Geometric complexity of embeddings

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Whitney embedding theorem: smooth n-manifolds embed into $\mathbb{R}^{2n}$.

General position: any n-dimensional simplicial complex embeds into $\mathbb{R}^{2n+1}$.

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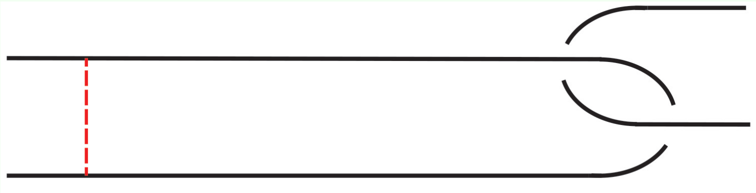
There have been recent interesting developments in quantifying the geometry of such embeddings.

I will focus on three measures of geometric complexity of embeddings: distortion, thickness, and refinement complexity.

Given a subset $K \subset \mathbb{R}^n$ and two points $x, y \in K$, let $d_K(x, y)$ denote the intrinsic metric: the infimum of the lengths of paths in K connecting x and y.

The *distortion* of K is defined as

$$\delta(K) := \sup_{x,y \in K} \frac{d_K(x, y)}{d_{\mathbb{R}^n}(x, y)}.$$



Question (Gromov, 1983): Does every isotopy class of knots in $\mathbb{R}^3$ have a representative in $\mathbb{R}^3$ with distortion < 100? Is it so for all torus knots $T_{p,q}$ for $p, q \longrightarrow \infty$?

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J. Pardon, *On the distortion of knots on embedded surfaces*, Ann. of Math. 174 (2011):

Theorem: $\delta(T_{p,q}) \geq \frac{1}{160} min(p, q)$.

M.Gromov - L. Guth (2011): An aternative proof that there are isotopy classes of knots requiring arbitrarily large distortion.

Given a simplicial complex K, its *intrinsic distortion* with respect to embeddings into $\mathbb{R}^n$ is defined as

$$D_n(K) \;:=\; \inf_i \;\delta(i(K)),$$

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An example of a family of spaces with unbounded intrinsic distortion is given by expander graphs.

A graph Γ is an *α-expander* if whenever S is a subset of the set of vertices V of Γ with $|S| \leq |V|/2$, the number of edges connecting vertices in S to vertices in $V \smallsetminus S$ is at least $\alpha|S|$.

Interesting examples are families of expander graphs Γ_k of bounded degree and fixed $\alpha > 0$, with the number of vertices $|V(\Gamma_k)|$ going to infinity.

Such families of examples are given by random bipartite graphs, and explicit constructions (G. Margulis) are based on groups with Kazhdan's property (T).

Consider embeddings $K^n \longrightarrow B_1^d \subset \mathbb{R}^d$ of an n-dimensional simplicial complex into the unit ball in $\mathbb{R}^d$.

Such an embedding has *Gromov-Guth thickness* at least T if the distance between the images of any two non-adjacent simplices is at least T.

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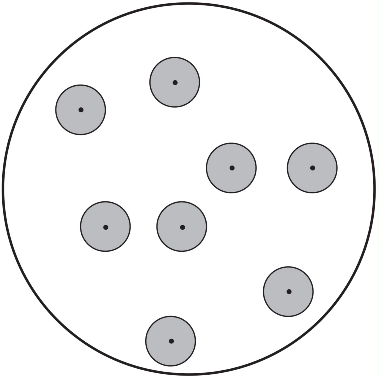
Kolmogorov-Barzdin (1967): predating a formal definition of expander graphs, established the sharp asymptotic bound $N^{-1/2}$ for embeddings of graphs with N vertices into 3-space.

(The definition of expander graphs is attributed to Pinsker (1973).)

Consider a plane in $\mathbb{R}^3$ dividing the vertices of an expander graph Γ in half:



Kolmogorov-Barzdin (1967): Asymptotic bound $N^{-1/2}$ for embeddings of graphs with N vertices into 3-space.

Let K be an n-dimensional simplicial complex with bounded local combinatorial complexity. Let $d \geq 2n+1$.

Gromov-Guth (2011):

$N^{-\frac{1}{d-n}}$ is a sharp (up to an ϵ summand in the exponent) bound for the thickness of K in $\mathbb{R}^d$, where N is the number of vertices of K.

A similar notion in the context of knot theory:

Ropelength of a knot:

The quotient of the length of a knot and the radius of the largest embedded normal tube around it.

There are estimates for the ropelength of links in terms of the linking number. (J. Cantarella, R. Kusner, J. Sullivan, *On the Minimum Ropelength of Knots and Links*)

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Codimensions $0, 1$:

Using Novikov's theorem that there can be no decision procedure for determining if a d-complex Σ^d, $d \geq 5$ is homeomorphic to S^d:

Theorem. For each $d \geq 5$ there exists a sequence $\{K_i\}$ of d-complexes ($(d-1)$-complexes) with n_i simplices which after subdivision PL embed in $\mathbb{R}^d$, but there is no recursive function of n_i which lower bounds the recursive complexity $\mathrm{rc}(n_i)$.

