# Macroeconomic Models of Heterogeneous Agents

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# Introduction to overall lecture

The myth of macroeconomics is that relations among aggregates are enlarged analogues of relations among corresponding variables for individual households, firms, industries, and markets. The myth is a harmless and useful simplification in many contexts, but sometimes it misses the essence of the phenomenon. (Tobin, 1972—as cited in Iwai, 1981)

- Two reasons that heterogeneity can be important.
  - Simple aggregation bias.
  - Heterogeneity in decisions and its impact on aggregates.

  Here, we emphasize the second aspect.

# I: Incomplete market models

# Why do I (personally) like incomplete market models?

- Realistic; Arrow-Debreu security markets don't exist in reality.
- Consistent with microeconomic models of consumption and saving. (Permanent income hypothesis.)
- Endogenous determination of wealth (and consumption) inequality.
- Can analyze the (heterogeneous) "insurance" aspect of macroeconomic policies.

# A digression: on inequality

- Three types of inequalities:
  - Wealth inequality,
  - Income inequality,
  - Earnings inequality.
- These inequalities have to be distinguished.
  - Gini coefficients are 0.78 (top 5% owns half, top 30% owns 90%), 0.57, and 0.63, respectively, in the U.S. (Díaz-Gimenéz, Quadrini, and Ríos-Rull 1997)
  - They are not perfectly correlated. In particular, one has to be careful about the life-cycle effect.
- In Japan, both wealth and income inequalities are much smaller than U.S. (0.56 and 0.30).
- What we care, in principle, is consumption (and leisure) inequality—difficult to measure.

# Complete market models

- Arrow-Debreu security for each state.
- Implications with CRRA utility:
  - $$\frac{c_{it}(s^t)}{C_t(s^t)} = \frac{c_{it}(\hat{s}^t)}{C_t(\hat{s}^t)}.$$
  - No mobility in consumption—everyone is insured similarly.
  - No mobility in wealth (initial wealth distribution remains forever).
  - $c_{it}$ is the linear function of the wealth level at time $t$. The slope of this function is common across people, while the intercept can be different.
  - Wealth distribution has no impact on aggregate dynamics (Gorman aggregation).
  - A "wealth-neutral" policy ($\tilde{W}_{i0}/W_{i0} = \tilde{W}_{j0}/W_{j0}$) has the same impact on everyone's welfare.

# (A bit of) history of "Bewley models"

- Earlier models
  - (Bewley)
  - İmrohoroğlu (1989 *JPE*) $\cdots$ storage economy.
- General equilibrium models
  - Huggett (1993 *JEDC*) $\cdots$ endowment economy with bond.
  - Aiyagari (1994 *QJE*) $\cdots$ neoclassical production with capital.
  - Krusell and Smith (1998 *JPE*) $\cdots$ with aggregate shocks.
- Survey: Krusell and Smith (2006, world congress volume)
- Basic structure $\cdots$ Consumers face idiosyncratic shocks, but cannot write insurance contracts for these shocks. Instead, they can accumulate one type of asset (money in Bewley, İmrohoroğlu; bond in Huggett; Capital in Aiyagari and Krusell-Smith). They also face a borrowing constraint.

# Applications of Krusell-Smith computational method

- Similar contexts:
  - Cost of business cycles: Krusell and Smith (1999 *RED*, 2002), Storesletten, Telmer, and Yaron (2001 *EER*), Mukoyama and Şahin (2006, *JME*)
  - Asset pricing: Krusell and Smith (1997 *MD*), etc.
  - Foreign exchange risk premium: Leduc (2002, *JIMF*)
  - Labor supply: Chang and Kim (2006 *IER*, 2007 *AER*)
  - Default: Nakajima and Ríos-Rull (2005)
  - Tax reform: Nishiyama and Smetters (2005 *JPE*)
- Different contexts:
  - Monetary policy: Cooley and Quadrini (2006 *ET*)
  - State-dependent pricing: Golosov and Lucas (2007 *JPE*, somewhat related)
  - $(S, s)$-investment: Khan and Thomas (2007)
  - Structural estimation: Lee and Wolpin (2006, *Econometrica*), Weintraub, Benkard, and Van Roy (2007, somewhat related)
- Some other topics
  - Housing market. (Chambers, Garriga, and Schlagenhauf)
  - International risk sharing (and crises). (Castro 2005)
  - Money/nominal asset holding. (Doepke and Schneider 2006)

