Notes for Cohabitation Model

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February 7, 1998

1. Basic Framework

Let $m_{1t} = 1$ iff single, $m_{2t} = 1$ iff cohabitating, and $m_{3t} = 1$ iff married, and let $m_t = k$ iff $m_{kt} = 1$.

Let c_t = child characteristics [c_{1t} = # children (with a cutoff of $\bar{n}_c$), c_{2t} = age of youngest child (with a cutoff of $\bar{n}_{ca}$)]. Let X_t be a set of other exogenous variables [$X_{1t} = 1$, other X variables include race, education, etc. They need to be limited to exogenous characteristics with known paths.]

Children are created from a Bernoulli process, $p_t(X_t, m_t) = \Pr$ [child concieved at t]. No deaths of children are allowed.

Let d_t = the length of a relationship (with a cutoff of $\bar{t}_d$).

Let θ be an unknown match quality, $\theta \sim N(\mu, \sigma_\theta^2)$. μ can be person specific (this involves using MSL). Note: [One must decide what happens at $\bar{t}_d$. In many models like this, it is assumed that θ is revealed to the couple at time $\bar{t}_d$. In this model, as in any empirical model, such an assumption would lead to an unusually large number of separations at $\bar{t}_d$ as couples discover big (negative) mistakes in their beliefs about θ. An alternative assumption without this unfortunate characteristic that the couple stops updating its beliefs about θ at $\bar{t}_d$ despite new information. While this is not "rational," it is a reasonable approximation to rationality if $\bar{t}_d$ is reasonably large, and it is no less unrealistic than assuming the truth about θ is revealed at $\bar{t}_d$.]

Couples draw ε_t which is observed by the agent and affects utility. It is assumed that

$$\begin{aligned} \varepsilon_t - \theta &= \rho(\varepsilon_{t-1} - \theta) + \eta_t \qquad \text{if } d_t > 1, \\ \varepsilon_t - \theta &= \eta_t \qquad \text{if } d_t = 1, \end{aligned} \tag{1.1}$$

$$\eta_t \sim N\left(0, \frac{\sigma_\eta^2}{1-\rho^2}\right) \text{ for } d_t = 1,$$
$$\eta_t \sim iidN\left(0, \sigma_\eta^2\right) \text{ for } d_t > 1.$$

ε_t is modeled as an AR(1) process to allow for divorces late in a relationship that can not be explained by learning; if ρ is large enough, then a bad draw of ε_t implies bad times for periods to come even if θ is known and good. Equation (1.1) implies a Bayesian updating rule for the estimate of θ at t, $\hat{\theta}_t$. Note that $\hat{\theta}_t$ will be a weighted average of μ and the GLS estimator of $\hat{\theta}_t$. For ease of notation, assume that the relationship starts at $t = 1$. The GLS estimator is

$$\left[\begin{pmatrix} 1 & 1 & .. & 1 \end{pmatrix} \begin{pmatrix} \frac{\sigma_\eta^2}{1-\rho^2} & \frac{\rho\sigma_\eta^2}{1-\rho^2} & .. & \frac{\rho^{d_t-1}\sigma_\eta^2}{1-\rho^2} \\ \frac{\rho\sigma_\eta^2}{1-\rho^2} & \frac{\sigma_\eta^2}{1-\rho^2} & .. & \frac{\rho^{d_t-2}\sigma_\eta^2}{1-\rho^2} \\ \vdots & \vdots & & \vdots \\ \frac{\rho^{d_t-1}\sigma_\eta^2}{1-\rho^2} & \frac{\rho^{d_t-2}\sigma_\eta^2}{1-\rho^2} & .. & \frac{\sigma_\eta^2}{1-\rho^2} \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}\right]^{-1} \bullet$$
$$\left[\begin{pmatrix} 1 & 1 & .. & 1 \end{pmatrix} \begin{pmatrix} \frac{\sigma_\eta^2}{1-\rho^2} & \frac{\rho\sigma_\eta^2}{1-\rho^2} & .. & \frac{\rho^{d_t-1}\sigma_\eta^2}{1-\rho^2} \\ \frac{\rho\sigma_\eta^2}{1-\rho^2} & \frac{\sigma_\eta^2}{1-\rho^2} & .. & \frac{\rho^{d_t-2}\sigma_\eta^2}{1-\rho^2} \\ \vdots & \vdots & & \vdots \\ \frac{\rho^{d_t-1}\sigma_\eta^2}{1-\rho^2} & \frac{\rho^{d_t-2}\sigma_\eta^2}{1-\rho^2} & .. & \frac{\sigma_\eta^2}{1-\rho^2} \end{pmatrix}^{-1} \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_t \end{pmatrix}\right].$$

