Some history on the subject: Ultrafilters on cardinals were probably first introduced by Riesz in 1909 and Ulam in 1929, and in general by Tarski in 1930. A similar notion was used by Gödel in his ontological proof of god.

Ultraproducts were used by Skolem in 1934 to study nonstandard models of arithmetic, Hewitt in 1948 (specially for fields in the construction of the hyperreals), and finally the general construction was introduced by Loś in 1955. In model theory, Loś’ theorem states that the ultrapower of a model is elementarily equivalent to the original model, that is, they satisfy the same first order sentences. This “transfer principal” is a rigorous replacement for Leibniz’s Law of Continuity to justify his proofs using infinitesimals. Robinson showed for instance that the hyperreals where consistent if and only if the real numbers were. Luxemburg also made advances using ultraproducts non-standard analysis. A related construction of the non-standard hull of a Banach spaces was introduced and studied by Luxemburg and Henson in 1969.

The Banach space ultraproduct construction was introduced in 1972 by Dacunha-Castelle and Krivine in order to study the “local” properties of Banach spaces which are determined by the finite dimensional subspaces.

We say \( F \) is a filter on a set \( I \) if \( F \subseteq \mathcal{P}(I) \) is a collection of subsets of \( I \) satisfying
1. \( I \in F, \emptyset \notin F \).
2. If \( A, B \in F \), then \( A \cap B \in F \).
3. If \( A \in F \), then \( A \subseteq B \) implies \( B \in F \).

One can think of elements of a filter \( F \) to be “big” sets in \( I \), and the complements of such sets as “small” sets: If we have a filter \( F \) on a set \( I \), then we get a finitely additive “measure” by

\[
m(A) = \begin{cases} 
1 & \text{if } A \in F \\
0 & \text{if } I \setminus A \in F \\
\text{undefined} & \text{otherwise}
\end{cases}
\]

Comment: For \( F \) a filter on \( I \), \( A \in F \) implies \( I \setminus A \notin F \), otherwise \( A \cap (I \setminus A) = \emptyset \in F \), a contradiction.

We say a filter \( F_1 \) dominates another filter \( F_2 \) on the same set \( I \) if for all \( A \in F_2 \), \( A \in F_1 \) as well, i.e. \( F_2 \subseteq F_1 \).
A filter $\mathcal{U}$ is called an ultrafilter if every filter dominating $\mathcal{U}$ coincides with $\mathcal{U}$.

Any filter $\mathcal{F}$ on a set $I$ is dominated by an ultrafilter $\mathcal{U}$ on $I$, since
\[
\{ \mathcal{G} \mid \mathcal{G} \text{ a filter on } I \text{ which dominates } \mathcal{F} \}
\]
is a non-empty family (as it contains $\mathcal{F}$) in which chains have upper bounds - namely if $\mathcal{G}_\alpha$ are a chain of filters on $I$ which all contain $\mathcal{F}$, then $\bigcup \mathcal{G}_\alpha$ is a filter on $I$ which contains $\mathcal{F}$. Thus by Zorn’s Lemma this family has a maximal filter, which is necessarily an ultrafilter.

From now on, $\mathcal{U}$ will always denote an ultrafilter.

Note that if $\mathcal{F}$ is a filter and $I_0 \subseteq I$ is a set with $I_0 \cup I \notin \mathcal{F}$ (which forces $I_0 \neq \emptyset, I$), consider
\[
\hat{\mathcal{F}} = \mathcal{F} \cup \{ A \cap F \mid A \supseteq I_0, F \in \mathcal{F} \}
\]
We will show that $\hat{\mathcal{F}}$ is a filter on $I$. It is clear that $I \in \mathcal{F} \subseteq \hat{\mathcal{F}}$, and since $\emptyset \notin \mathcal{F}$, it suffices to show that $\emptyset \notin \{ A \cap F \mid A \supseteq I_0, F \in \mathcal{F} \}$. Otherwise, we have $A \cap F = \emptyset$, so in particular, $I_0 \cap F = \emptyset$. However, this means that $I_0 \subseteq I \setminus F$, so $I \setminus I_0 \supseteq F$, but since $F \in \mathcal{F}$ this forces $I \setminus I_0 \in \hat{\mathcal{F}}$, contradicting our initial assumption. Thus $\emptyset \notin \hat{\mathcal{F}}$.

Suppose $A, B \in \hat{\mathcal{F}}$, we want to show $A \cap B \in \hat{\mathcal{F}}$. Case 1: $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F} \subseteq \hat{\mathcal{F}}$. Case 2: $F_1 \in \mathcal{F}$, $B \cap F_2$ where $B \supseteq I_0$, $F_2 \in \mathcal{F}$, then $F_1 \cap (B \cap F_2) = B \cap (F_1 \cap F_2) \in \hat{\mathcal{F}}$ since $B \supseteq I_0$ and $F_1 \cap F_2 \in \mathcal{F}$. Case 3: $F_1 \in \mathcal{F}$, $B \cap F_2$ where $A, B \supseteq I_0$, $F_1, F_2 \in \mathcal{F}$, then their intersection is $(A \cap B) \cap (F_1 \cap F_2) \in \hat{\mathcal{F}}$, since $A \cap B \supseteq I_0$ and $F_1 \cap F_2 \in \mathcal{F}$.

Finally, suppose $A \in \mathcal{F}$, $A \subseteq B$, we want to show that $B \in \hat{\mathcal{F}}$. Case 1: If $A \in \mathcal{F}$, then $A \subseteq B$ implies $B \in \mathcal{F} \subseteq \hat{\mathcal{F}}$. Case 2: If $A \cap B \in \mathcal{F}$ where $A \supseteq I_0$ and $F \in \mathcal{F}$, then $B = (A \cup B) \cap (F \cup B)$, since if $b \in B$, then $b \in A \cup B$ and $b \in F \cup B$, and conversely if $x \in A \cup B$ and $x \notin F \cup B$, if $x \notin B$ we are done, otherwise $x \in A$ and $x \in F$, but then $x \in A \cap F \subseteq B$. Thus $B = (A \cup B) \cap (F \cup B) \in \mathcal{F}$, since $(A \cup B) \supseteq I_0$ and $(F \cup B) \in \mathcal{F}$ so it contains $F \in \hat{\mathcal{F}}$. Thus $\mathcal{F}$ is a filter dominating $\hat{\mathcal{F}}$.

An ultrafilter $\mathcal{U}$ is called principal, or trivial, or non-free if it is determined by a single element $i_0 \in I$, i.e. $A \in \mathcal{U}$ if and only if $i_0 \in A$. An ultrafilter $\mathcal{U}$ is free or non-trivial if and only if
\[
\bigcap_{A \in \mathcal{U}} A = \emptyset
\]
An ultrafilter $\mathcal{U}$ is countably incomplete if there is a sequence of elements of $\mathcal{U}$
\[
I_1 \supseteq I_2 \supseteq I_3 \supseteq \ldots \bigcap_{k=1}^{\infty} I_k = \emptyset
\]
For example each non-principal ultrafilter on the set of natural numbers if countably incomplete: if $\mathcal{U}$ is a non-principal ultrafilter on $\mathbb{N}$, then for all $n \in \mathbb{N}$, there is a set $X_n \in \mathcal{U}$ with $n \notin X_n$, but then $\mathbb{N} \setminus \{n\} \supseteq X_n$ so $\mathbb{N} \setminus \{n\} \in \mathcal{U}$, so cofinite sets are in $\mathcal{U}$ since $\mathcal{U}$ is closed under finite intersections. Thus $I_n = \mathbb{N} \setminus \{1, 2, \ldots, n\}$ form the desired nested sequence of elements in $\mathcal{U}$ with empty intersection.
Now we can define the set-theoretic ultraproduct. Let $A_i$ be an arbitrary collection of sets indexed by $I$, and let $\mathcal{U}$ be an ultrafilter on $I$. Consider the cartesian product $\prod_{i \in I} A_i$. Define an equivalence relation $\sim_{\mathcal{U}}$ on $\prod_{i \in I} A_i$ by
\[
(a_i) \sim_{\mathcal{U}} (b_i) \text{ if and only if } \{i \in I | a_i = b_i\} \in \mathcal{U}
\]
Clearly $(a_i) \sim_{\mathcal{U}} (a_i)$ since $\{i \in I | a_i = a_i\} = I \in \mathcal{U}$, and if $(a_i) \sim (b_i)$, then $\{i \in I | b_i = a_i\} = \{i \in I | a_i = b_i\} \in \mathcal{U}$. Finally if $(a_i) \sim_{\mathcal{U}} (b_i)$ and $(b_i) \sim_{\mathcal{U}} (c_i)$, then
\[
\{i \in I | a_i = c_i\} \supseteq \{i \in I | a_i = b_i\} \cap \{i \in I | b_i = c_i\} \in \mathcal{U}
\]
Thus $\sim_{\mathcal{U}}$ is an equivalence relation as claimed. Now we define the set-theoretic ultraproduct. Let $\prod_{i \in I} A_i$ as above be an arbitrary collection of sets $\{A_i\}_{i \in I}$ with respect to the ultrafilter $\mathcal{U}$ to be $(\prod_{i \in I} A_i) / \sim_{\mathcal{U}}$, which is sometimes denoted $\prod_{i \in I} A_i / \mathcal{U}$ or $(a_i)_{\mathcal{U}}$.

Note that if $A_i \subseteq B_i$ for all $i \in I$, then $(A_i)_{\mathcal{U}} \subseteq (B_i)_{\mathcal{U}}$.

Proposition: Set theoretic operations of ultraproducts: if $(A_i), (B_i)_{i \in I}$ are families of sets indexed by $I$, then
1. $(A_i)_{\mathcal{U}} \cup (B_i)_{\mathcal{U}} = (A_i \cup B_i)_{\mathcal{U}}$

Proof. $\subseteq$: if $x \in (A_i)_{\mathcal{U}} \cup (B_i)_{\mathcal{U}}$, then either $x \in (A_i)_{\mathcal{U}}$ or $x \in (B_i)_{\mathcal{U}}$. Suppose $x \in (A_i)_{\mathcal{U}}$, then $x = [(a_i)]$ is an equivalence class of a sequence $(a_i) \in \prod_{i \in I} A_i$. The other case is similar.
$\supseteq$: if $x \in (A_i \cup B_i)_{\mathcal{U}}$, then $x = [(x_i)]$ is an equivalence class of a sequence $(x_i) \in \prod_{i \in I} A_i \cup B_i$, so $x_i \in A_i \cup B_i$ for all $i \in I$, i.e. $x_i \in A_i \setminus B_i$ or $x_i \in B_i$ for all $i \in I$. Then
\[
I = \bigcup_{i \in I} \left\{ i \in I | x_i \in A_i \setminus B_i \right\} \cup \bigcup_{i \in I} \left\{ i \in I | x_i \in B_i \right\}
\]
Since $\mathcal{U}$ is an ultrafilter, either $I_A \in \mathcal{U}$ or $I_B \in \mathcal{U}$. If $I_A \in \mathcal{U}$, then $(x_i)$ is equivalent (in $\prod_{i \in I} A_i \cup B_i$) to a sequence $(a_i)$ in $\prod_{i \in I} A_i$. The other case is similar.
\[\square\]
2. $(A_i)_{\mathcal{U}} \cap (B_i)_{\mathcal{U}} = (A_i \cap B_i)_{\mathcal{U}}$
3. $(A_i)_{\mathcal{U}} \setminus (B_i)_{\mathcal{U}} = (A_i \setminus B_i)_{\mathcal{U}}$

Proposition: Let $K$ be a compact Hausdorff space. Then for each $(x_i)_{i \in I}$ in $\prod_{i \in I} K$ the limit
\[
\lim_{\mathcal{U}} x_i = x
\]
exists in $K$. That is, there is a unique point $x \in K$ such that for each neighborhood $U$ of $x$, the set $\{i \in I | x_i \in U\} \in \mathcal{U}$.
Proof. Existence: suppose for sake of contradiction no such $x$ exists. Then for all $x \in K$, there is a neighborhood $U_x$ of $x$ such that $I_x = \{ i \in I \mid x_i \in U_x \} \notin \mathcal{U}$. Then since $K$ is compact, there is a finite subcover $U_{x_1}, \ldots, U_{x_n}$ of $K$, but then for all $i \in I$, $x_i \in U_{x_j}$ for some $1 \leq j \leq n$, so

$$I = \bigcup_{j=1}^{n} \{ i \in I \mid x_i \in U_{x_j} \} = \bigcup_{j=1}^{n} U_{x_j} \notin \mathcal{U}$$

Which is a contradiction.