# A digression: Is it a good thing to be able to solve a heterogeneous-agent model for a monetary policy analysis?

- Various applications are possible.
  - State-dependent pricing model with idiosyncratic shocks. (Golosov-Lucas)
    - Inventory holding, customer market, and micro-founded model of "sales."
    - Heterogeneous information, heterogeneous expectation, experimentation.
  - Models with nominal and real assets. (Portfolio choice?)
  - Monetary policy affects the financial conditions of firms via "liquidity effect." Financial constraints are more important for small firms. (Cooley-Quadrini)
- Even if we limit attention to "cash"...
  - Cash holding behavior as $(S, s)$-behavior. (Baumol-Tobin)
  - Transaction demand for cash—randomness leads to heterogeneous cash holding. (Kiyotaki-Wright)

# Krusell-Smith model: Setup

- Consumers:

$$\max \quad E_0 \left[ \sum_{t=0}^{\infty} \beta^t \log(c_t) \right]$$

  subject to

$$a_{t+1} = (1 + r_t - \delta) a_t + w_t \epsilon_t + h(1 - \epsilon_t) - c_t$$

  and

$$a_{t+1} \geq \underline{a}.$$

  There is only one kind of asset $a$: capital stock.

- Firms:

$$\max \quad z_t K_t^{\alpha} L_t^{1-\alpha} - r_t K_t - w_t L_t$$

  $\rightarrow r_t = \alpha z_t (K_t / L_t)^{\alpha - 1}$ and $w_t = (1 - \alpha) z_t (K_t / L_t)^{\alpha}$.

# Krusell-Smith model: Exogenous variables

- The aggregate state $z_t \in \{g, b\}$ evolves according to the probabilities $\phi_{zz'}$.
- The idiosyncratic state $\epsilon_t \in \{0, 1\}$ evolves according to the conditional probabilities $\pi_{\epsilon'|\epsilon zz'}$.
    - To make the aggregate employment $L_t = \int \epsilon_{it} di$ as just a function of $z_t$ (i.e. two points), the following is satisfied:

      $$L_z \pi_{1|1zz'} + (1 - L_z)\pi_{1|0zz'} = L_{z'},$$

      for all $(z, z')$.

# Krusell-Smith model: Market equilibrium

$$K_t = \int a_t d\Gamma_t(a_t, \epsilon_t)$$

$$L_t = \int \epsilon_t d\Gamma_t(a_t, \epsilon_t).$$

# Krusell-Smith model: Recursive competitive equilibrium

Thinking "recursively" $\cdots$

- All the decisions are *functions* of the current state variables.
- All the prices are *functions* of the current state variables.
- All the next-period state variables are *functions* (which can be stochastic) of the current state variables.

The last two include equilibrium objects—individuals take these functions as given. Let's start from the individual decision problem.