The weights are σ_θ^{-2} for μ and

$$\begin{pmatrix} 1 & 1 & .. & 1 \end{pmatrix} \begin{pmatrix} \frac{\sigma_\eta^2}{1-\rho^2} & \frac{\rho\sigma_\eta^2}{1-\rho^2} & .. & \frac{\rho^{t-1}\sigma_\eta^2}{1-\rho^2} \\ \frac{\rho\sigma_\eta^2}{1-\rho^2} & \frac{\sigma_\eta^2}{1-\rho^2} & .. & \frac{\rho^{t-2}\sigma_\eta^2}{1-\rho^2} \\ \vdots & \vdots & & \vdots \\ \frac{\rho^{t-1}\sigma_\eta^2}{1-\rho^2} & \frac{\rho^{t-2}\sigma_\eta^2}{1-\rho^2} & .. & \frac{\sigma_\eta^2}{1-\rho^2} \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

for the GLS estimator. Thus the Bayesian updating rule is

$$\hat{\theta}_t = \frac{\sigma_\theta^{-2}\mu + \begin{pmatrix} 1 & 1 & .. & 1 \end{pmatrix} \begin{pmatrix} \frac{\sigma_\eta^2}{1-\rho^2} & \frac{\rho\sigma_\eta^2}{1-\rho^2} & .. & \frac{\rho^{t-1}\sigma_\eta^2}{1-\rho^2} \\ \frac{\rho\sigma_\eta^2}{1-\rho^2} & \frac{\sigma_\eta^2}{1-\rho^2} & .. & \frac{\rho^{t-2}\sigma_\eta^2}{1-\rho^2} \\ \vdots & \vdots & & \vdots \\ \frac{\rho^{t-1}\sigma_\eta^2}{1-\rho^2} & \frac{\rho^{t-2}\sigma_\eta^2}{1-\rho^2} & .. & \frac{\sigma_\eta^2}{1-\rho^2} \end{pmatrix}^{-1} \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_t \end{pmatrix}}{\sigma_\theta^{-2} + \begin{pmatrix} 1 & 1 & .. & 1 \end{pmatrix} \begin{pmatrix} \frac{\sigma_\eta^2}{1-\rho^2} & \frac{\rho\sigma_\eta^2}{1-\rho^2} & .. & \frac{\rho^{t-1}\sigma_\eta^2}{1-\rho^2} \\ \frac{\rho\sigma_\eta^2}{1-\rho^2} & \frac{\sigma_\eta^2}{1-\rho^2} & .. & \frac{\rho^{t-2}\sigma_\eta^2}{1-\rho^2} \\ \vdots & \vdots & & \vdots \\ \frac{\rho^{t-1}\sigma_\eta^2}{1-\rho^2} & \frac{\rho^{t-2}\sigma_\eta^2}{1-\rho^2} & .. & \frac{\sigma_\eta^2}{1-\rho^2} \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}}$$

$$= \frac{\sigma_\theta^{-2}\mu + \frac{1}{\sigma_\eta^2}\begin{pmatrix} 1 & 1 & .. & 1 \end{pmatrix} \begin{pmatrix} 1 & -\rho & .. & 0 \\ -\rho & 1+\rho^2 & .. & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & .. & 1 \end{pmatrix} \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_t \end{pmatrix}}{\sigma_\theta^{-2} + \frac{1}{\sigma_\eta^2}\begin{pmatrix} 1 & 1 & .. & 1 \end{pmatrix} \begin{pmatrix} 1 & -\rho & .. & 0 \\ -\rho & 1+\rho^2 & .. & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & .. & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}}$$