Uniqueness: Suppose for sake of contradiction there are two distinct points $x_1$ and $x_2$ with the desired property. Since $K$ is Hausdorff, there are disjoint neighborhoods $U_1$ and $U_2$ of $x_1$ and respectively, but then

$$\emptyset = \{ i \in I \mid x_i \in U_1 \} \cap \{ i \in I \mid x_i \in U_2 \} \in \mathcal{U}$$

Which is a contradiction. \[ \square \]

Properties of Ultralimits: Say $(a_i), (b_i) \in \ell_\infty(I, \mathbb{R})$, in particular, $\|(a_i)\|_\infty = \sup_{i \in I} |a_i| < \infty$, $\|(b_i)\|_\infty = \sup_{i \in I} |b_i| < \infty$, such that $\lim_{\mathcal{U}} a_i = A$ and $\lim_{\mathcal{U}} b_i = B$. Then

1. If $(a_i) \sim_{\mathcal{U}} (b_i)$, then $A = \lim_{\mathcal{U}} a_i = \lim_{\mathcal{U}} b_i$.
   
   Proof. 
   
   $$\{ i \in I \mid |b_i - A| < \varepsilon \} \supseteq \{ i \in I \mid |a_i - A| < \varepsilon \} \cap \{ i \in I \mid a_i = b_i \} \in \mathcal{U}$$

   \[ \square \]

2. 

   $$\lim_{\mathcal{U}} a_i + \lim_{\mathcal{U}} b_i = \lim_{\mathcal{U}} (a_i + b_i)$$

   Proof. Since

   $$|a_i + b_i - (A + B)| \leq |a_i - A| + |b_i - B|$$

   we have that

   $$\{ i \in I \mid |a_i + b_i - (A + B)| < \varepsilon \} \supseteq \{ i \in I \mid |a_i - A| < \frac{\varepsilon}{2} \} \cap \{ i \in I \mid |b_i - A| < \frac{\varepsilon}{2} \} \in \mathcal{U}$$

   \[ \square \]

3. If $\lim_{\mathcal{U}} a_i = 0$, then $\lim_{\mathcal{U}} a_i b_i = 0$
   
   Proof. If $\|(b_i)\|_\infty = 0$, then

   $$\{ i \in I \mid |a_i b_i| = 0 \} = \{ i \in I \mid |a_i| = 0 \} \in \mathcal{U}$$

   Otherwise, $\|(b_i)\|_\infty > 0$, and we use

   $$\{ i \in I \mid |a_i b_i| < \varepsilon \} \supseteq \{ i \in I \mid |a_i| < \varepsilon \} \in \mathcal{U}$$

   \[ \square \]

4. 

   $$\left( \lim_{\mathcal{U}} a_i \right) \left( \lim_{\mathcal{U}} b_i \right) = \lim_{\mathcal{U}} (a_i \cdot b_i)$$
Proof. By the first part, we will assume \( \|(a_i)\|_\infty \neq 0 \) and \( \lim_\mathcal{U} b_i \neq 0 \). Since

\[
|a_i b_i - AB| \leq |a_i b_i - a_i B| + |a_i B - AB| \leq \|(a_i)\|_\infty |b_i - B| + |a_i - A||B|
\]

we have that

\[
\{ i \in I \mid |a_i b_i - (AB)| < \varepsilon \} \supseteq \{ i \in I \mid |a_i - A| < \frac{\varepsilon}{|b_i|} \} \cap \{ i \in I \mid |b_i - B| < \frac{\varepsilon}{\|(a_i)\|_\infty} \} \in \mathcal{U}
\]

\[
\square
\]

We also get as a special case that

\[
\lim_\mathcal{U} \lambda a_i = \lambda \lim_\mathcal{U} a_i
\]

since \( \lim_\mathcal{U} \lambda a_i = (\lim_\mathcal{U} \lambda) (\lim_\mathcal{U} a_i) \) and clearly \( \lim_\mathcal{U} \lambda = \lambda \).

5. If \( a_i \geq 0 \) for all \( i \in I \), then \( \lim_\mathcal{U} a_i \geq 0 \) (assuming that it exists).

Proof. Suppose not, then \( \lim_\mathcal{U} a_i = A < 0 \), then

\[
\mathcal{U} \ni \{ i \in I \mid |a_i - A| < |A| \} \subseteq \{ i \in I \mid a_i = 0 \} = \emptyset
\]

Which is a contradiction. \( \square \)

We also get from the linearity of the ultralimit that if \( a_i \geq b_i \), and \( \lim_\mathcal{U} a_i \) and \( \lim_\mathcal{U} b_i \) both exist, then

\[
\lim_\mathcal{U} a_i - \lim_\mathcal{U} b_i = \lim_\mathcal{U} (a_i - b_i) \geq 0
\]

so \( \lim_\mathcal{U} a_i \geq \lim_\mathcal{U} b_i \).

Comment: it suffices for the above properties to hold \( \mathcal{U} \)-almost everywhere, since we can change a sequence on a small set without changing the ultralimit.

In general, the set-theoretic ultraproduct of Banach spaces does not constitute a Banach space, so we need a modified construction. This notion of a Banach space ultraproduct was introduced by Dacunha-Castelle and Krivine. Let \( (E_i)_{i \in I} \) be a family of Banach spaces. Consider the set

\[
\ell_\infty(I, E_i) = \{ (x_i) \in \prod_{i \in I} E_i \mid \|(x_i)\| = \sup_{i \in I} \|x_i\|_{E_i} < \infty \}
\]

Then \( \ell_\infty(I, E_i) \) is a Banach space under componentwise addition and scalar multiplication:

\[
(x_i)_{i \in I} + (y_i)_{i \in I} = (x_i + y_i)_{i \in I} \quad \lambda(x_i)_{i \in I} = (\lambda x_i)_{i \in I}
\]

Note that if \( (\tilde{x}_n) = ((x_{n,i})) \) is a Cauchy sequence in \( \ell_\infty(I, E_i) \), then for any \( \varepsilon > 0 \), there is an \( N_\varepsilon \in \mathbb{N} \) so for all \( n, m \geq N_\varepsilon \), \( \|\tilde{x}_n - \tilde{x}_m\| = \sup_{i \in I} \|x_{n,i} - x_{m,i}\| < \varepsilon \). Thus in each component, i.e. for each \( i \in I \), \( (x_{n,i}) \) is a Cauchy sequence in \( E_i \), which is a Banach space, so it has a limit \( x_i \). Note that for all \( n \geq N_1 \), \( \|\tilde{x}_n - \tilde{x}_{N_1}\| \leq 1 \), so for all \( i \in I \), \( \|x_{n,i} - x_{N_1,i}\| = 1 \). Fix an index \( i \in I \). Since \( (x_{n,i}) \rightarrow x_i \) there is an \( M_i, \varepsilon \in \mathbb{N} \) such that for all \( n \geq M_i, \varepsilon, \|x_i - x_{n,i}\| < 1 \). Now for \( n \geq \max(N_1, M_i, 1) \), we have

\[
\|x_i\| \leq \|x_i - x_{n,i}\| + \|x_{n,i}\| \leq 1 + \|\tilde{x}_{N_1}\| \leq 2 + \|\tilde{x}_{N_1}\|
\]
Thus \( \sup_{i \in I} ||x_i|| \leq 2 + ||\bar{x}_{N_i}|| < \infty \). Then consider \( \bar{x} = (x_i) \). We claim \( \bar{x}_n \to \bar{x} \). Let \( \varepsilon > 0 \) be given, then for \( n \geq N_{\varepsilon/2} \) we claim

\[
||\bar{x}_n - \bar{x}|| = \sup_{i \in I} ||x_{n,i} - x_i|| < \varepsilon
\]

For each \( i \in I \), let \( m_i = \max(N_{\varepsilon/2}, M_{i, \varepsilon/2}) \), then for all \( i \in I \), \( n \geq N_{\varepsilon/2} \)

\[
||x_{n,i} - x_i|| \leq ||x_{n,i} - x_{m_i,i}|| + ||x_{m_i,i} - x_i|| < ||\bar{x}_m - \bar{x}_{m,i}|| + \frac{\varepsilon}{2} < \varepsilon
\]

Thus \( \ell_{\infty}(I, E_i) \) is a Banach space.

For \( \mathcal{U} \) an ultrafilter on \( I \), let

\[
N_{\mathcal{U}} = \{ (x_i) \in \ell_{\infty}(I, E_i) \mid \lim_{\mathcal{U}} ||x_i||_{E_i} = 0 \}
\]

I.E. the set of sequences \((x_i)\) such that for all \( \varepsilon > 0 \),

\[
\{ i \in I \mid ||(x_i)||_{E_i} < \varepsilon \} \in \mathcal{U}
\]

It is easy to see \( N_{\mathcal{U}} \) is a linear subspace of \( \ell_{\infty}(I, E_i) \), since if \((x_i), (y_i) \in N_{\mathcal{U}}, \lambda \neq 0 \), then for \( \varepsilon > 0 \) given,

\[
\{ i \in I \mid ||\lambda x_i||_{E_i} < \varepsilon \} = \{ i \in I \mid ||x_i||_{E_i} < \frac{\varepsilon}{|\lambda|} \} \in \mathcal{U}
\]

so \((\lambda x_i) \in N_{\mathcal{U}}\), and

\[
\{ i \in I \mid ||x_i + y_i||_{E_i} < \varepsilon \} \subseteq \{ i \in I \mid ||x_i||_{E_i} < \frac{\varepsilon}{2} \} \cap \{ i \in I \mid ||y_i||_{E_i} < \frac{\varepsilon}{2} \} \in \mathcal{U}
\]

Thus \((x_i + y_i) \in N_{\mathcal{U}}\).

Furthermore, \( N_{\mathcal{U}} \) is closed: if \((x_n) \notin N_{\mathcal{U}}\), then since \((x_n) \in \ell_{\infty}(I, E_i), ||x_n|| = M\), then since \(||x_i|| \in [0, M] \subseteq \mathbb{R} \) is compact Hausdorff, we have that

\[
\lim_{\mathcal{U}} ||x_i|| = \varepsilon > 0
\]

We claim \( B((x_i), \frac{\varepsilon}{2}) \subseteq \ell_{\infty}(I, E_i) \setminus N_{\mathcal{U}} \). If \(||(y_i) - (x_i)|| = \sup_{i \in I} ||y_i - x_i|| < \frac{\varepsilon}{2} \), then

\[
I_{\varepsilon} = \{ i \in I \mid ||y_i|| < \frac{\varepsilon}{4} \} \in \mathcal{U}
\]

Then for \( i \in I_{\varepsilon} \),

\[
||y_i|| \geq ||x_i|| - ||y_i - x_i|| \geq \frac{3\varepsilon}{4} - \frac{\varepsilon}{4} = \frac{\varepsilon}{2}
\]

In particular,

\[
\{ i \in I \mid ||y_i|| < \frac{\varepsilon}{2} \} = I \setminus I_{\varepsilon} \notin \mathcal{U}
\]

So \( \lim_{\mathcal{U}} ||(y_i)|| \neq 0 \) so \((y_i) \notin N_{\mathcal{U}} \) as claimed.

Finally, we can define the ultraproduct of the \((E_i)_{i \in I}\) with respect to \( \mathcal{U} \) an ultrafilter on \( I \) by

\[
(E_i)_{\mathcal{U}} = \ell_{\infty}(I, E_i)/N_{\mathcal{U}}
\]
We will denote the equivalence class of an element \((x_i) \in \ell_\infty(I, E_i)\) in the ultraproduct as \((x_i)_U\).

We equip the ultraproduct \((E_i)_U\) with the canonical quotient norm:

\[
\|(x_i)_U\| = \inf_{(a_i) \in N_U} \|(x_i - a_i)\|_\infty = \inf_{(a_i) \in N_U} \sup_{i \in I} \|x_i - a_i\|_{E_i}
\]

In fact, the norm can be computed as

\[
\|(x_i)_U\| = \inf_{(a_i) \in N_U} \sup_{i \in I} \|x_i - a_i\|_{E_i} = \lim_U \|x_i\|
\]

**Proof.** The limit on the right hand side always exists, since if we consider the representative \((x_i)_{i \in I}\), \(\sup_{i \in I} \|x_i\|_{E_i} = M < \infty\), so each \(\|x_i\|_{E_i} \in [0, M]\), a compact Hausdorff space. Let \(L = \lim_U \|x_i\|\).

Now note that for any \(\varepsilon > 0\),

\[
I_\varepsilon = \{i \in I \mid \|x - L\| < \varepsilon\} \in \mathcal{U}
\]

Define the sequence \(\alpha_i\) by

\[
\alpha_i = \begin{cases} 
  x_i & \text{if } i \notin I_\varepsilon, \text{ i.e. } \|x_i - \lim \| \geq \varepsilon \\
  0 & \text{if } i \in I_\varepsilon
\end{cases}
\]

Then note

\[
\{i \in I \mid \alpha_i = 0\} \supseteq I_\varepsilon \in \mathcal{U}
\]

In particular,

\[
\lim_U \alpha_i = 0
\]

Thus \((\alpha_i) \in N_U\), and for all \(i \in I\),

\[
\|x_i - \alpha_i\| = \begin{cases} 
  0 & \text{if } i \notin I_\varepsilon \\
  \|x_i\| & \text{if } i \in I_\varepsilon
\end{cases} < L + \varepsilon
\]

Thus

\[
\|(x_i)_U\| = \inf_{(a_i) \in N_U} \|x_i - a_i\| \leq \|x_i - a_i\| \leq \sup_i \|x_i - a_i\| \leq L + \varepsilon
\]

Conversely, if \((a_i) \in N_U\), then

\[
I_{(a_i), \varepsilon} = \{i \in I \mid \|a_i\| < \frac{\varepsilon}{2}\} \cap \{i \in I \mid \|x_i - L\| < \frac{\varepsilon}{2}\} \in \mathcal{U}
\]

Then for all \(i \in I_{(a_i), \varepsilon} \neq \emptyset\),

\[
\|x_i - a_i\| \geq \|x_i\| - \|a_i\| \geq L - \varepsilon - \frac{\varepsilon}{2} = L - \varepsilon
\]

Thus for any \((a_i) \in N_U\),

\[
\sup_{i \in I} \|x_i - a_i\| \geq L - \varepsilon
\]

So

\[
\|(x_i)_U\| = \inf_{(a_i) \in N_U} \sup_{i \in I} \|x_i - a_i\| \geq L - \varepsilon
\]

Thus we have shown for all \(\varepsilon > 0\),

\[
\lim_U \|x_i\| - \varepsilon \leq \|(x_i)_U\| = \inf_{(a_i) \in N_U} \|(x_i - a_i)\| = \inf_{(a_i) \in N_U} \sup_{i \in I} \|x_i - a_i\| \leq \lim_U \|x_i\| + \varepsilon
\]

And we conclude that

\[
\|(x_i)_U\| = \lim_U \|x_i\|
\]

as claimed. \(\square\)
Note that if $\mathfrak{U}$ is the trivial or principal ultrafilter generated by $i_0 \in I$ then $(E_i)_{\mathfrak{U}} \cong E_{i_0}$.