# Krusell-Smith model: Individual Markov decision

- What are the state variables?
  - Clearly, $a_t$ and $\epsilon_t$ matter for the consumption/saving decision. What else?
  - One would want to know the return from saving: $r_{t+1}$. Also the future income matters—so, $w_{t+1}$, $w_{t+2}$, ..., $r_{t+1}$, $r_{t+2}$, ... have to be predicted. Also $\epsilon_{t+1}$, $\epsilon_{t+2}$, ... should be predicted.
  - That's enough to make decisions in a competitive market (all the prices and income). So, how do we predict future prices?
    - The future prices are influenced by future $z$'s and $L_z$'s. Future $\epsilon$'s are influenced by future $z$'s.
      $\rightarrow$ $z_t$ has to be a state variable.
    - Also future $K$'s influence the future prices. $K_{t+1}$ is influenced by everyone's decision of $a_{t+1}$, which is influenced by each person's state variables, which contain $(a_t, \epsilon_t)$. Therefore, it is necessary that the distribution of these individual state variables $\Gamma_t$ is a state variable.
  - Thus, at least, $(a_t, \epsilon_t, z_t, \Gamma_t)$ are state variables.
  - Are they sufficient? Yes.

# Krusell-Smith model: Individual Markov decision

- Bellman equation:

$$v(a, \epsilon, z, \Gamma) = \max_{c, a'} \log(c) + \beta E[v(a', \epsilon', z', \Gamma')|\epsilon, z]$$

subject to

$$a' = (1 + r(K, z) - \delta)a + w(K, z)\epsilon + h(1 - \epsilon) - c,$$

$$a' \geq \underline{a},$$

and

$$\Gamma' = H(\Gamma, z, z').$$

Here, $H$ is the law of motion for $\Gamma$.

# Krusell-Smith model: Recursive competitive equilibrium

- Consumers optimize.
- Firms optimize.
- Markets clear:

$$K_t = \int a_t d\Gamma_t(a_t, \epsilon_t)$$

$$L_t = \int \epsilon_t d\Gamma_t(a_t, \epsilon_t).$$

- Consistency:
  - $\Gamma'$ is generated by $a'(a, \epsilon, z, \Gamma)$ and $\epsilon'$ with $\Gamma$ and $z'$.

# Krusell-Smith model: Computation

- Main issues:
  - What do we do with the state variable $\Gamma$?
  - What do we do with the law of motion $H$?
- Krusell and Smith's solution:
  - Instead of considering the entire distribution, consider a limited set of moments.
  - Consider a simple forecasting rule.
- In particular, consider only the first moment $K$ and the simple linear rule:

$$\log(K') = \begin{cases} a_0 + b_0 \log(K) & \text{if } z = g, \\ a_1 + b_1 \log(K) & \text{if } z = b. \end{cases}$$

# Krusell-Smith model: Computation

- These consumers are "boundedly rational" in the sense that
  - they do not fully utilize the information of $\Gamma$, and
  - they limit themselves to the simple linear forecasting rule.

  However, if it turns out that this simple forecasting rule actually predict $K'$, these consumers are actually *fully rational*, because
  - the other information actually is not necessary for forecasting, and
  - the actual process follows the simple linear law of motion.

  Therefore, the resulting equilibrium is a rational expectations equilibrium.
- It turns out, in this class of models, that the prediction with this forecasting rule is very accurate: "approximate aggregation."

# Krusell-Smith model: Computation

1. Guess the law of motion for $K$:

$$\log(K') = \begin{cases} a_0 + b_0 \log(K) & \text{if } z = g, \\ a_1 + b_1 \log(K) & \text{if } z = b. \end{cases}$$

2. Solve the individual optimization problem:

$$v(a, \epsilon, z, K) = \max_{c,a'} \log(c) + E[v(a', \epsilon', z', K')|\epsilon, z]$$

subject to

$$a' = (1 + r(K, z) - \delta)a + w(K, z)\epsilon + h(1 - \epsilon) - c,$$

$$a' \geq \underline{a},$$

and the law of motion given above. ($r$ and $w$ are the MPs.)

