{aligned}
 \boxed{\text{Diagram: two vertical lines with a box between them}} &= \boxed{\text{Diagram: two vertical lines}} - \frac{1}{q + q^{-1}} \boxed{\text{Diagram: two lines crossing}} \\
 &= \boxed{\text{Diagram: two vertical lines}} + \sum_{i=1}^{\infty} (-1)^i q^{2i-1} \boxed{\text{Diagram: two lines crossing}}
 \end{aligned}$$



## Categorification:

Bar-Natan's formulation of the Khovanov categorification of the Temperley-Lieb algebra:

Additive category  $\text{Pre-Cob}(n)$ :

**Objects:** isotopy classes of formally  $q$ -graded Temperley-Lieb diagrams with  $2n$  boundary points.

**Morphisms:** the free  $\mathbb{Z}$ -module spanned by isotopy classes of orientable cobordisms bounded in  $\mathbb{R}^3$  between two planes containing diagrams.


**Notation:**


$$\begin{array}{c} \text{cap} \end{array} = 2 \begin{array}{c} \text{cup} \end{array} = 2 \begin{array}{c} \text{crossing} \end{array} \quad \text{and} \quad \begin{array}{c} \text{cup} \\ \text{cap} \end{array} = \begin{array}{c} \text{thick line} \end{array}$$


The diagram shows the following equivalences:


- A cap (a square with a curved top) is equal to 2 times a cup (a square with a curved bottom).
- A cup is equal to 2 times a crossing (two lines crossing at a point).
- A cup followed by a cap (a square with a hole) is equal to a thick horizontal line.




Form a new category  $\text{Cob}(n) = \text{Cob}_{\hbar}^3(n)$  obtained as a quotient of the category  $\text{Pre-Cob}(n)$  by the relations


 $= 0$


 $= 1$


 $= 0$


 $= \alpha$


 $=$ 

 $+$ 


$\Sigma_3 = 8\alpha$  is a parameter.

**Recall:** the Jones-Wenzl projectors are elements of the Temperley-Lieb algebra that are uniquely characterized by

- 1  $p_n = 1 + \dots$
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**Categorification of the Jones-Wenzl projectors:**

**Theorem:** (Cooper - K.) For each  $n > 0$ , there exists a chain complex  $P_n$  (positively graded with degree zero differential) such that

- The identity diagram appears only in homological degree zero and only once.
- The chain complex  $P_n$  is contractible “under turnbacks”: for any generator  $e_i \in \text{TL}_n$ ,  $0 < i < n$ ,  $P_n \otimes e_i \simeq 0$ ,  $e_i \otimes P_n \simeq 0$

Example: the second projector:

$$p_2 = \left| \begin{array}{c} | \\ | \\ | \end{array} \right\rangle - \frac{1}{q + q^{-1}} \left| \begin{array}{c} \cup \\ \cap \end{array} \right\rangle = \left| \begin{array}{c} | \\ | \\ | \end{array} \right\rangle + \sum_{i=1}^{\infty} (-1)^i q^{2i-1} \left| \begin{array}{c} \cup \\ \cap \end{array} \right\rangle$$

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Categorified second projector  $P_2$ :

$$\begin{array}{|c|} \hline \text{---} \\ \hline \end{array} \xrightarrow{\begin{array}{|c|} \hline \text{---} \\ \hline \end{array}} q \begin{array}{|c|} \hline \text{---} \\ \hline \end{array} \xrightarrow{\begin{array}{|c|} \hline \text{---} \\ \hline \end{array} - \begin{array}{|c|} \hline \text{---} \\ \hline \end{array}} q^3 \begin{array}{|c|} \hline \text{---} \\ \hline \end{array} \xrightarrow{\begin{array}{|c|} \hline \text{---} \\ \hline \end{array} + \begin{array}{|c|} \hline \text{---} \\ \hline \end{array}} q^5 \begin{array}{|c|} \hline \text{---} \\ \hline \end{array} \dots$$

Contractibility under turnbacks:

$$\begin{array}{c} \boxed{\text{cap}} \xrightarrow{\boxed{A}} q \boxed{\text{cup}} \xrightarrow{\boxed{\text{cup}} - \boxed{\text{cup}}} q^3 \boxed{\text{cup}} \xrightarrow{\boxed{\text{cup}} + \boxed{\text{cup}}} q^5 \boxed{\text{cup}} \dots \end{array}$$