$$= \frac{\sigma_\theta^{-2}\mu + \frac{1}{\sigma_\eta^2}\begin{pmatrix} 1-\rho & (1-\rho)^2 & .. & (1-\rho)^2 & 1-\rho \end{pmatrix} \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_{t-1} \\ \varepsilon_t \end{pmatrix}}{\sigma_\theta^{-2} + \frac{1}{\sigma_\eta^2}\begin{pmatrix} 1-\rho & (1-\rho)^2 & .. & (1-\rho)^2 & 1-\rho \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{pmatrix}}$$

$$= \frac{\sigma_\theta^{-2}\mu + \frac{1}{\sigma_\eta^2}\left[(1-\rho)(\varepsilon_1+\varepsilon_t) + (1-\rho)^2 \sum_{i=2}^{t-1} \varepsilon_i\right]}{\sigma_\theta^{-2} + \frac{1}{\sigma_\eta^2}\left[2(1-\rho) + (t-2)(1-\rho)^2\right]}.$$

The problem here is to write $\hat{\theta}_t$ recursively in terms of ε_t and $\hat{\theta}_{t-1}$. The above formula is difficult to use because the weights for ε_i, $i \leq t$ vary over i. However, it is possible to write a simple, recursive formula in terms of a newly defined state variable that determines $\hat{\theta}_t$. In particular, define $\varsigma_1 = 0$

$$\begin{aligned} \varsigma_2 &= (1-\rho)\,\varepsilon_1, \\ \varsigma_t &= (1-\rho)\,\varepsilon_1 + (1-\rho)^2 \sum_{i=2}^{t-1} \varepsilon_i \text{ for } t > 2. \end{aligned}$$

Then

$$\begin{aligned} \hat{\theta}_t &= \frac{\sigma_\theta^{-2}\mu + (1-\rho^2)\,\sigma_\eta^{-2}\varepsilon_t}{\sigma_\theta^{-2} + (1-\rho^2)\,\sigma_\eta^{-2}} \text{ if } d_t = 1, \\ &= \frac{\sigma_\theta^{-2}\mu + \frac{1}{\sigma_\eta^2}\left[\varsigma_t + (1-\rho)\,\varepsilon_t\right]}{\sigma_\theta^{-2} + \frac{1}{\sigma_\eta^2}\left[2\,(1-\rho) + (t-2)\,(1-\rho)^2\right]} \text{ if } d_t > 1, \end{aligned} \tag{1.2}$$

and

$$\begin{aligned} \varsigma_2 &= \varsigma_1 + (1-\rho)\,\varepsilon_1, \\ \varsigma_t &= \varsigma_{t-1} + (1-\rho)^2\,\varepsilon_{t-1}. \end{aligned} \tag{1.3}$$

Thus at time t, we can define the state variables as ς_t and ε_t, we can write $\hat{\theta}_t$ in terms of the state variables (and μ), and we can define a recursive formula for ς_t.

Now we can define a recursive rule for $\hat{\theta}_t$ in terms of the state variables from the period before, $\hat{\theta}_{t-1}$ and ε_{t-1}, and ε_t. From equation (1.2), we get

$$\varsigma_{t-1} = \sigma_\eta^2\hat{\theta}_{t-1}\left[\sigma_\theta^{-2} + \frac{1}{\sigma_\eta^2}\left[2\,(1-\rho) + (t-3)\,(1-\rho)^2\right]\right] - \sigma_\eta^2\sigma_\theta^{-2}\mu - (1-\rho)\,\varepsilon_{t-1} \text{ if } t-1 > 1,$$

and, from equations (1.2) and (1.3), we get

$$\begin{aligned} \hat{\theta}_t &= \frac{\sigma_\theta^{-2}\mu + (1-\rho^2)\,\sigma_\eta^{-2}\varepsilon_t}{\sigma_\theta^{-2} + (1-\rho^2)\,\sigma_\eta^{-2}} \text{ if } d_t = 1, \\ &= \frac{\sigma_\theta^{-2}\mu + \sigma_\eta^{-2}\left[(1-\rho)\,\varepsilon_{t-1} + (1-\rho)\,\varepsilon_t\right]}{\sigma_\theta^{-2} + 2\,(1-\rho)\,\sigma_\eta^{-2}} \text{ if } d_t = 2, \end{aligned}$$