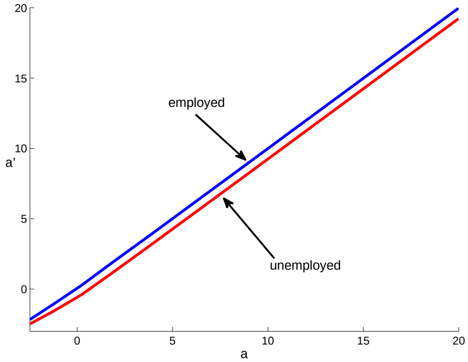
3. Simulate the economy, using the policy functions obtained from the optimization. Obtain the simulated time series of $K$ and $z$.
4. Compare the time series with the law of motion we guessed first. If the time series satisfy the law of motion, we found the REE. If not, revise the law of motion and repeat.

# Krusell-Smith model: Computation

More in detail:

- The initial guess of the law of motion is important. (In practice, one can obtain a good guess from the representative-agent model.)
- Since there are many state variables, it is important to use appropriate approximation methods in optimization. (I usually use cubic spline, linear interpolation, or polynomial interpolation for approximating the value function.) See Numerical Recipes.
- There are several alternative methods for simulation (see, for example, Heer and Maussner's book or Young 2005).
- We can use $R^2$, or the maximum forecast error to check the accuracy of the forecasting rule. (Den Haan 2007)
- If the approximate aggregation doesn't work, one can try to add the second moment, or departing from the linear specification.

# The economics of "approximate aggregation"




- The decision rule of the individual is very close to a linear function.
- It is non-linear when *a* is very small, but these people's behavior does not have much effect on the aggregate.

# Looking ahead: what is the important heterogeneity?

- Consumers:
  - Asset. (liquid/illiquid, nominal/real, financial conditions)
  - Income.
  - Skills.
  - Occupation/industry.
  - Age, sex, marital status, kids, health.
  - Preferences.
  - Information/expectation.
  - Insurance/networks/culture/language?
- Firms:
  - Capital.
  - Labor.
  - Technology.
  - Industry/sector.
  - Exporting/outsourcing.
  - Financial conditions.
- Heterogeneity across communities/nations.

# II: Firm/establishment dynamics

# Why do I (personally) like the models of firm dynamics?

- Realistic. There are big firms and small firms, and they behave quite differently.
- Can analyze the gains from reallocation of resources. Foster, Haltiwanger, and Krizan (2001) decompose the sources of multifactor productivity growth in U.S. manufacturing into
  - productivity gain within each plant,
  - change in output shares, and
  - entry and exit.

  The second accounts for 34%, and the third accounts for 24% of the total productivity gain.

# (A bit of) History

- Models of establishment(plant)/firm/industry dynamics
  - Perfect competition
    - Jovanovic (1982 *Econometrica*)
    - Hopenhayn (1992 *Econometrica*)
  - Imperfect competition
    - Ericson and Pakes (1995 *REStud*)
  - General equilibrium
    - Hopenhayn and Rogerson (1993 *JPE*)
    - Melitz (2003 *Econometrica*)
    - Cooley, Marimon, and Quadrini (2004 *JPE*)
    - Klette and Kortum (2004 *JPE*)
  - More on microfoundation
    - Albuquerque and Hopenhayn (2004 *REStud*)
    - Clementi and Hopenhayn (2006 *QJE*)

# Establishment size distribution (US, total private)

| | Number (%) | Emp. Share (%) |
|---|---|---|
| $0-4$ | 59.86 | 6.66 |
| $5-9$ | 16.81 | 8.26 |
| $10-19$ | 11.04 | 11.06 |
| $20-49$ | 7.62 | 17.08 |
| $50-99$ | 2.60 | 13.29 |
| $100-249$ | 1.48 | 16.49 |
| $250-499$ | 0.37 | 9.42 |
| $500-999$ | 0.14 | 6.82 |
| $1000-$ | 0.07 | 10.92 |