Contractibility under turnbacks:

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 \text{A} \\
 \boxed{\phantom{A}} \\
 \xrightarrow{\quad} q \boxed{\text{A}} \xrightarrow{\boxed{\text{A}} - \boxed{\text{A}}} q^3 \boxed{\text{A}} \xrightarrow{\boxed{\text{A}} + \boxed{\text{A}}} q^5 \boxed{\text{A}} \dots
 \end{array}$$

The diagram illustrates a sequence of transformations of a cup-shaped diagram. The first diagram is a simple cup. An arrow labeled with a boxed 'A' points to a cup with a small circle on top, labeled 'q'. A second arrow, labeled with a boxed 'A' minus a boxed 'A', points to a cup with a small circle on top, labeled 'q^3'. A third arrow, labeled with a boxed 'A' plus a boxed 'A', points to a cup with a small circle on top, labeled 'q^5'. The sequence ends with an ellipsis '...'.

Applications of categorified projectors:

- Categorification of spin networks
- In particular,  $6j$ -symbols
- The colored Jones polynomial
- The  $SO(3)$  Kauffman polynomial
- The chromatic polynomial of planar graphs



Algebra.

Temperley - Lieb algebra:  $\mathrm{TL}_n$

$$p_n \in \mathrm{TL}_n$$

$$p_n \cdot p_n = p_n$$

$p_n$  is unique

Category.

Khovanov - Bar-Natan Category

$$P_n \in \mathrm{Kom}(n), K_0(P_n) = p_n$$

$$P_n \otimes P_n \simeq P_n$$

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Alternative constructions:

Lev Rozansky, Igor Frenkel - Catharina Stroppel - Joshua Sussan

More recent developments:

Ben Cooper - Matt Hogancamp: generalized projectors (for  $\mathfrak{sl}_2$ )David Rose:  $\mathfrak{sl}_3$ Sabin Cautis:  $\mathfrak{sl}_n$ 

Matt Hogancamp: functoriality under cobordisms

## Recall:

Summary of the construction of the Turaev-Viro theory  $TV_n(\Sigma)$   
(without evaluation at a root of unity):

$$\overline{TV}_n(\Sigma) = TL(\Sigma) / \langle p_{n-1} \rangle$$

## Recall:

Summary of the construction of the Turaev-Viro theory  $TV_n(\Sigma)$  (without evaluation at a root of unity):

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Categorify this construction?

$$TL(\Sigma) \rightsquigarrow Kom(Cob(\Sigma)) \quad p_{n-1} \rightsquigarrow P_{n-1}$$

Categorical analogue of taking a quotient  $TL(\Sigma) / \langle p_{n-1} \rangle$ :

Localization

## Definition

Given  $P$  in  $\text{Ho}(\text{Kom}(D^2))$ , consider  $\ll P \gg$ : the smallest full subcategory of  $\text{Ho}(\text{Kom}(\Sigma))$  which contains all objects obtained by gluing  $P$  to  $B$  where  $\Sigma = D^2 \cup (\Sigma \setminus D^2)$ ,  $B$  is any object of  $\text{Ho}(\text{Kom}(\Sigma \setminus D^2))$ , and which is closed under cones and grading shifts.

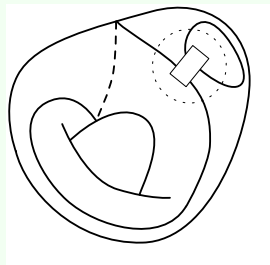


Figure: An object in  $\ll P \gg$

In a category  $\mathcal{C}$ , localized at an object  $P$ , the cone on any morphism  $P \rightarrow Q$  should be isomorphic to  $Q$ .

The classical Verdier localization is defined as the “quotient” of the category  $\mathcal{C}$  by the smallest **thick** subcategory containing  $\langle\langle P \rangle\rangle$ .

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It turns out that the smallest thick subcategory containing  $\langle\langle P_n \rangle\rangle$  is the entire category  $Ho(Kom(\Sigma))$ . (The chain complex for the trace of the projector is chain-homotopic to its homology). Therefore the Verdier localization is trivial.

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Towards 3-manifold invariants: we categorify the “magic element”  $\omega$  at low levels, and show that the localized construction is invariant under handle slides.