and

$$\hat{\theta}_t = \frac{\sigma_\theta^{-2}\mu + \frac{1}{\sigma_\eta^2}\left[\varsigma_{t-1} + (1-\rho)^2\,\varepsilon_{t-1} + (1-\rho)\,\varepsilon_t\right]}{\sigma_\theta^{-2} + \frac{1}{\sigma_\eta^2}\left[2\,(1-\rho) + (t-2)\,(1-\rho)^2\right]} \text{ if } d_t > 2$$

$$= \frac{\sigma_\theta^{-2}\mu + \frac{1}{\sigma_\eta^2}\left\{\sigma_\eta^2\hat{\theta}_{t-1}\left[\sigma_\theta^{-2} + \frac{1}{\sigma_\eta^2}\left[2\left(1-\rho\right) + \left(t-3\right)\left(1-\rho\right)^2\right]\right] - \sigma_\eta^2\sigma_\theta^{-2}\mu - \left(1-\rho\right)\varepsilon_{t-1} + \left(1-\rho\right)^2\right.}{\sigma_\theta^{-2} + \frac{1}{\sigma_\eta^2}\left[2\left(1-\rho\right) + \left(t-2\right)\left(1-\rho\right)^2\right]}$$

$$= \frac{\hat{\theta}_{t-1}\left[\sigma_\theta^{-2} + \frac{1}{\sigma_\eta^2}\left[2\left(1-\rho\right) + \left(t-3\right)\left(1-\rho\right)^2\right]\right] - \frac{1}{\sigma_\eta^2}\rho\left(1-\rho\right)\varepsilon_{t-1} + \frac{1}{\sigma_\eta^2}\left(1-\rho\right)\varepsilon_t}{\sigma_\theta^{-2} + \frac{1}{\sigma_\eta^2}\left[2\left(1-\rho\right) + \left(t-2\right)\left(1-\rho\right)^2\right]}$$

$$\left[= \frac{\hat{\theta}_{t-1}\left[\sigma_\theta^{-2} + \frac{1}{\sigma_\eta^2}\left(t-1\right)\right] + \frac{1}{\sigma_\eta^2}\varepsilon_t}{\sigma_\theta^{-2} + \frac{t}{\sigma_\eta^2}} \text{ when } \rho = 0\right].$$

So we can think of the state variables as being $\hat{\theta}_t$ and ε_t.

2. Value Function

Let $S_t = \left(m_t, m_{t-1}, c_t, d_t, \bar{\theta}_t, \varepsilon_t\right)$ be the state variables at time t that are either endogenous or stochastic. Then the value function can be written as

$$\begin{aligned} V_t\left[S_t, X_t\right] &= f_t\left(m_t, c_t, d_t, X_t\right) + 1\left(m_t > 1\right)\varepsilon_t - D_{m_{t-1}}1\left(m_t = 1\right) \\ &\quad + \beta E_{c_{t+1},\varepsilon_{t+1}}\left\{\max_{m_{t+1}\in F(m_t)}\left(V_{t+1}\left[S_{t+1}, X_{t+1}\right]\right) \mid \bar{\theta}_t, \varepsilon_t, c_t\right\} \end{aligned} \tag{2.1}$$

where $F\left(m_t\right)$ is the feasible set of choices,

$$\begin{aligned} f_t\left(m_t, c_t, d_t, X_t\right) &= \alpha_0\left(m_t\right) + c_t\alpha_c\left(m_t\right) + d_t\alpha_d\left(m_t\right) \\ &\quad + t\alpha_t\left(m_t\right) + X_t\alpha_x\left(m_t\right) \end{aligned} \tag{2.2}$$

is the deterministic flow given S_t, and D_m is a divorce cost from state m ($D_1 = 0$, $D_2 > 0$, $D_3 > 0$):

$$D_m = X_t\delta_m \tag{2.3}$$

where the X variables that affect D_m are a constant, religion, # children, and age of the youngest child.[1] Let $b_t = 1$ iff a child is conceived in t.The probability of a child being conceived in period t is

$$p_t = e^{\pi_t} / \left[1 + e^{\pi_t}\right] \tag{2.4}$$

[1] X_t can include splines in age.

where

$$\pi_t = \gamma_0 (m_t) + c_t \gamma_c (m_t) + d_t \gamma_d (m_t) + t\gamma_t (m_t) + X_t \gamma_x (m_t) .^2 \tag{2.5}$$