If all the spaces $E_i = E$ then we speak of the ultrapower $E^I/\mathfrak{U}$ or $(E)_{\mathfrak{U}}$. There is a canonical isometric embedding $J$ of $E$ into its ultrapower $(E)_{\mathfrak{U}}$ which is defined by $J(x) = (x_i)_{\mathfrak{U}}$ where $x_i = x$ for all $i \in I$.

Now that we have introduced the ultraproduct of Banach spaces, we can define the ultraproduct of operators. If $(E_i)_{i \in I}$ and $(F_i)_{i \in I}$ are families of Banach spaces indexed by the same set $I$, and for each $i \in I$, $T_i \in B(E_i, F_i)$ is a bounded linear map from $E_i$ to $F_i$, such that

\[
\sup_{i \in I} ||T_i|| < \infty
\]

The ultraproduct of the family of operators $(T_i)_{i \in I}$ with respect to the ultrafilter $\mathfrak{U}$ on $I$ is $(T_i)_{\mathfrak{U}}$ defined by

\[
(x_i) \mapsto (T_ix_i)_{\mathfrak{U}}
\]

This map is well-defined: if $||T_i|| = 0$ for all $i \in I$, then $(T_i)_{\mathfrak{U}}((x_i)_{\mathfrak{U}}) = (0)_{\mathfrak{U}} = (T_i)_{\mathfrak{U}}((y_i)_{\mathfrak{U}})$. If $0 < \sup_{i \in I} ||T_i|| = M < \infty$, and if $(x_i) \sim_{\mathfrak{U}} (y_i)$, then $||(x_i - y_i)_{\mathfrak{U}}|| = \lim_{\mathfrak{U}} ||x_i - y_i|| = 0$, so for all $\varepsilon > 0$, \( \{i \in I \mid ||x_i - y_i|| < \varepsilon \} \in \mathfrak{U} \), thus

\[
\{i \in I \mid ||T_i(x_i) - T_i(y_i)|| < \varepsilon \} \supseteq \{i \in I \mid ||x_i - y_i|| < \frac{\varepsilon}{M} \} \in \mathfrak{U}
\]

And in particular

\[
\lim_{\mathfrak{U}} ||T_i(x_i) - T_i(y_i)|| = \lim_{\mathfrak{U}} ||T_i(x_i) - T_i(y_i)|| = ||(T_i(x_i))_{\mathfrak{U}} - (T_i(y_i))_{\mathfrak{U}}|| = 0
\]

So $(T_i(x_i))_{\mathfrak{U}} = ((T_i(y_i))_{\mathfrak{U}}$.

Moreover, we see that $||T_i||$ takes values in $[0, M]$ which is compact Hausdorff, so $\lim_{\mathfrak{U}} ||T_i|| = L$ exists.

**Proof.** It is clear $\lim_{\mathfrak{U}} ||T_i||$ exists, since

\[
\sup_{i \in I} ||T_i|| = M < \infty
\]

So by a previous proposition, $||T_i||$ takes values in $[0, M]$ which is compact Hausdorff, so

\[
\lim_{\mathfrak{U}} ||T_i|| = L \text{ exists.}
\]

Let $\varepsilon > 0$ be given. Let $||x_i|| = 1$, then

\[
I_0 = \{i \in I \mid 1 - ||x_i|| < \varepsilon \} \subseteq \{i \in I \mid ||x_i|| < 1 + \varepsilon \} \in \mathcal{F}
\]

So we can pick an equivalent sequence $(x'_i) \sim_{\mathfrak{U}} (x_i)$ with $||x'_i|| < 1 + \varepsilon$ for all $i \in I$. Now

\[
||T_i(x'_i)|| \leq ||T_i|| ||x_i|| \leq ||T_i|| (1 + \varepsilon)
\]

So in particular,

\[
||T((x_i)_{\mathfrak{U}})|| = \lim_{\mathfrak{U}} ||T_i(x_i)|| < (1 + \varepsilon) \lim_{\mathfrak{U}} ||T_i||
\]

Likewise, for all $i \in I$ we can find an $\alpha_i \in E_i$ with $||\alpha_i|| = 1$ and $||T_i(x_i)|| > ||T_i|| - \varepsilon$. Thus

\[
||T((x_i)_{\mathfrak{U}})|| = \lim_{\mathfrak{U}} ||T_i(x'_i)|| \geq \lim_{\mathfrak{U}} ||T_i|| (1 + \varepsilon)
\]

\[]
Proposition:
1. Banach algebras are stable under ultraproducts.
2. $C^*$-algebras are stable under ultraproducts.

Proof. Let $E_i$ be Banach algebras or $C^*$-algebras, so for $x_i, y_i \in E_i$
$$||x_i \cdot y_i|| \leq ||x_i|| \cdot ||y_i||$$
and (if the $E_i$ are $C^*$-algebras),
$$||x_i^* x_i|| = ||x_i||^2$$
We define
$$(x_i)_U \cdot (y_i)_U = (x_i \cdot y_i)_U$$
$$(x_i)_U^* = (x_i^*)_U$$
First we show that these are well-defined. If $(x_i)_U = (x'_i)_U$ and $(y_i)_U = (y'_i)_U$, then
$$\lim_U ||x_i - x'_i|| = 0 \quad \lim_U ||y_i - y'_i|| = 0$$
Thus
$$\lim_U ||x_i \cdot y_i - x'_i y'_i|| \leq \lim_U ||x_i \cdot y_i - x'_i \cdot y'_i|| + \lim_U ||x_i \cdot y'_i - x'_i y'_i||$$
$$\leq \lim_U ||x_i|| \cdot ||y_i - y'_i|| + \lim_U ||x_i - x'_i|| \cdot ||y'_i||$$
$$\leq ||(x_i)_U|| \lim_U ||y_i - y'_i|| + ||(y_i)_U|| \lim_U ||x_i - x'_i|| = 0$$
Thus
$$(x_i \cdot y_i)_U = (x'_i \cdot y'_i)_U$$
Likewise, if $(x_i)_U = (y_i)_U$, then
$$\lim_U ||(x_i^*)_U - (y_i^*)_U|| = \lim_U ||(x_i - y_i)^*|| = \lim_U ||x_i - y_i|| = 0$$
Thus
$$(x_i^*)_U = (y_i^*)_U$$
Next we check that the norm on $(E_i)_U$ is submultiplicative:
$$||(x_i \cdot y_i)_U|| = \lim_U ||x_i \cdot y_i|| \leq \lim_U ||x_i|| \cdot ||y_i||$$
$$\leq \lim_U ||x_i|| \lim_U ||y_i|| = ||(x_i)_U|| ||(y_i)_U||$$
Next we check that the norm on $(E_i)_U$ satisfies the $C^*$ identity:
$$||(x_i)_U^* \cdot (x_i)_U|| = ||(x_i^* \cdot x_i)_U|| = \lim_U ||x_i^* \cdot x_i||$$
$$= \lim_U ||x_i||^2 = \left( \lim_U ||x_i|| \right) \left( \lim_U ||x_i|| \right) = ||(x_i)_U||^2$$
Algebraic properties:
$$(x_i)_U((y_i)_U + (z_i)_U) = (x_i)_U \cdot (y_i + z_i)_U = (x_i \cdot (y_i + z_i))_U = (x_i y_i + x_i z_i)_U = (x_i)_U (y_i)_U + (x_i)_U (z_i)_U$$
$$(x_i)_U + \lambda (y_i)_U = ((x_i + \lambda y_i)_U) = ((x_i^* + \lambda y_i^*)_U) = (x_i^*)_U + \lambda (y_i^*)_U$$
$$(x_i)_U = (x_i^*)_U + \lambda (y_i)_U$$
Recall that a non-empty set \(M\) with a relation \(\leq\) is said to be an ordered set if

1. \(x \leq x\) for all \(x \in M\) (\(\leq\) reflexive).
2. \(x \leq y\) and \(y \leq x\) implies \(x = y\) (\(\leq\) antisymmetric).
3. \(x \leq y\) and \(y \leq z\) implies \(x \leq z\) (\(\leq\) is transitive).

If \(A \subset M\) an ordered set, elements \(x, z \in M\) are an upper bound or lower bound respectively of \(A\) if \(a \leq x\) for all \(a \in A\) and \(z \leq a\) for all \(a \in A\). If there is an upper bound or lower bound of \(A\), we say \(A\) is bounded from above or from below respectively. If \(A\) is bounded from above and from below, then \(A\) is called order bounded. Let \(x, y \in M\) with \(x \leq y\). We let the order interval between \(x\) and \(y\) be \([x, y] = \{z \in M \mid x \leq z \leq y\}\)

It is clear than a subset \(A \subseteq M\) is order bounded if and only if it is contained in some order interval. A real vector space \(E\) which is ordered by a relation \(\leq\) is called a vector lattice or Riesz space if any two elements \(x, y \in E\) have a least upper bound (or ‘join’) denoted \(x \lor y = \sup(x, y)\) and a greatest lower bound (or ‘meet’) denoted by \(x \land y = \inf(x, y)\). Furthermore, the order relation \(\leq\) must satisfy

1. \(x \leq y\) implies \(x + z \leq y + z\) for all \(x, y, z \in E\).
2. \(0 \leq x\) implies \(0 \leq tx\) for all \(x \in E\) and \(t \in \mathbb{R}_+\).

Let \(E\) be a vector lattice. The positive cone of \(E\) is \(E_+ = \{x \in E \mid 0 \leq x\}\)

For any \(x \in E\), we let

\[
\begin{align*}
    x_+ &= x \lor 0 \\
    x_- &= (-x) \lor 0 \\
    |x| &= x \lor (-x)
\end{align*}
\]

be the positive part, negative part, and absolute value of \(x\) respectively.

Two elements \(x, y \in E\) are called orthogonal or lattice disjoint, denoted by \(x \perp y\) is \(|x| \land |y| = 0\).

For a vector lattice \(E\), we have the following properties:

Proposition: For \(t \in \mathbb{R}\),

\[
(tx) \land (ty) = \begin{cases} 
    t(x \land y) & t \geq 0 \\
    t(x \lor y) & t \leq 0
\end{cases}
\]

In particular, for \(t = -1\) \(x \lor y = -((x) \land (-y))\).

**Proof.** Case 0: \(t = 0\), then it is clear that \(tx = 0 = ty\), so \((tx) \land (ty) = 0 = t(x \lor y) = t(x \land y)\).

Case 1: \(t > 0\). Note that

\[x \land y \leq x, y\]

Thus

\[t(x \land y) \leq tx, ty\]
In particular, \(tx \land y\) is a lower bound for both \(tx\) and \(ty\), so it is less than the greatest lower bound of \(tx\) and \(ty\), and we have
\[
(tx \land y) \leq (tx) \land (ty)
\]
Conversely,
\[
(tx) \land (ty) \leq tx, ty
\]
Multiplying both sides by \(1/t > 0\) we have
\[
\frac{1}{t} [(tx) \land (ty)] \leq x, y
\]
In particular, \(\frac{1}{t} [(tx) \land (ty)]\) is a lower bound for both \(x\) and \(y\), so it is less than the greatest lower bound of \(x\) and \(y\), and we have
\[
\frac{1}{t} [(tx) \land (ty)] \leq x \land y
\]
Multiplying both sides by \(t > 0\) yields
\[
(tx) \land (ty) \leq t(x \land y)
\]
Thus for \(t > 0\),
\[
(tx) \land (ty) = t(x \land y)
\]
as claimed.

Case 2: \(t < 0\), so \(-t > 0\). Note that for any \(w, z \in E\),
\[
w \leq z \text{ implies } -tw \leq -tz \implies tz \leq tw \text{ or } tw \geq tz
\]
Now
\[
x \lor y \geq x, y
\]
Multiplying both sides by \(t < 0\) gives
\[
t(x \lor y) \leq tx, ty
\]
In particular \(t(x \lor y)\) is a lower bound of \(tx\) and \(ty\), so it is less than the greatest lower bound of \(tx\) and \(ty\), so
\[
t(x \lor y) \leq (tx) \lor (ty)
\]
Conversely,
\[
(tx) \land (ty) \leq tx, ty
\]
Multiplying both sides by \(1/t < 0\) we get
\[
\frac{1}{t} [(tx) \land (ty)] \geq x, y
\]
In particular \(\frac{1}{t} [(tx) \land (ty)]\) is an upper bound of \(x\) and \(y\), so it is greater than the least upper bound of \(x\) and \(y\), so
\[
\frac{1}{t} [(tx) \land (ty)] \geq x \lor y
\]
Multiplying both side by \(t < 0\) gives
\[
(tx) \land (ty) \leq t(x \lor y)
\]
Thus for \(t < 0\)
\[
(tx) \land (ty) = t(x \lor y)
\]
as claimed.  

\[\square\]
Proposition: for all \( x, y, \text{ and } z \),

\[
(x \lor y) + z = (x + z) \lor (y + z)
\]

and

\[
(x \land y) + z = (x + z) \land (y + z)
\]

**Proof.**

\( x \lor y \geq x, y \)

Adding \( z \) to both sides

\[
(x \lor y) + z \geq x + z, y + z
\]

In particular, \((x \lor y) + z\) is an upper bound for \(x + z\) and \(y + z\), so it must be greater than the least upper bound, so

\[
(x \lor y) + z \geq (x + z) \lor (y + z)
\]

Conversely,

\[
(x + z) \lor (y + z) \geq x + z, y + z
\]

Thus

\[
[(x + z) \lor (y + z)] - z \geq x \lor y
\]

In particular, \([(x + z) \lor (y + z)] - z\) is an upper bound of \(x \) and \(y\), so it must be greater than the least upper bound of \(x\) and \(y\), so

\[
[(x + z) \lor (y + z)] - z \geq x \lor y
\]

Adding \( z \) to both sides gives

\[
(x + z) \lor (y + z) \geq (x \lor y) + z
\]

Thus

\[
(x + z) \lor (y + z) = (x \lor y) + z
\]

as claimed.