Data Source: Quarterly Census of Employment and Wages (BLS)
2001-2006

# Establishment size distribution (U.S. and Japan)

| | U.S. (%) | Japan (%) |
|---|---|---|
| 1 – 4 | 48.52 | 60.94 |
| 5 – 9 | 21.52 | 19.16 |
| 10 – 19 | 14.24 | 10.89 |
| 20 – 49 | 9.77 | 6.30 |
| 50 – 99 | 3.32 | 1.63 |
| 100 – 499 | 2.35 | 0.91 |
| 500 – 999 | 0.17 | 0.06 |
| 1000– | 0.10 | 0.02 |

Data Source: Statistics of U.S. Businesses (Census) 2003-2004
Japan: Establishment and Enterprise Census, Statistics Bureau, 2006, Table 11

# Firm size distribution (US and Japan)

| | U.S. (%) | Japan (%) |
|---|---|---|
| 0 – 4 | 60.82 | 53.24 |
| 5 – 9 | 17.72 | 19.23 |
| 10 – 19 | 10.75 | 12.83 |
| 20 – 99 | 8.94 | 11.90 |
| 100– | 1.76 | 2.81 |

| | U.S. | Japan |
|---|---|---|
| number of firms | 5,885,784 | 1,515,835 |
| number of establishments | 7,387,724 | 5,722,559 |

Data Source:
US:Statistics of U.S. Businesses (Census) 2004
Japan: Establishment and Enterprise Census, Statistics Bureau, 2006, Table 2

# Measures of reallocation

- Entry rate:
$$\frac{B_{t-1,t}}{T_t}.$$
- Exit rate:
$$\frac{D_{t,t+1}}{T_t}.$$
- Job creation rate:
$$\frac{\sum_{n_t > n_{t-1}} (n_t - n_{t-1})}{\sum n_{t-1}}.$$
- Job destruction rate:
$$\frac{\sum_{n_{t-1} > n_t} (n_{t-1} - n_t)}{\sum n_{t-1}}.$$

# Establishment dynamics: U.S.-Japan comparison

| | U.S. | Japan |
|---|---|---|
| Entry rate (annual, %) | 13.0 | 4.4 |
| Exit rate (annual, %) | 12.0 | 4.4 |
| JC by expanding establishments (annual, %) | 8.2 | 4.2 |
| JC by opening establishments (annual, %) | 5.2 | 4.5 |
| JD by contracting establishments (annual, %) | 8.8 | 3.9 |
| JD by closing establishments (annual, %) | 4.9 | 3.7 |

- Data Source:
  - Japan:
    - Entry/exit$\cdots$White Paper on Small and Medium Enterprises in Japan, Fig 1-2-4 (for 2005).
    - JC/JD$\cdots$Genda (1998), Figures 2 and 3, 1991-1994.
- US: Pinkston and Spletzer (2004) Tables 2 and 4 (for 1998-2001): constructed from BED.

# Establishment dynamics: U.S.-Japan comparison

Since

$$\text{JC by opening establishments} \approx \frac{\sum_{\text{entrant}} n_t}{\sum_{\text{total}} n_t}$$

and

$$\text{Entry rate} = \frac{\text{Number of opening establishments}}{\text{Total number of establishments}},$$

the ratio of the average size of opening establishments and the average size of total establishment is approximately equal to

$$\frac{[\text{JC by opening establishments}]}{[\text{Entry rate}]}.$$

The size ratio for closing establishments can be calculated similarly.

| | U.S. (SUSB) | Japan (EEC) |
|---|---|---|
| Size ratio for opening establishments | 0.40 (0.48) | 1.02 (0.94) |
| Size ratio for closing establishments | 0.41 (0.51) | 0.84 |

# A big picture

Compared to the U.S., in Japan,

- Establishments are smaller.
- Firms are bigger (and own more establishments).
- Opening establishments are bigger (relative to existing establishments). $\cdots$ Large entry costs?
- Closing establishments are bigger (relative to existing establishments). $\cdots$ Inefficient exit?
- Reallocations of labor across the existing plants are smaller. $\cdots$ Large mobility costs?