Our work is motivated by the question of what kinds of complexity functions can be realized "close" to the stable range.

We show that the bound for thickness for embeddings $K^n \hookrightarrow \mathbb{R}^d$ shifts from polynomial to exponential when the dimension d passes from $2n+1$ to $2n$.

Theorem 1. (Freedman-K., 2013) For each $n \geq 2$ there exist families of n-complexes K^n with bounded local combinatorial complexity which embed into $\mathbb{R}^{2n}$, such that the refinement complexity is at least c^m (and the thickness of any embedding is at most C^{-m}) where $c, C > 1$ and m is the number of simplices of K.

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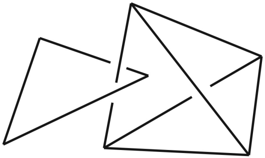
Theorem 2. Let K be an n-complex with N simplices which topologically embeds in $\mathbb{R}^{2n}$, $n \geq 3$. Then the refinement complexity of K is $O(e^{N^{4+\epsilon}})$ for any $\epsilon > 0$.

Let K_0^2 denote the 2-skeleton of the 6-simplex, with a single 2-cell removed.

Denote the boundary circle of the missing 2-cell by C, and let S be the 2-sphere spanned in K by the 4 vertices which are not in C.

Proposition. The 2-complex K_0^2 embeds into $\mathbb{R}^4$. Moreover, for any embedding $i\colon K_0^2 \longrightarrow \mathbb{R}^4$ the mod-2 linking number $\mathrm{lk}_{\mathrm{mod}\,2}(i(C), i(S))$ is non-zero.



For each $l \geq 1$ define the 2-complex K_l to be the l-fold mapping telescope

$$S^1 \xrightarrow{\times 2} S^1 \xrightarrow{\times 2} \dots \xrightarrow{\times 2} S^1 \xrightarrow{\times 2} C \subset K_0$$

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Denote the left-most circle by $\overline{C}$.

By construction for any embedding $i\colon K_l \subset \mathbb{R}^4$

$$|\mathrm{lk}\,(i(\overline{C}), i(S))| \;>\; 2^l.$$

Claim: Exponential linking number implies an exponential bound on refinement complexity and thickness.

Use the Gauss integral formula for the linking number to get an estimate on thickness.

The classical Gauss integral in 3-space computes the linking number of a 2-component link (α, β) in terms of the triple scalar product of $\dot{\alpha}$, $\dot{\beta}$ and $v = \alpha(s) - \beta(t)$:

$$\mathrm{Lk}\,(\alpha, \beta) \;=\; \frac{1}{4\pi} \int \frac{[\dot{\alpha}(s), \dot{\beta}(t), v]}{|v|^3}\, ds\, dt$$

The higher-dimensional Gauss integral computing the degree of the map $S^{n-1} \times S^n \longrightarrow S^{2n-1}$, equal (up to a sign) to the linking number:

$$2^l \;\leq\; \mathrm{Lk}\,(\alpha, \beta) \;=\; c \int_{S^{n-1} \times S^n} \frac{\det\,[\,d\alpha,\, d\beta,\, v\,]}{|v|^{2n}} \tag{1}$$

An upper bound in terms of the $(n-1)$-volume V_α, the n-volume V_β and the thickness T:

$$2^l \;\leq\; \mathrm{Lk}\,(\alpha, \beta) \;<\; \frac{V_\alpha\, V_\beta}{T^{2n}}.$$

To get an estimate on the thickness of a given embedding $i\colon\; K_l \longrightarrow B_1^{2n}$, suppose T-normal bundles of the simplices of K_l are embedded and are disjoint if the simplices are not adjacent.

There is a polynomial expression, the classical Weyl tube formula, for the volume of an ϵ-normal bundle over a submanifold of a Euclidean space in terms of the volume of the submanifold, for small ϵ. For example, the Weyl formula for closed surfaces in $\mathbb{R}^3$ states:

$$\mathrm{Vol}(N_\epsilon\Sigma) \;=\; 2\,\mathrm{Area}(\Sigma)\,\epsilon + \frac{4\pi}{3}\,\chi(\Sigma)\,\epsilon^3,$$

A crude bound:

$$\frac{T^{n+1}}{2^n}\,V'_\alpha < \frac{\pi^n}{n!}, \quad \frac{T^n}{2^n}\;V'_\alpha < \frac{\pi^n}{n!}$$

Combine:

$$T^{2n-1}\,2^l \;<\; (n+1)(n+2)\,\frac{(2\pi)^{2n}}{T^{2n+1}\,(n!)^2}$$

It follows that

$$T \;<\; 2^{-l/4n}\,n^{-1/2}.$$

This is the sought exponential thinness in terms of l.

Questions:

- Are there examples with exponential bound on distortion?
- Superexponential distortion or thickness?

- Let L be a Brunnian q-component link in S^3. Let M be the maximum value among Milnor's μ-invariants with distinct indices $|\mu_{i_1,\ldots,i_q}(L)|$. Is there a bound thickness$(L) < c_q M^{-1}$ for some constant $c_q > 0$ independent of the link L? Is there a bound on the crossing number of L in terms of M?