The other equality is proved the same way. \( \square \)

Proposition: For all \( x \) in a vector lattice, \( x = x_+ - x_- \text{ and } |x| = x_+ + x_- \).

**Proof.** By (a) i.,

\[
x = x + 0 = (x \lor 0) + (x \land 0) = x_+ + (x \land 0)
\]

by (a) ii., \( x_- = (x) \lor 0 = -[(x) \land 0] = -(x \land 0) \), so \( x \land 0 = -x_- \). Thus

\[
x = x_+ + (x \land 0) = x_+ - x_-
\]

Alternatively:

\[
2x_+ = 2(x \lor 0) = (2x) \lor 0 = (x + x) \lor (x - x) = x + [x \lor (-x)] = x + |x|
\]

and

\[
2x_- = 2((-x) \lor 0) = (-2x) \lor 0 = (-x - x) \lor (-x + x) = -x + [-x \lor x] = -x + |x|
\]
Thus
\[ 2(x_+ + x_-) = x + |x| - x + |x| = 2|x| \]
and similarly
\[ 2(x_+ - x_-) = x + |x| - (-x + |x|) = 2x \]
Thus
\[ x_+ - x_- = x \quad \text{and} \quad x_+ + x_- = |x| \]
as claimed.

Proposition: If \( x, y \) are elements of a Banach lattice and \( z = x \land y \), then \( x - z \) and \( y - z \) are positive and lattice disjoint or orthogonal, i.e. \( (x - z) \perp (y - z) \) or \( |x - z| \lor |y - z| = 0 \).

Proof. Note that \( x \geq x \land y = z \) and \( y \geq x \land y = z \), so \( x - z \geq 0 \) and \( y - z \geq 0 \). In particular,
\[ |x - z| = (x - z) \lor (z - x) = x - z \]
since \( x - z \geq 0 \geq -(x - z) \) and similarly, \( |y - z| = y - z \). Now
\[ |x - z| \land |y - z| = (x - z) \land (y - z) = (x \land y) - z = 0 \]
As claimed.

Proposition: \( x = 0 \) if and only if \( |x| = 0 \).

Proof. Suppose \( x = 0 \). Then
\[ |x| = x \land -x = 0 \land 0 = 0 \]
Conversely, if \( |x| = 0 \), then \( 0 = |x| = x \land (-x) \geq x, -x \), so \( x = 0 \).

Proposition: \( x \) and \( y \) are lattice disjoint, i.e. \( x \perp y \) or \( |x| \land |y| = 0 \) if and only if \( |x| \lor |y| = |x| + |y| \).

Proof. Note that
\[ |x| \lor y = |x| + |y| \iff |x| \lor |y| - (|x| + |y|) = 0 \]
\[ \iff (|x| - (|x| + |y|)) \lor (|y| - (|x| + |y|)) = 0 \]
\[ \iff (-|y|) \lor (-|x|) = 0 \]
\[ \iff -(|x| \land |y|) = 0 \]
\[ \iff |x| \land |y| = 0 \]

Proposition: if \( x \) and \( y \) are elements of a vector lattice, then
\[ x + y = x \lor y + x \land y \]
Proof. Note that
\[ x \lor y = [x - (x \land y)] \lor [y - x \land y] + x \land y \]
And by the previous propositions, \( x - (x \land y) \) and \( y - x \land y \) are positive and lattice disjoint, so we have that
\[ [x - (x \land y)] \lor [y - x \land y] = [x - (x \land y)] + [y - x \land y] = x + y - 2(x \land y) \]
Thus we have that
\[ x \lor y = x + y - x \land y \]
Thus
\[ x + y = x \lor y + x \land y \]
as claimed. \( \square \)

Proposition: For any \( x \) in a vector lattice, \( x = x_+ - x_- \) and this decomposition of \( x \) into postive lattice disjoint elements is unique: if \( x = y - z \) with \( y, z \geq 0 \) and \( y \perp z \), i.e. \( y \lor z = 0 \) then \( y = x_+ \) and \( z = x_- \).

Proof.
\[ x = x + 0 = x \lor 0 + x \land 0 = x \lor 0 - (-x \lor 0) = x_+ - x_- \]
It is clear \( x_+, x_- \geq 0 \). Furthermore,
\[ x - (x_+ \lor x_-) + x_- = (x - x_+ + x_-) \lor (x - x_- + x_-) = 0 \lor x = x_+ = x + x_- \]
Thus \( x_+ \lor x_- = 0 \), and these elements are lattice disjoint.

To show this decomposition is unique, suppose \( x_+ - x_- = x = y - z \) with \( y, z \geq 0 \) and \( y \lor z = 0 \). Then
\[ y - x_+ = x + z - x_+ = z + x_- \geq 0 \]
And similarly since \( z = y - x \),
\[ z - x_- = y - x - x_- = y - x_+ \geq 0 \]
Thus \( y - x_+ = z - x_- \geq 0 \). Furthermore since \( 0 \leq y - x_+ \leq y \), and \( 0 \leq z - x_- \leq z \)
\[ 0 \leq (y - x_+) \land (z - x_-) \leq y \land z = 0 \]
so
\[ (y - x_+) \land (z - x_-) = 0 \]
However we’ve shown \( y - x_+ = z - x_- \) so
\[ 0 = y - x_+ = (y - x_+) \land (z - x_-) = z - z_- = 0 \]
so we have \( y = x_+ \) and \( z = x_- \), as claimed. \( \square \)

Proposition: For \( \lambda \in \mathbb{R}, |\lambda x| = |\lambda| |x| \) (note that \( |\cdot| \) is being used for two different opperations here).
Proof. If \( \lambda \geq 0 \), \( \lambda = |\lambda| \), then
\[
|\lambda x| = (\lambda x) \lor (-\lambda x) = \lambda(x \lor -x) = |\lambda| |x|
\]
Similarly, if \( \lambda \leq 0 \), \( |\lambda| = -\lambda \), and
\[
|\lambda x| = (\lambda x) \lor (-\lambda x) = -\lambda(-x \lor x) = |\lambda| |x|
\]

Lemma: if \( |x| \lor |y| = 0 \) then \( x \land y = 0 \) and \( x \land y = 0 \).

Proof. Since \( 0 \leq x_+, x_-, y_+, y_- \) and \( x_+, x_- \leq |x| \) and \( y_+, y_- \leq |y| \),
\[
0 \leq x_+ \land y_+ \leq |x| \land |y| = 0 \quad \text{and} \quad 0 \leq x_- \land y_- \leq |x| \land |y| = 0
\]

Thus \( x_+ \land y_+ = 0 = x_- \land y_- \) as claimed.

Proposition: for all \( x, y \) in a vector lattice, \(|x + y| \leq |x| + |y|\), and equality holds if and only if \( x \) and \( y \) are lattice disjoint.

Proof. Note that \( 0 \leq x_+, x_-, y_+, y_- \) so in particular,
\[
0 \leq x_+ \land y_+, x_- \land y_-
\]
Also, since \( x_+ \perp x_-, y_+ \perp y_- \) are positive lattice disjoint elements, we have that
\[
|x| = x_+ \lor x_- = x_+ \lor x_- \quad \text{and} \quad |y| = y_+ \lor y_- = y_+ \lor y_-
\]
Thus
\[
|x + y| = (x + y) \lor (-x - y)
\]
\[
= (x + |x| + y + |y|) \lor (-x + |x| - y + |y|) - |x| - |y|
\]
\[
= (2x_+ + 2y_+) \lor (2x_- + 2y_-) - |x| - |y|
\]
\[
= 2[(x_+ \lor y_+) \lor (x_- \lor y_-)] - |x| - |y|
\]
\[
= 2[(x_+ \lor y_+) \lor (x_- \lor y_-)] - |x| - |y|
\]
\[
\leq 2[(x_+ \lor y_+) \lor (x_- \lor y_-)] - |x| - |y|
\]
\[
= 2[(x_+ \lor y_+) \lor (y_+ \lor y_-)] - |x| - |y|
\]
\[
= 2[(x_+ \lor x_-) \lor (y_+ \lor y_-)] - |x| - |y|
\]
\[
= 2[|x| \lor |y|] - |x| - |y|
\]
\[
= 2[|x| + |y| - |x| \land |y|] - |x| - |y|
\]
\[
= |x| + |y| - 2[|x| \land |y|]
\]
\[
\leq |x| + |y|
\]

From the previous lemma, we see equality holds throughout exactly when \( |x| \lor |y| = 0 \), i.e. when \( x \) and \( y \) are lattice disjoint.

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A norm $|| \cdot ||$ on a vector lattice $E$ is called a lattice norm if

$$ |x| \leq |y| \text{ implies } ||x|| \leq ||y|| \quad \text{for all } x, y \in E $$

A Banach lattice is a real Banach $E$ endowed with an ordering $\leq$ such that $(E, \leq)$ is a vector lattice and the norm on $E$ is a lattice norm.

Proposition: If $E$ is a Banach lattice, then

1. The lattice operations $\lor, \land, \cdot_+, \cdot_-$, and $| \cdot |$ are continuous.
2. The positive cone $E_+$ is closed.
3. Order intervals are closed and bounded.

Proposition: The class of Banach lattices is stable under ultraproducts.

Proof. Given Banach lattices $E_i$, we define the positive cone of $(E_i)_\mathcal{U}$ to be the set of all $(x_i)_\mathcal{U}$ with $x_i \geq 0$ for all $i \in I$. Thus $(x_i)_\mathcal{U} \geq (y_i)_\mathcal{U}$ if and only if there is a sequence $(z_i)$ with $x_i \geq y_i + z_i$ and $\lim_{i \to \infty} z_i = 0$. It follows that

$$(x_i)_\mathcal{U} \lor (y_i)_\mathcal{U} = (x_i \lor y_i)_\mathcal{U}$$
$$(x_i)_\mathcal{U} \land (y_i)_\mathcal{U} = (x_i \land y_i)_\mathcal{U}$$
$$||(x_i)_\mathcal{U}|| = ||x_i||_{\mathcal{U}}$$

where $\lor$ denotes the least upper bound, $\land$ denotes the greatest lower bound, and $|| \cdot ||$ denotes the absolute value, respectively. We can now check the axioms of Banach lattices... \qed

Note that $C(K)$ for $K$ compact Hausdorff and $L_p(X, \mu)$ are Banach lattices. We say that $f(x) \leq g(x)$ if and only if $\Re(f) \leq \Re(g)$ and $\Im(f) \leq \Im(g)$ ($\mu$-a.e. if applicable). It is straightforward to check that these spaces are vector lattices under this ordering. Then

$$f \lor g(x) = \max(\Re(f(x)), \Re(g(x))) + i \max(\Im(f(x)), \Im(g(x)))$$
$$f \land g(x) = \min(\Re(f(x)), \Re(g(x))) + i \min(\Im(f(x)), \Im(g(x)))$$
$$f_+(x) = \Re(f(x))_+ + i\Im(f(x))_+$$
$$f_-(x) = \Re(f(x))_- + i\Im(f(x))_-$$

We will use $[\cdot]$ for the absolute value determined by the ordering and $| \cdot |$ for the absolute value on $\mathbb{C}$.

$$[f](x) = |\Re(f(x))| + i|\Im(f(x))|$$

Note that

$$|[f](x)| = |\Re(f(x))| + i|\Im(f(x))| = \sqrt{|\Re(f(x))|^2 + |\Im(f(x))|^2}$$
$$= |\Re(f(x)) + i\Im(f(x))| = |f(x)|$$

In particular,

$$|||f||| = ||f||$$

Also, if two functions $f$ and $g$ are lattice disjoint, $[f] \land [g] \equiv 0$ (the zero function), so

$$\min(|\Re(f(x))|, |\Re(g(x))|) + \min(|\Im(f(x))|, |\Im(g(x))|) = 0 \quad (\mu - \text{a.e. if applicable})$$
So \( f \) and \( g \) have (\( \mu \)-a.e.) disjoint ‘supports’ \( \{x|f(x) \neq 0\} \), not the closure. We need to check that the norms on these spaces are lattice norms, i.e. \( |f| \leq |g| \) implies \( ||f|| \leq ||g|| \). Suppose \( |f| \leq |g| \) for \( f, g \in C(K) \). Then \( |g| - |f| \geq 0 \), so \( |\Re(g)| \geq |\Re(f)| \) and \( |\Im(g)| \geq |\Im(f)| \). In particular, for all \( x \in K \) (or \( \mu \)-a.e. in \( X \))

\[
g(x) = \sqrt{|\Re(g(x))|^2 + |\Im(g(x))|^2} \geq \sqrt{|\Re(f(x))|^2 + |\Im(f(x))|^2} = |f(x)|
\]

Thus \( ||g|| \geq ||f|| \).