# Modeling strategies (without aggregate shocks)

- At the firm level,
  - Decreasing returns to scale $\cdots$ the concept of "firm" is meaningful.
  - Exogenous idiosyncratic shocks $\cdots$ productivity shocks, etc.
  - Endogenous idiosyncratic states $\cdots$ capital, employment, equity, etc.
  - Entry: sunk entry costs+free entry or entrepreneurs+occupational choice
  - Exit: A firm exits when the value of exiting exceeds the value of staying. "Selection," "evolution."
- At the aggregate level,
  - With the sufficient amount of "mixing," there is a unique stationary distribution of the individual states.
  - Consumers supply the factors and consume.

# Entry and exit (Hopenhayn and Rogerson 1993)

- Incumbent firms:
  - Start the period with the state variable $s_{t-1}$.
  - Decides whether to exit.
  - If continue, pay the fixed operation cost $c_f$, draw new productivity level $s_t$, decide on $n_t$, and produce.
- Entrants:
  - Pay the entry cost $c_e$.
  - Pay the fixed operation cost, draw the new productivity level $s_t$, decide on $n_t$, and produce.

# Individuals

Look at the steady-state with constant $w$.

- Incumbents:

  $$W(s;w) = \max_n \left\{ F(s,n) - wn - c_f + \beta \max\langle E[W(s';w)|s], 0\rangle \right\}.$$

  $\rightarrow$ the decision rule of $n$, $\psi(s)$ and the decision rule of exit, $\chi(s)$. ($\chi(s) = 1$ when exit.)
- Entrants:

  $$W^e(w) = \sum_s W(s;w)\nu(s),$$

  where $\nu(s)$ is the probability of initial draw being $s$.
- Consumers:

  $$\max_{C_t, N_t} \sum_{t=0}^{\infty} [u(C_t) - aN_t]$$

  subject to

  $$C_t = wN_t + \Pi_t.$$

  Thus, FOC

  $$wu'(C_t) = a.$$

# Equilibrium

First, we determine two equilibrium objects, $w$ and $C$.

- $w$ is determined from the free-entry condition

$$W^e(w) = c_e.$$

- $C$ is determined from the consumer's first-order condition

$$wu'(C) = a.$$

- How about the other quantities?

# Equilibrium

- First, assume that there is one unit of entry, and calculate the stationary distribution of firms (including the new entrant and excluding this period's exit) as a vector $\mu(s)$. Then, the labor demand in the unit entry economy is

$$n = \sum_s \psi(s)\mu(s),$$

  and since the economy is "replicable," $N = Mn$ holds where $M$ is the amount of entry.
- The total output $Y$ is

$$Y = M \sum_s F(\psi(s))\mu(s).$$

- Thus, $M$ can be solved from

$$C = Y - Mc_e = M \sum_s F(s, \psi(s))\mu(s) - Mc_e,$$

  and $Y$ and $N$ can be calculated from $M$.
- How do we obtain $\mu(s)$?

# Equilibrium

- $\mu(s)$ is the stationary distribution of the productivity across firms, calculated by

$$\mu(s') = \sum_{s}(1 - \chi(s))\pi(s, s')\mu(s) + \nu(s'),$$

  where $\pi(s, s')$ is the transition probability from $s$ to $s'$.
- This is a fixed-point problem—usually (with appropriate conditions) the above mapping is contraction and can be solved by an iterative procedure. (See SLP.)

# Applications

- Firing costs. (Hopenhayn and Rogerson, Veracierto)
- Size restrictions. (Guner, Ventura, and Yi)
- Monetary policy and entry. (Bilbiie, Ghironi, and Melitz)
- Financial frictions. (Cooley, Marimon, and Quadrini)
- Search. (Hobijn and Şahin)
- Business cycles. (Lee and Mukoyama)
- Innovation. (Klette and Kortum, Lentz and Mortensen)
- Development. (Restuccia and Rogerson, Moscoso Boedo and Mukoyama)