Now we will give descriptions of the ultraproducts of \( C(K) \) (for compact Hausdorff \( K \)) and \( L_p(X, \mu) \) spaces.

First, a definition: an abstract \( L^p \) space is a Banach lattice whose norm is \( p \)-additive \((1 \leq p \leq \infty)\), i.e. for any positive, lattice disjoint elements \( x, y \geq 0 \),

\[
||x + y||^p = ||x||^p + ||y||^p
\]

if \( 1 \leq p < \infty \) and

\[
||x + y|| = \max(||x||, ||y||)
\]

when \( p = \infty \).

Theorem (Bohnenblust, Nakano): Let \( Y \) be a abstract \( L_p \) space, then \( Y \) is linearly isometric and lattice isomorphic to \( L_p(\mu, \mathbb{C}) \) for some measure \( \mu \).

Comment: The following are equivalent: non-zero multiplicative linear functionals on a unital \( C^* \)-algebra \( A \), non-zero algebra homomorphisms from \( A \) to \( \mathbb{C} \), and non-zero *-homomorphisms from \( A \) to \( \mathbb{C} \) (which are called characters). This is because a non-zero algebra homomorphism \( \Phi \) is necessarily continuous as \( \ker(\Phi) \) is codimension one, and since \( \Phi \) is multiplicative \( \ker(\Phi) \) is an ideal, thus it is a maximal ideal. Thus either it is closed, or its closure is the whole space, but this is impossible, since if \( ||x - I|| < 1 \) then \( x \) is invertible in \( A \), so \( 1 = \Phi(I) = \Phi(xx^{-1}) = \Phi(x)\Phi(x^{-1}) \).

so \( x \notin \ker(\Phi) \). Thus \( \ker(\Phi) \) cannot be the whole space. Since \( \ker(\Phi) \) is closed, \( \Phi \) is continuous. If not, then \( \Phi \) is unbounded, so we could find \( x_n \) with \( ||x_n|| = 1 \) and \( |\Phi(x_n)| \geq n \) for all \( n \in \mathbb{N} \), then \( y_n = x_n/\Phi(x_n) \) have \( \Phi(y_n) = 1 \), so \( y_n \in I + \ker(\Phi) \), which is a closed set (as translation by \( I \) is a homeomorphism), but \( y_n \to 0 \notin I + \ker(\Phi) \), which is a contradiction.

Next, a non-zero algebra homomorphism \( \Phi \) from \( A \) to \( \mathbb{C} \) has to respect the star operations since \( x \in A \) has a decomposition as

\[
x = \frac{x + x^*}{2} + i \frac{x - x^*}{2i} = R(x) + i \Im(x)
\]

Where \( \Re(x) \) and \( \Im(x) \) are self-adjoint. Thus

\[
x^* = (\Re(x) + i\Im(x))^* = \Re(x) - i\Im(x)
\]

In particular

\[
\Phi(x^*) = \Phi(\Re(x) - i\Im(x)) = \Phi(\Re(x)) - i\Phi(\Im(x)) = \Phi(\Re(x)) + i\Phi(\Im(x)) = \Phi(\Re(x) + i\Im(x)) = \Phi(x)
\]

So \( \Phi \) is *-preserving.

Fact: if \( \Phi \) is a multiplicative, (non-zero) linear functional on \( C(K) \) with \( K \) compact Hausdorff, then \( \Phi \) is evaluation at a point \( x_0 \in K \).
Proof. It is clear that for all $x \in K$, evaluation at $x$, denoted $e_x$ is a multiplicative linear functional on $\mathcal{C}(K)$. Thus we have an embedding $K \hookrightarrow \widehat{\mathcal{C}(K)}$, the characters on $\mathcal{C}(K)$, that is, the non-zero $*$-homomorphisms from $\mathcal{C}(K)$ to $\mathbb{C}$.

Suppose that $\psi$ is a character which is not of the form $i_x$ for any $x \in K$. Then for each $x \in K$, there is a function $g_x \in \mathcal{C}(X)$ with $g_x(x) = i_x(g_x) \neq \psi(g_x)$, in particular the function $f_x = g_x - \psi(g_x)$ has $f_x(x) \neq 0$ and $\psi(f_x) = 0$. The cozero set of $f_x$, say $N_x = \{ t \in K \mid f_x(t) \neq 0 \} = f_x^{-1}(\mathbb{C} \setminus \{0\})$ is an open set by the continuity of $f_x$, and $U_x$ contains $x$. In particular, $\{U_x \mid x \in K\}$ is an open cover of $K$, which is compact, so we get a finite subcover $\{U_{x_1}, \ldots, U_{x_n}\}$. Consider the corresponding functions $f_{x_1}, \ldots, f_{x_n}$, which satisfy $f_{x_j} \neq 0$ on $U_{x_j}$ for $1 \leq j \leq n$.

Consider the function $f = \sum_{j=1}^n f_{x_j}f_{x_j} = \sum_{j=1}^n |f_{x_j}|$. Then $f(x) > 0$ for all $x \in K$, so $f^{-1} \in \mathcal{C}(K)$, and $\psi(f) = 1$, but then

$$1 = \psi(1) = \psi(ff^{-1}) = \psi(f)\psi(f^{-1}) = 0$$

Which is a contraction. Thus every multiplicative linear functional on $\mathcal{C}(K)$ is evaluation at a point $x \in K$.

Theorem:

1. Let $K_i, i \in I$ be compact Hausdorff spaces. Then there is a compact Hausdorff space $K$ such that the ultraproduct $(\mathcal{C}(K_i))_U$ is linearly isometric to $\mathcal{C}(K)$. The isometry preserves the multiplicative and lattice structure.

2. Let $1 \leq p < \infty$, and let $\mu_i (i \in I)$ be arbitrary $\sigma$-additive measures. Then $(L_p(\mu_i))_U$ is order isometric to $L_p(\nu)$ for a certain measure $\nu$.

Proof. Note that $(\mathcal{C}(K_i))_U$ is a commutative $C^*$-algebra with identity. Thus by the Gelfand representation theorem, it is isomorphic to continuous functions on the character space of this $C^*$-algebra, which is a compact Hausdorff space in the weak* topology.

Let $x, y \in (L_p(\mu_i))_U$ be positive disjoint elements. Choose representations $x = (x_i)_U$ and $y = (y_i)_U$ and put $z_i = x_i \wedge y_i$. Then $z_i \geq 0$ as 0 is a lower bound for $x_i$ and $y_i$. In particular,

$$|x_i| = x_{i,+} \vee x_{i,-} = (x_i \vee 0) \vee (-x_i \vee 0) = x_i \vee 0 = x_i$$

And similar equalities hold for $y_i$ and $z_i$. Now

$$(0)_U = |x| \wedge |y| = |(x_i)_U| \wedge |(y_i)_U|$$

$$= \left( |x_i| \wedge |y_i| \right)_U$$

$$= \left( |x_i| \wedge |y_i| \right)_U$$

$$= (x_i \wedge y_i)_U = |(x_i \wedge y_i)|_U$$

$$= |(x_i \wedge y_i)|_U = |(z_i)|_U$$

Thus

$$|(z_i)| = |0|$$

and

$$\lim_U |z_i| = |(z_i)| = |(0)| = 0$$
It follows that $\lim \|z_i\| = 0$. On the other hand, $(x_i - z_i)$ and $(y_i - z_i)$ are disjoint, since $x_i \geq z_i = x_i \wedge y_i$ so $x_i - z_i \geq 0$, and $|x_i - z_i| = x_i - z_i$, and similarly $|y_i - z_i| = y_i - z_i$.

$$|x_i - z_i| \wedge |y_i - z_i| = (x_i - z_i) \wedge (y_i - z_i) = |x_i - (x_i \wedge y_i)| \wedge |y_i - (x_i \wedge y_i)| = |x_i \wedge y_i| - (x_i \wedge y_i) = 0$$

Therefore

$$|x_i - z_i| + |y_i - z_i| = |x_i + y_i - 2z_i|$$

Also the functions $|x_i - z_i|$ and $|y_i - z_i|$ have $\mu_i$-a.e. disjoint supports, and

$$\int_{X_i} |x_i - z_i|^p d\mu_i + \int_{X_i} |y_i - z_i|^p d\mu_i = \int_{X_i} |x_i + y_i - 2z_i|^p d\mu_i$$

$$||x_i - z_i||^p + ||y_i - z_i||^p = ||x_i + y_i - 2z_i||^p$$

Taking limits with respect to $\mathcal{U}$, and using the fact that $\lim \|z_i\| = 0$, so that $(x_i)_{\mathcal{U}} = (x_i - z_i)_{\mathcal{U}}$, etc. we see that

$$\lim_{\mathcal{U}} (||x_i - z_i||^p + ||y_i - z_i||^p) = \lim_{\mathcal{U}} ||x_i + y_i - 2z_i||^p$$

$$(\lim_{\mathcal{U}} ||x_i - z_i||)^p + (\lim_{\mathcal{U}} ||y_i - z_i||)^p = (\lim_{\mathcal{U}} ||x_i + y_i - 2z_i||)^p$$

$$||x_i - z_i||^p + ||y_i - z_i||^p = ||(x_i + y_i - 2z_i)||^p$$

Thus $(L_p(\mu_i))_{\mathcal{U}}$ is an abstract $L_p$ space, so by Bohnenblust’s theorem, it is isometric and lattice isomorphic to a (concrete) $L_p$ space.

Next, we would like to study the structure of the space $K$ that satisfies

$$(\mathcal{C}(K_i))_{\mathcal{U}} \cong \mathcal{C}(K)$$

We will connect $K$ with the set-theoretic ultraproduct of the spaces $K_i$ topologized in a specific canonical way.

Given a familiar of topological compact Hausdorff spaces $K_i (i \in I)$ and an ultrafilter $\mathcal{U}$ on $I$, we form the set-theoretic ultraproduct $(K_i)_{\mathcal{U}}$ and consider the following family of subsets

$$B = \{(U_i)_{\mathcal{U}} \subseteq (K_i)_{\mathcal{U}} : U_i \text{ is an open subset of } K_i \text{ for all } i \in I\}$$

This family forms a basis of a topology on $(K_i)_{\mathcal{U}}$: certainly it covers $(K_i)_{\mathcal{U}}$ as the whole space is in this family, and if $(U_i)_{\mathcal{U}}, (V_i)_{\mathcal{U}} \in B$, then

$$(U_i)_{\mathcal{U}} \cap (V_i)_{\mathcal{U}} = (U_i \cap V_i)_{\mathcal{U}} \in B$$

We will equip $(K_i)_{\mathcal{U}}$ with the topology generated by this basis.

Recall a Boolean algebra is a set $A$, equipped with two binary operations $\wedge$ (called “meet” or “and”), $\vee$ (called “join” or “or”), a unary operation $\neg$ (called “complement” or “not”) and two
Recall that a topological Hausdorff space \( X \) component of each point is the point itself. In particular, if \( X \) is compact, then \( X \) is closed under \( \cap \), \( \cup \), and \( \cdot \) operations.

Prototypical examples are the algebras of subsets of a set \( S \), \( A = \mathcal{P}(S) \) the power set of \( S \) which is closed under \( \cup, \cap \), and \( \Delta = S \setminus \cdot \), with \( 0 = \emptyset \) and \( 1 = S \). Also can be thought of the propositional calculus with and, or, and not, \( 0 \) as false and \( 1 \) as true. Comment: Boolean algebras are bounded, distributive lattices with complements.

Let \( B(K_i) \) be the Boolean algebra of clopen sets of \( K_i \). Then the ultraproduct of these algebras is

\[
(B(K_i))_\mathcal{U} = \{(A_i)_\mathcal{U} : A_i \in B(K_i) \text{ for all } i \in I\}
\]

Then \( (B(K_i))_\mathcal{U} \) is a Boolean algebra of subset of \( (K_i)_\mathcal{U} \) in the obvious way:

\[
(A_i)_\mathcal{U} \cap (B_i)_\mathcal{U} = (A_i \cap B_i)_\mathcal{U} \\
(A_i)_\mathcal{U} \cup (B_i)_\mathcal{U} = (A_i \cup B_i)_\mathcal{U} \\
(K_i)_\mathcal{U} \setminus (A_i)_\mathcal{U} = (K_i \setminus A_i)_\mathcal{U}
\]

Recall that a topological Hausdorff space \( X \) is totally disconnected if and only if the connected component of each point is the point itself. In particular, if \( X \) is compact, then \( B(X) \) must be a basis of the topology of \( X \) (this is a general topology fact).

**Theorem:** Let \( K_i \ (i \in I) \) be compact Hausdorff spaces and let \( C(K) \cong (C(K_i))_{\mathcal{U}} \) topologized as above. Then

1. \( (K_i)_{\mathcal{U}} \) is canonically homeomorphic to a dense subset of \( K \).

**Proof.** Let \( \vec{t} = (t_i)_{\mathcal{U}} \in (K_i)_{\mathcal{U}} \). Then we get a linear function \( \Phi_t : C(K_i)_{\mathcal{U}} \rightarrow \mathbb{C} \) for functions \( f_i \in C(K_i) \ (i \in I) \) defined by

\[
\Phi_t((f_i)_{\mathcal{U}}) = \lim_{\mathcal{U}} f_i(t_i)
\]

Note that the limit exists since the functions \( f_i \) are uniformly bounded. Then

\[
\Phi_t((f_i)_{\mathcal{U}}(g_i)) = \lim_{\mathcal{U}} g_i(t_i)f_i(t_i) = \lim_{\mathcal{U}} g_i(t_i)f_i(t_i) = \Phi_t((f_i)_{\mathcal{U}}) \Phi_t((g_i)_{\mathcal{U}})
\]

Thus \( \Phi_t \) defines a multiplicative functional on \( C(K) \cong (C(K_i))_{\mathcal{U}} \), so it must be a point evaluation, so we associate

\[
t = (t_i)_{\mathcal{U}} \leftrightarrow \Phi_t = e_{k(t)}, \ k(t) \in K
\]

This defines a map \( h : (K_i)_{\mathcal{U}} \rightarrow K \). This map is one-to-one: given two distinct elements \( s = (s_i)_{\mathcal{U}} \) and \( t = (t_i)_{\mathcal{U}} \), there is a set \( I_0 \in \mathcal{U} \) with \( s_i \neq t_i \), then since \( K_i \) is compact Hausdorff thus normal, by Urysohn’s lemma there are functions \( f_i \in C(K_i) \) with \( \|f_i\|_{\infty} = 1 \),

\[
f_i(s_i) = 0 \quad \text{and} \quad f_i(t_i) = 1 \quad i \in I_0
\]
Letting $f = (f_i)_U \in \mathcal{C}(K)$, we see that 

\[ f(h(s)) = \lim_{U} f_i(x_i) = 0 \quad \text{and} \quad f(h(t)) = \lim_{U} f_i(t_i) = 1 \]

Thus $h(s) \neq h(t)$.

The density of $h((K_i)_U)$ in $K$ follows from similar considerations: Suppose for sake of contradiction that the closure of $h((K_i)_U) \subseteq K$. Then since $K$ is compact Hausdorff, so normal, by Urysohn’s lemma there is a function $f \in \mathcal{C}(K)$ corresponding to $(f_i)_U \in (\mathcal{C}(K_i))_U$ with 

\[ ||f|| = 1 = \lim_{U} ||f_i|| \quad \text{and} \quad f(h(t)) = 0 \quad \text{for all} \quad t \in (K_i)_U. \]

Then we have that 

\[ ||f|| = \lim_{U} ||f_i|| = 1 \]

Thus there exist $t_i \in K_i$ with $|f_i(t_i)| = ||f_i||$, so $\lim_{U} |f_i(t_i)| = \lim_{U} ||f_i|| = 1$. Let $t = (t_i)_U$, then 

\[ 0 = f(h(t)) = \lim_{U} f_i(t_i) \]

But since $\lim_{U} |f_i(t_i)| = 1$ and the limit on the right exists, it must have norm 1. This is a contraction.

We can show that $h$ is a homeomorphism by noting that by Urysohn’s lemma that sets of the form 

\[ \{ t \in K : |f(t)| < \gamma \} \]

with $||f|| = 1$ and $0 < \gamma < 1$ are a basis of the topology of $K$. Similar, the sets 

\[ \{ t_i \in K_i : |f_i(t_i)| < \gamma \}_U \]

with $||f_i|| = 1$ for all $i \in I$ and $0 < \gamma < 1$ constitutes a basis of the topology of $(K_i)_U$. Since these families are interchanged by the action of $h$ and $h^{-1}$ we get the desired result.

2. A subset $A \subseteq K$ belongs to the algebra of clopen sets $B(K)$ if and only if it is the closure in $K$ of an element of $(B(K_i))_U$.

**Proof.** Let $A \in B(K)$ be a clopen set in $K$. Then the characteristic function $\xi_A$ belongs to $\mathcal{C}(K)$ (obvious as the inverse image of an open subset of the reals under $\xi_A$ is either $\emptyset, A, K \setminus A,$ or $K$, which are open). Let $\xi_A = (f_i)_U$, and let $A_0$ be the set of all $t \in (K_i)_U$ with $h(t) \in A$. Since $\xi(A)$ only takes on the values 0 and 1, it follows that 

\[ A_0 = \{ t_i \in K_i : f_i(t_i) > 1/2 \}_U \]

and 

\[ A_0 = \{ t_i \in K_i : f_i(t_i) \geq 1/2 \}_U \]

Thus the set of $i \in I$ such that 

\[ \{ t_i \in K_i : f_i(t_i) > 1/2 \} = \{ t_i \in K_i : f_i(t_i) \geq 1/2 \} \]

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is in $\mathcal{U}$, in particular, $A_0$ is a clopen set, so it is in $(B(K_i)_\mathcal{U})$. Since $h((K_i)_\mathcal{U})$ is dense in $K$, it follows that $h(A_0)$ is dense in $A$.

If, conversely, $A_0 \in (B(K_i))_\mathcal{U}$, then $A_0 = (A_i)_\mathcal{U}$ with $A_i \in B(K_i)$ for all $i \in I$, then the ultraproduct of the characteristic functions $\xi = (\xi_{A_i})_\mathcal{U}$ is a function in $C(K)$ which equals 1 on $h(A_0)$ and 0 on $h((K_i)_\mathcal{U} \setminus A_0)$. Since $h((K_i)_\mathcal{U})$ is dense in $K$ it follows that $\xi$ is the characteristic function of the closure $A$ of $h(A_0)$ in $K$. \hfill \square

3. If the spaces $K_i$ are totally disconnected, then so is $K$.

Proof. It suffices to prove that each two distinct points $s$, $t \in K$ can be separated by disjoint clopen sets. Via Urysohn’s lemma, choose $f \in C(K)$ with $f(s) = 0$ and $f(t) = 1$, and let $f = (f_i)_\mathcal{U}$. The assumption implies that there are clopen sets $A_i \in B(K_i)$ with

$$\{t_i \in K_i : f_i(t_i) \leq 1/3\} \subseteq A_i, \quad \text{and} \quad \{t_i \in K_i : f_i(t_i) \geq 2/3\} \subseteq K \setminus A_i$$

Let $A_0 = (A_i)_\mathcal{U}$, and let $A$ be the closure of $h(A_0)$ in $K$. By the previous part, $A$ is clopen. As above, it follows that $K \setminus A$ is the closure of the set $h((K_i \setminus A_i)_\mathcal{U})$. Since $h((K_i)_\mathcal{U})$ is dense in $K$, we can conclude that $s \in A$ and $t \in K \setminus A$. \hfill \square

Recall a topological space is extremally disconnected if the closure of every open set is itself open. An extremely disconnected compact Hausdorff space is a Stonean space. For example, the Stone-Čech compactification of a discrete space is Stonean, in particular, $\beta(\mathbb{N})$ is a Stonean space.

Recall (in the category of Banach spaces with contractive linear maps) a Banach space $I$ is injective if for all Banach spaces $A$ and $B$ with $A \subseteq B$ a subspace and $\phi : A \to I$ a contractive linear map there is an extension of $\phi$ to $\Phi : B \to I$ a contractive linear map. For example, $\mathbb{C}$ is injective by the Hahn-Banach theorem, and $\ell^\infty$ is injective by using Hahn-Banach component-wise (the direct sum of injective spaces is injective). Since every Banach space embeds isometrically into an injective space, $B \hookrightarrow \ell^\infty(B^*_2)$, there are sufficiently many injective, and being injective is equivalent to being complemented in any super Banach algebra, which is equivalent to being complemented in a super Banach algebra.

It is a fact that if $K$ is Stonean (compact, Hausdorff, and extremally disconnected) then $C(K)$ is an injective Banach space, and these are all the injective Banach spaces.

We set that many properties of $K$ are determined by the properties of the $K_i$. On the other hand, if $\mathcal{U}$ is a non-principal ultrafilter on $\mathbb{N}$, then there is a natural isometric embedding of

$$C(K) = (\ell^\infty)_\mathcal{U} \hookrightarrow \ell^\infty((\mathbb{N})_\mathcal{U})$$

The map sends $f = (f_i)_\mathcal{U}$ to $\Phi_f : (\mathbb{N})_\mathcal{U} \to \mathbb{C}$ a bounded map so $\Phi_f \in \ell^\infty((\mathbb{N})_\mathcal{U})$ defined by

$$\Phi_f((n_i)_\mathcal{U}) = \lim_\mathcal{U} f_i(n_i)$$

Note that $\ell^\infty = C(\beta(\mathbb{N}))$, where the $\beta(\mathbb{N})$ are extremally disconnected. One can show via contradiction that $K$ as above is not extremally disconnected, so this property is not preserved. Were $K$ extremally disconnected, the we would have $C(K)$ injective, so it would have to be contractively
complemented in $\ell_\infty(\mathbb{N})_U$ which is an injective super Banach algebra.

Proposition: If $U$ is a free ultrafilter on $\mathbb{N}$, ultraproducts of the form $A = (A_i)_U$, $\emptyset \neq A_i \subseteq \mathbb{N}$ are either finite or uncountable.

Proof. A couple of comments: This proposition fails if $U = U_{i_0}$ is a principal ultrafilter and $A_i$ is countable. Thus we must take $U$ to be free. In particular, $U$ contains the cofinite sets.

For all $n \in \mathbb{N}$, let $I_{\leq n} = \{ i \in \mathbb{N} \mid \#(A_i) \leq n \}$.
Case 1: There is a minimal $N \in \mathbb{N}$ such that $I_{\leq N} \in U$. Note that since $I_{\leq N - 1} \notin U$

$$I_{= N} = \{ i \in \mathbb{N} \mid \#(A_i) = N \} = I_{\leq N} \setminus I_{\leq N - 1} \in U$$

Suppose you have $K$ distinct elements $\vec{a}(j)$ for $1 \leq j \leq K$ with representations $\vec{a}^{(i)} = (a_n^{(i)})$. Then let

$$J_{r, s} = \{ i \in I \mid a_i^{(r)} \neq a_i^{(s)} \} \in U$$

Then

$$I' = I_{= N} \cap \bigcap_{1 \leq r < s \leq n} J_{r, s} = \{ i \in \mathbb{N} \mid a_i^{(j)} \text{ are distinct in } A_i, 1 \leq j \leq K \} \in U$$

since it is a finite intersection of sets in $U$, so it is non-empty, so there is some $i_o \in I'$, for which

$$a_i^{(j)} \text{ are distinct in } A_i, 1 \leq j \leq K$$

are $K$ distinct elements of the $N$-element set $A_i$, thus $K \leq N$. Then $\#(A) = \#(A_i)_U \leq N$.

In fact, we have equality. Let $I_{= N}$ be as above, with out loss of generality suppose for $i \in I_{= N}$ that $A_i = \{1, \ldots, n\}$. Then let $\vec{a}(j) = (j \chi_{(i=N)(n)})_U$ for $1 \leq j \leq N$. These are clearly $N$ distinct elements of $A = (A_i)_U$.

Case 2: For all $n \in N$, $I_{\leq n} \notin U$, so for all $n \in \mathbb{N}$,

$$I_{> n} = I \setminus I_{\leq n} = \{ i \in \mathbb{N} \mid \#(A_i) > n \} \in U$$

We will construct an imbedding $\Phi : [0, 1) \hookrightarrow A = (A_i)_U$, so $A$ is uncountable. For each $n \in \mathbb{N}$, let $K_n = \min(n, \#(A_n))$, and let

$$a_i^{(1)}, \ldots, a_i^{(K_n)} \text{ be } K_n \text{ distinct elements of } A_n$$

For $0 \leq r < 1$, for each $i \in I$ there is a unique integer $1 \leq r_i \leq K_i$ such that

$$\frac{r_i - 1}{K_i} \leq r < \frac{r_i}{K_i}$$

Define $\Phi(r) = (a_i^{(r_i)})_U$. We claim that $\Phi$ is injective. Suppose $0 \leq r < s < 1$, then there is an $N \in \mathbb{N}$ such that $|r - s| < 1/N$. In particular, for all $n$ with $K_n \geq N$, $r_n \neq s_n$ (they are in different elements of the partition of $[0, 1)$ into subintervals of length $1/K_n < 1/N$) Thus

$$r_n \neq s_n \text{ for all } n \in \{ i \in \mathbb{N} \mid \#(A_i) \geq K_N \} = I_{\geq K_N} \in U$$

Thus

$$a_n^{(r_n)} \neq a_n^{(s_n)} \text{ for all } n \in I_{\geq K_N} \in U$$
There must be an $n \in \mathbb{N}$ such that $a_n (r_n) \not\in \langle a_n (s_n) \rangle$, i.e. $\Phi(r) \neq \Phi(s)$.

Thus $\Phi$ is in fact injective.

Using this fact, Henson and Moore showed that the existence of a bounded projection from $\ell_\infty((\mathbb{N})_\mathbb{U})$ onto $(\ell_\infty)_\mathbb{U}$ would imply the existence of a bounded projection from $\ell_\infty$ to $e_0$. We now show this is impossible.

Lemma: If $S$ is a countably infinite set, there is an uncountable family (with the cardinality of $\mathbb{R}$) $\{A_i\}_{i \in I}$ of (countably) infinite subsets of $S$ which are “almost disjoint”, i.e. for $i \neq j$, $A_i \cap A_j$ is finite.

Proof. Without loss of generality, assume $S = \mathbb{Q} \cap [0, 1]$. Let $I = [0, 1] \setminus S$, which is an uncountable set. Since $S$ is dense in $[0, 1]$, for each $i \in I$ we can pick a sequence

$$a_n^{(i)} \in S \text{ for all } n \in \mathbb{N}, \quad \lim_{n \to \infty} a_n^{(i)} = i$$

Let $A_i = \{a_n^{(i)} | n \in \mathbb{N}\}$. It is clear that each set $A_i$ is infinite (take balls of decreasing diameter about $i$), and $A_i \cap A_j$ is finite for $i \neq j$ (take disjoint balls $U_i, U_j$ around $i$ and $j$ respectively, then for all sufficient large $n, m$, $a_n^{(i)} \in U_i$ and $a_m^{(j)} \in U_j$, so these cannot be equal). Since there are uncountably such $A_i$, we are finished.

Lemma: Let $q$ is a bounded projection from $\ell_\infty \to \ell_\infty$ such that $q(\vec{x}) = \vec{0}$ for all $\vec{x} \in e_0$. Then there is an infinite set $A \subseteq \mathbb{N}$ such that if $\vec{x} \in \ell_\infty$ and $\text{supp}(\vec{x}) \subseteq A$, then $q(\vec{x}) \neq \vec{0}$.

Proof. For sake of contradiction, suppose there is no such infinite set $A$. Then for every infinite set $A \subseteq \mathbb{N}$ there is a $\vec{x} \in \ell_\infty$ with $\text{supp}(\vec{x}) \subseteq A$ and $q(\vec{x}) = \vec{0}$.

Let $\{A_i\}_{i \in \mathbb{N}}$ be a countable family of infinite subsets of $\mathbb{N}$ which are almost disjoint, as guaranteed by the previous lemma. Then for each $A_i$, there is a $\vec{x}^{(i)} = (x_n^{(i)})_{n \in \mathbb{N}} \in \ell_\infty$ with $\text{supp}(\vec{x}^{(i)}) \subseteq A_i$, and $q(\vec{x}^{(i)}) \neq \vec{0}$ Let $q(\vec{x}^{(i)}) = \vec{y}^{(i)} = (y_n^{(i)})$.

Now since $I$ is an uncountable set, and for each $i \in I$, $q(\vec{x}^{(i)}) = \vec{y}^{(i)} \neq \vec{0}$, there is an $n \in \mathbb{N}$ such that the $n^{th}$ entry is non-zero, so we have

$$I = \bigcup_{n \in \mathbb{N}} I_n \quad I_n = \{i \in I | y_n^{(i)} \neq 0\}$$

There must be an $n \in \mathbb{N}$ such that $I_n$ is uncountable. Then since

$$I_n = \bigcup_{k \in \mathbb{N}} I_{n,k} \quad I_{n,k} = \{i \in I | |y_n^{(i)}| > \frac{1}{k}\}$$

Thus there is a $k \in \mathbb{N}$ such that $I_{n,k}$ is uncountable.

Now, for all finite subsets $F$ of $I_{n,k}$, we can define $\vec{w}$ by

$$\vec{w} = \sum_{j \in F} \text{sgn}(y_n^{(j)}) \vec{x}^{(j)}$$

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(Recall that $q(\vec{x}^{(j)}) = \vec{y}^{(j)} = (y^{(j)}_n)$. Note that the $n^{th}$ coordinate of $\vec{w}$ is

$$w_n = \sum_{j \in F} \text{sgn}(y^{(j)}_n)x^{(j)}_n$$

Since $||\vec{x}||_\infty = \sup_{n \in \mathbb{N}} |x^{(j)}_n| = 1$ and supp$(\vec{x}^{(j)}) \subset A_j$ and the $A_i$ are almost disjoint

$$|w_n| = \left| \sum_{j \in F} \text{sgn}(y^{(j)}_n)x^{(j)}_n \right| \leq \sum_{j \in F} |x^{(j)}_n| \leq \sum_{j \in F} ||\vec{x}^{(j)}|| \leq \# \{ j \in F : x^{(j)}_n \neq 0 \} \leq \bigcap_{j \in F} A_j$$

In particular, if $|w_n| > 1$ for only finitely many $n \in \mathbb{N}$ since we are only considering finitely many $\vec{x}^{(j)}$ with $j \in F$ and the supports of the $\vec{x}^{(j)}$ are almost disjoint. Thus we can write

$$\vec{w} = \vec{f} + \vec{z}$$

Where $\vec{f}$ has finite support (thus $\vec{f} \in c_0$, and $q(\vec{f}) = 0$) and $||\vec{z}|| \leq 1$. Thus we have

$$q(\vec{w}) = q(\vec{f} + \vec{z}) = q(\vec{f}) + q(\vec{z}) = q(\vec{z})$$

On the other hand

$$q(\vec{w}) = q \left( \sum_{j \in F} \text{sgn}(y^{(j)}_n)\vec{x}^{(j)} \right) = \sum_{j \in F} \text{sgn}(y^{(j)}_n)q(\vec{x}^{(j)}) = \sum_{j \in F} \text{sgn}(y^{(j)}_n)y^{(j)}$$

In particular, the $n^{th}$ coordinate of $q(\vec{w})$ is

$$\sum_{j \in F} \text{sgn}(y^{(j)}_n)y^{(j)} = \sum_{j \in F} |y^{(j)}_n| \geq |F| \min_{j \in F} |y^{(j)}_n| \geq \frac{|F|}{k}$$

Since $j \in F \subseteq I_{n,k} = \{ i \in I \mid |y^{(i)}_n| > \frac{1}{k} \}$. Now we see that

$$||q(\vec{w})||_\infty \geq \frac{|F|}{k}$$

But this means that

$$\frac{|F|}{k} = ||q(\vec{w})|| = ||q(\vec{z})|| \leq ||q|| ||\vec{z}|| \leq ||q||$$

and this means $|F| \leq k||q||$, which is ridiculous since $F$ can be any finite subset of the uncountable set $I_{n,k}$, a contradiction. We conclude that there must be an infinite set $A \subseteq \mathbb{N}$ such that for all $\vec{x} \in \ell^\infty$ with supp$(\vec{x}) \subseteq A$ has $q(\vec{x}) = 0$.

\[ \square \]

Theorem: There is no bounded projections $p : \ell^\infty \to c_0$.

Proof. Were there such a bounded projection $p$, then $I - q : \ell^\infty \to \ell^\infty$ would be a bounded projection which vanishes at all $\vec{x} \in c_0$. By the previous lemma, there is an infinite subset $A \subseteq \mathbb{N}$ such that whenever supp$(\vec{x}) \subseteq A$, $q(\vec{x}) = 0$. However, this means that for any $\vec{x} \in \ell^\infty$ with supp$(\vec{x}) \subseteq A$,

$$\vec{x} = I(\vec{x}) = p(\vec{x}) + q(\vec{x}) = p(\vec{x})$$

Thus all such $\vec{x}$ are in $c_0$. This is clearly not the case as $A$ is infinite: $(\chi_A(n))$ is a bounded sequence supported in $A$ which is not in $c_0$.

\[ \square \]
In particular, \((\ell_\infty)_\mathcal{U}\) is not an injective space, so the ultraproduct of injective spaces need not be injective. Also, this supplies an example of compact extremally disconnected \(K_i\) (for which the \(K\) satisfying \(C(K) \cong (C(K_i))_\mathcal{U}\) is not extremally disconnected.

### The Local Structure of Ultraproducts

#### Finite Representability and Ultrapowers

This section is devoted to the study of finite dimensional subspaces of ultraproducts. In the case of ultrapowers, this leads to the connection with the notion of finite representability. We discuss the subject of local properties of Banach spaces which are determined by finite dimensional subspaces, in particular the notion of “localizing” a Banach space property. Finally, an ultrapower version of the principle of local reflexivity is given.

Our first result show that the finite dimensional subspaces of an ultraproduct \((E_i)_\mathcal{U}\) are arbitrarily close to suitable subspaces of the spaces \(E_i\).

Recall that an operator \(T : E \to F\) is a \((1 + \varepsilon)\)-isomorphism (where \(0 < \varepsilon < 1\)) if \(T\) is an isomorphism and for all \(x\),

\[
\left| |Tx| - |x| \right| \leq \varepsilon |x| \quad (♠)
\]

Equivalently, \(|T| \leq 1 + \varepsilon\) and \(|T^{-1}| \leq \frac{1}{1 - \varepsilon}\):  

**Proof.** First notice \((♠)\) when \(|x| = 1\) is equivalent to

\[-\varepsilon \leq |Tx| - |x| \leq \varepsilon
\]

\[1 - \varepsilon \leq |Tx| \leq 1 + \varepsilon\]

And we know that for \(|x| = 1\),

\[1 = |x| = ||T^{-1}T(x)|| \leq ||T^{-1}|| ||T||
\]

Thus

\[
\frac{1}{||T^{-1}||} \leq ||T(x)|| \leq ||T|| ||x|| = ||T|| \quad (♥)
\]

Thus \((♠)\) holds.

\((⇒):\) suppose \(|T| \leq 1 + \varepsilon\), and \(|T^{-1}| \leq \frac{1}{1 - \varepsilon}\). Then for \(|x| = 1\) we have by \((♥)\) that

\[1 - \varepsilon = \frac{1}{1 - \varepsilon} \leq \frac{1}{||T^{-1}||} \leq ||T(x)|| \leq ||T|| \leq 1 + \varepsilon
\]

\((⇐):\) Suppose for all \(|x| = 1\), we have that

\[1 - \varepsilon \leq |Tx| \leq 1 + \varepsilon\]

So

\[|T| = \sup_{||x||=1} ||Tx|| \leq 1 + \varepsilon\]
Note that if $||T(x)|| = 1$, then

$$1 = ||T(x)|| = ||x|| ||T(x/||x||)|| \geq ||x|| (1 - \varepsilon)$$

thus in this case

$$||x|| \leq \frac{1}{1 - \varepsilon}$$

Thus

$$||T^{-1}|| = \sup_{||y = T(x)|| = 1} ||T^{-1}(y)|| = \sup_{||T(x)|| = 1} ||T^{-1}T(x)|| = \sup_{||T(x)|| = 1} ||x|| \leq \frac{1}{1 - \varepsilon}$$

Proposition: Let $X$ be a Banach space and let $\mathcal{B}$ be a family of Banach spaces which models the finite subspaces of $X$. That is, for each $\varepsilon > 0$, and each finite dimensional subspace $M$ of $X$ there is a space $E = E(M, \varepsilon)$ and a map $T(M, \varepsilon) : M \to E(M, \varepsilon)$, a $(1 + \varepsilon)$-isomorphism. Then there is an ultrafilter $\mathcal{U}$ on an index set $I$ and a map from $I$ into $\mathcal{B}$ sending $i \mapsto E_i \in \mathcal{B}$, so $X$ is isometrically isomorphic to a subspace of $(E_i)_{\mathcal{U}}$.

Proof. Let $I$ be the collection of all pairs $(M, \varepsilon)$ where $M$ is a finite dimensional subspace of $X$s and $\varepsilon > 0$. There is a canonical partial ordering on $I$ given by

$$(M_1, \varepsilon_1) \prec (M_2, \varepsilon_2)$$

if and only if $M_1 \subseteq M_2$ and $\varepsilon_1 \geq \varepsilon_2$

We get an order filter generated by all the “upsets” of any $(M_0, \varepsilon_0)$, i.e. all indices greater than or equal to a particular fixed index:

$$\uparrow \{(M_0, \varepsilon_0)\} = \{\{M, \varepsilon\) : (M_0, \varepsilon_0) \prec (M, \varepsilon)\}$$

The set of subsets of $I$ which contain an upset is a filter: upsets are non-empty, and $I$ clearly contains upsets, containing a set that contains an upset means you contain an upset, and the intersection of two upsets is an upset:

$$\uparrow \{(M_1, \varepsilon_1)\} \cap \uparrow \{(M_2, \varepsilon_2)\} = \{\{M, \varepsilon\) : M_1, M_2 \subseteq M, \varepsilon_1, \varepsilon_2 \leq \varepsilon\} = \uparrow \{\{M_1 + M_2, \max(\varepsilon_1, \varepsilon_2)\}\}$$

Now, let $\mathcal{U}$ be an ultrafilter dominating the order filter, denote an element $i \in I$ by $(M_i, \varepsilon_i)$.

By assumption, for each $i \in I$ there is a space $E_i \in \mathcal{B}$ and a $(1 + \varepsilon_i)$-isomorphism $T_i : M_i \to N_i \subseteq E_i$. We define a map $J : X \to (E_i)_{\mathcal{U}}$ for $x \in X$ by

$$J(x) = (y_i)_{\mathcal{U}} \quad y_i = \begin{cases} T_i(x) & \text{if } x \in M_i \\ 0 & \text{otherwise} \end{cases}$$

Then $J$ is clearly linear, and for all $\varepsilon_0 > 0$, the set

$$I_0 = \uparrow \{\text{span}(x), \varepsilon_0\} = \{(M, \varepsilon) : x \in M, \varepsilon \leq \varepsilon_0\} \in \mathcal{U}$$

Then since $y_i = 0$ or $y_i = T_i(x)$, we have

$$||y_i|| = ||T_i(x)|| \leq ||T_i|| ||x|| \leq \varepsilon_0 ||x||$$

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and
\[ \|x\| = \|T_i^{-1}(y_i)\| \leq \|T_i^{-1}\| \|y_i\| \]
so
\[ \|x\| \leq \frac{1}{1 + \varepsilon_0} \|x\| \leq \|y_i\| \leq \varepsilon_0 \|x\| \]
Thus
\[ \|x\| + \varepsilon_0 \leq \|Jx\| = \lim_{\mathfrak{U}} \|y_i\| \leq (1 + \varepsilon_0) \|x\| \]
Since this holds for all \( \varepsilon_0 > 0 \), we can conclude \( J \) is an isometry of \( X \) into \((E_i)_\U\) as claimed. \( \square \)

This type of approach is typical: the finite substructures of a given object are used to build up an ultrafilter which then allows one to reproduce the whole object as a substructure of a corresponding ultraproduct.

There is a partial converse to this proposition as well:

**Proposition:** Let \( M \) a finite dimensional subspace of \((E_i)_\U\), and \( \varepsilon > 0 \) be given. Then there is a set \( I_\varepsilon \in \U \) such that for all \( i \in I_\varepsilon \) there are maps \( T_i : M \to E_i \) which are \((1 + \varepsilon)\)-isomorphisms.

**Proof.** Let \( \bar{x}^{(1)}, \ldots, \bar{x}^{(n)} \in M \) be a basis of \( M \), choose representations \( \bar{x}^{(k)} = (x_i^{(k)})_\U \), say with \( \|x_i^{(k)}\| \leq 2 \) (since \( \|\bar{x}^{(k)}\| = \lim_{\mathfrak{U}} \|x_i^{(k)}\| = 1 \), \( \|x_i^{(k)}\| \leq 2 \) on a large set, so we can change the rest of the terms if needed), and let \( M_i = \text{span}(x_i^{(k)})_{1 \leq k \leq n} \subseteq E_i \). Define operators \( T_i : M \hookrightarrow M_i \) by
\[ T_i(\bar{x}^{(k)}) = x_i^{(k)} \quad \text{for } 1 \leq k \leq n, i \in I \]
It follows that \( \|T_i\| \leq C \) for all \( i \in I \), where \( C \) is a constant depending on the basis and the representations: for example if we pick an Auerbach basis of \( M \), \( \|\bar{x}^{(k)}\| = 1 \) and the corresponding dual functionals \( \phi^{(k)}(\bar{x}^{(j)}) = \delta_{i,j} \)
\[ \bar{x} = \sum_{k=1}^{n} \lambda_k \bar{x}^{(k)}, \quad \|\bar{x}\| = 1 \]
Thus for \( 1 \leq k \leq n \),
\[ |\lambda_k| = |\phi^{(k)}(\bar{x})| \leq ||\phi^{(k)}|| \|\bar{x}\| = 1 \]
So, since \( \|(x_k^{(k)})_i\|_\infty = \sup_i \|x_k^{(k)}\| \|\bar{x}\| \leq 2 \) for each \( k \),
\[ \|T_i(\bar{x})\| = \left\| \sum_{k=1}^{n} \lambda_k x_i^{(k)} \right\| \leq \sum_{k=1}^{n} |\lambda_k| \|x_i^{(k)}\| \leq n \|x_i^{(k)}\|_\infty \leq 2n \]
Thus we have have that \( (T_i(x))_{i \in I} \) is a bounded family, say \( \sup_i \|T_i\| = C < \infty \). Now for \( \bar{x} = \sum_{k=1}^{n} \lambda_k \bar{x}^{k} \in M \), we have
\[ \lim_{\mathfrak{U}} \|T_i(\bar{x})\| = \lim_{\mathfrak{U}} \left\| \sum_{k=1}^{n} \lambda_k x_i^{(k)} \right\| = \left\| \left( \sum_{k=1}^{n} \lambda_k \bar{x}^{(k)} \right)_\U \right\| = \left\| \sum_{k=1}^{n} \lambda_k (x_i^{(k)})_\U \right\| = \left\| \sum_{k=1}^{n} \lambda_k \bar{x}^{(k)} \right\| = \|x\| \]
Thus for each \( \bar{x} \in M \),
\[ I_{\bar{x}} = \{ i \in I \mid \|T_i(\bar{x})\| - \|\bar{x}\| < \varepsilon \} \]
Now since $M_{\leq 1}$ is compact as $M$ is finite dimensional, we can pick a finite $\delta$-net $\vec{y}^{(1)}, \ldots, \vec{y}^{(m)}$ of the unit ball of $M$, and let

$$I_\varepsilon = \bigcap_{k=1}^{m} I_{\vec{y}^{(k)}} \in \mathcal{U}$$

Then for any $\vec{x}$ in $M_{\leq 1}$ there is a $\vec{y} = \vec{y}^{(k)}$ with $||\vec{x} - \vec{y}|| < \delta$, so for $i \in I_\varepsilon$

$$||T_i(\vec{x})|| - ||\vec{x}|| = ||T_i(\vec{x}) - T_i(\vec{y}) + T_i(\vec{y})|| - ||\vec{y} + \vec{x} - \vec{y}|| \leq ||T_i(\vec{x} - \vec{y})|| + ||T_i(\vec{y})|| - ||\vec{y}|| + ||\vec{y} - \vec{x}||$$

$$\leq ||T_i(\vec{x} - \vec{y})|| + ||T_i(\vec{y})|| - ||\vec{y}|| + ||\vec{y} - \vec{x}|| \leq C\delta + \frac{\varepsilon}{2}||\vec{y}|| + \delta \leq (C + 1)\delta + \frac{\varepsilon}{2}$$

Letting $\delta = \frac{\varepsilon}{2(1+C)} > 0$ completes the proof.

We are ready to establish the connection between finite representability and ultrapowers: Recall that a Banach space $F$ is finitely representable in a Banach space $E$ if for each finite dimensional subspace $M \subseteq F$ and each $\varepsilon > 0$ there is a $(1 + \varepsilon)$-isomorphism $T : M \to E$. In other words, $E$ models the finite dimensional subspaces of $F$.

**Theorem:** $F$ is finitely representable in $E$ if and only if there is an ultrafilter $\mathcal{U}$ such that $F$ is isometric to a subspace of $(E)_\mathcal{U}$.

**Proof.** ($\Rightarrow$) If $F$ is finitely representable in $E$, then $E$ itself is a family of Banach spaces which models the finite subspaces of $F$ so by our previous construction there is an index set $I$ and an ultrafilter $\mathcal{U}$ on $I$ such that $Y$ is isometrically isomorphic to a subspace of $(E)_\mathcal{U}$.

($\Leftarrow$) If $F$ is isometrically isomorphic to $\hat{F} \subseteq (E)_\mathcal{U}$ for some $\mathcal{U}$ then for all $M$ finite dimensional in $F$, $M$ is isometrically isomorphic to a finite dimensional subspace of $\hat{F}$, so it is a finite dimensional subspace of $(E)_\mathcal{U}$. Thus for every $\varepsilon > 0$, $M$ is $(1 + \varepsilon)$-isomorphic to a subspace of $X$. Thus $F$ is finitely represented in $E$.

Moreover, if $F$ is separable and finitely representable in $E$, then $F$ embeds isometrically into each ultrapower $(E)_\mathcal{U}$ where $\mathcal{U}$ is a countably incomplete ultrafilter.

**Proof.** The first part follows directly from the last two results: if $F$ is finitely represented in $E$, then we can construct an index set and ultrafilter so that $F$ is isometric to a subspace of the ultrapower of $E$ by the previous construction. On the other hand, if $F$ is isometric to a subspace of an ultrapower of $E$, then by the penultimate result every finite dimensional subspace of $F$ is $(1 + \varepsilon)$-isometric to a subspace of $E$.

If $\mathcal{U}$ is countably incomplete, then we have a countable chain

$$I = I_1 \supseteq I_2 \supseteq \ldots, \quad I_n \in \mathcal{U}, \quad \bigcap_{n \in \mathbb{N}} I_n = \emptyset$$
Let \( \{x_n\}_{n \in \mathbb{N}} \) be a countable and linearly independent subset of \( F \) whose space in dense in \( F \). By assumption, we can find \((1 + 1/n)\)-isometries \( T_n \) from \( \text{span}_{1 \leq k \leq n}(x_k) \) to \( E \). Then define

\[
J(x_m) = (y_i)_U, \quad y_i = \begin{cases} 0 & \text{if } i \in I \setminus I_m \\ T_n(x_m) & \text{if } n \geq m \text{ and } i \in I_n \setminus I_{n+1} \end{cases}
\]

Then one can check that \( J \) is a linear isometry.

If \( \Psi \) is a Banach space property, recall that a Banach space \( E \) is said to be super-\( \Psi \) if and only if each Banach space \( F \) which is finite representable in \( E \) is \( \Psi \). This should be thought of as a local variant of the property \( \Psi \) which is determined by behavior on finite dimensional subspaces.

A precise knowledge about the structure of ultrapowers allow one to localize infinite dimensional results - to transform connections between infinite dimensional properties to connections between the local properties. Here is an example:

**Proposition:** If \( X_i \) are Banach spaces and \( Y_i \subseteq X_i \) are closed subspaces with common index set \( I \) and \( \mathfrak{U} \) is an ultrafilter on \( I \), then there is a canonical isometric isomorphism

\[(X_i/Y_i)_\mathfrak{U} \cong (X_i)_{\mathfrak{U}}/(Y_i)_{\mathfrak{U}}\]

**Proof.** Note that \((X_i/Y_i)_\mathfrak{U}\) is canonically isometric to \((X_i)_{\mathfrak{U}}/(Y_i)_{\mathfrak{U}}\), since if

\[
(\bar{x}_i) \in (E/F)_{\mathfrak{U}}, \quad ||(\bar{x}_i)|| = \lim_{\mathfrak{U}} ||\bar{x}_i|| = \lim_{\mathfrak{U}} \inf_{y_i \in Y_i} ||x_i - y_i||
\]

On the other hand,

\[
(x_i) \in (X_i)_{\mathfrak{U}}/(Y_i)_{\mathfrak{U}}, \quad ||(x_i)|| = \inf_{(y_i) \in (F)_{\mathfrak{U}}} ||(x_i) - (y_i)|| = \inf_{(y_i) \in (Y_i)_{\mathfrak{U}}} \lim_{\mathfrak{U}} ||x_i - y_i||
\]

Which is the same as the above.

We say a Banach space property \( P \) satisfies the “three space condition” if whenever \( Y \) is a closed subspace of a Banach space \( X \), and both \( Y \) and \( X/Y \) have \( P \) then \( X \) also has \( P \). For instance, reflexivity.

**Theorem:** If \( P \) is a Banach space property which is inherited by subspaces and it satisfies the three space condition, then super-\( P \) also satisfies the three space condition.

**Proof.** Suppose \( P \) is above, \( Y \) is a closed subspace of \( X \), and both \( Y \) and \( X/Y \) are super \( P \). Then for any ultrafilter \( \mathfrak{U} \), \((Y)_{\mathfrak{U}}\) and \((X/Y)_{\mathfrak{U}} \cong (X)_{\mathfrak{U}}/(Y)_{\mathfrak{U}} \) have \( P \). Then by the three space property for \( P \), \((X)_{\mathfrak{U}}\) has \( P \). Thus \( X \) has super-\( P \) as claimed.

**Theorem:** Let \( P \) be a Banach space property. A Banach space \( X \) has super-\( P \) if and only if every ultrapower \((X)_{\mathfrak{U}}\) has \( P \).

**Proof.** \((\Leftarrow)\) Note that if \( X \) has super-\( P \), then since \((X)_{\mathfrak{U}}\) is finitely represented in \( X \) by Theorem 2, \((X)_{\mathfrak{U}}\) must have \( P \).

\((\Rightarrow)\) Note that if each ultrapower \((X)_{\mathfrak{U}}\) has \( P \), then if \( Y \) is finitely represented in \( X \), then by Theorem 3, \( Y \) is isometrically isomorphic to a subspace of an ultrapower \((X)_{\mathfrak{U}}\) for some ultrafilter \( \mathfrak{U} \), which by assumption has \( P \). Now since \( P \) is inherited by subspaces, \( Y \) has \( P \). Thus \( X \) has super-\( P \).
Let us express a relation between a Banach space and its second dual using ultrapowers and the principle of local reflexivity. Recall any Banach space $E$ embeds canonically into its double dual by the map $\iota$ which sends an element to evaluation at that element.

Proposition: (The Principal of Local Reflexivity) If $E$ is a Banach space, then for all $M$ a finite dimensional subspace of the double dual $E^{**}$, all finite subsets $\{f_1, \ldots, f_n\}$ in the dual $E^*$, and each $\varepsilon > 0$, there is an operator $T : M \to E$ which is a $(1 + \varepsilon)$-isomorphism onto its image, which satisfies

$$T|_{M \cap \iota(E)}(\iota(x)) = x \quad \langle T(u)|f_k \rangle = \langle u, f_k \rangle \text{ for all } u \in M, 1 \leq k \leq n$$

An immediate corollary of the Principal of Local Reflexivity is that $E^{**}$ is finitely representable in $E$, and in particular, it follows from our previous work that $E^{**}$ embeds isometrically into the ultrapower $(E)_\mathcal{U}$. In fact, more is true:

Proposition: For each Banach space $E$ there exists an ultrafilter $\mathcal{U}$ and an isometric embedding $J : E^{**} \to (E)_\mathcal{U}$. The restriction of $J$ to $\iota(E)$ is the canonical diagonal embedding of $E$ into its ultrapower. Furthermore, there is a projection of norm $1$ $\pi : (E)_\mathcal{U} \to J(E^{**})$.

The proof is similar to our previous construction.