Value functions are solved by assuming that there is some time t^* such that no decisions are made after t^* and another time t^{**} such that the person dies right after t^{**}. This implies that

$$V_{t^*} [S_{t^*}, X_{t^*}] = \sum_{s=t^*}^{t^{**}} \beta^{s-t^*} f_s (m_{t^*}, c_s, d_s, X_s) + 1 (m_{t^*} > 1) E \left[\sum_{t=t^*}^{t^{**}} \beta^{t-t^*} \varepsilon_t \mid \bar{\theta}_{t^*}, \varepsilon_{t^*} \right] - D_{m_{t^*-1}} 1 (m_{t^*} = 1) \tag{2.6}$$

where $c_{1s} = c_{1t^*}$, $c_{2s} = \min (c_{2t^*} + s - t^*, \bar{n}_{ca})$, $d_s = \min (d_{t^*} + s - t^*, \bar{t}_d)$, and X_s changes in a nonstochastic, exogenous way. Then $V_t [S_t, X_t]$ can be evaluated iteratively for all $t < t^*$.

3. Implications of the Model

Theorem 1. *$\partial V_t / \partial \varepsilon_t = 0$ and $\partial V_t / \partial \bar{\theta}_t = 0$ if $m_t = 1$, $B_t \geq \partial V_t / \partial \varepsilon_t \geq 1$ and $B_t \geq \partial V_t / \partial \bar{\theta}_t \geq 0$ if $m_t > 1$ where $1/(1-\beta) > B_t = \left(1 - \beta^{t^{**}+1-t}\right) / (1-\beta)$, and $V_t[m_t, m_{t-1}, c_t, d_t, 0, 0]$ is finite.*

Proof.

$$\begin{aligned}
& V_{t^*}[m_{t^*}, m_{t^*-1}, c_{t^*}, d_{t^*}, \bar{\theta}_{t^*}, \varepsilon_{t^*}, X_{t^*}] \\
= \; & E \left[\sum_{t=t^*}^{t^{**}} \beta^{t-t^*} \left(f_t (m_t, c_t, d_t, X_t) + 1 (m_t > 1) \varepsilon_t \right) \mid \bar{\theta}_{t^*}, \varepsilon_{t^*} \right] \\
= \; & \sum_{t=t^*}^{t^{**}} \beta^{t-t^*} f_t (m_t, c_t, d_t, X_t) + \\
& 1 (m_{t^*} > 1) \varepsilon_{t^*} + 1 (m_{t^*} > 1) E \left[\sum_{t=t^*+1}^{t^{**}} \beta^{t-t^*} \varepsilon_t \mid \bar{\theta}_{t^*}, \varepsilon_{t^*} \right]
\end{aligned}$$

which implies that

$$\frac{\partial V_{t^*}[m_{t^*}, m_{t^*-1}, c_{t^*}, d_{t^*}, \bar{\theta}_{t^*}, \varepsilon_{t^*}, X_t]}{\partial \varepsilon_{t^*}} = 1 (m_{t^*} > 1) + 1 (m_{t^*} > 1) \frac{\partial E \left[\sum_{t=t^*+1}^{t^{**}} \beta^{t-t^*} \varepsilon_t \mid \bar{\theta}_{t^*}, \varepsilon_{t^*} \right]}{\partial \varepsilon_{t^*}}. \tag{3.1}$$

[2] Again, X_t can include a spline in age.

Note that $m_t = m_{t^*}$ for all $t \geq t^*$ by the definition of t^*.If $m_{t^*} = 1$, then equation (3.1) is identically zero. If $m_{t^*} > 1$, then, since ε_t is positively serially correlated, equation (3.1) is positive. But, since $\partial E\left[\varepsilon_t \mid \bar{\theta}_{t^*}, \varepsilon_{t^*}\right] / \partial\varepsilon_{t^*} \leq 1$,

$$\begin{aligned}
\frac{\partial V_{t^*}[m_{t^*}, m_{t^*-1}, c_{t^*}, d_{t^*}, \bar{\theta}_{t^*}, \varepsilon_{t^*}, X_t]}{\partial\varepsilon_{t^*}} &= 1\left(m_{t^*} > 1\right) + 1\left(m_{t^*} > 1\right) \sum_{t=t^*+1}^{t^{**}} \beta^{t-t^*} \frac{\partial E\left[\varepsilon_t \mid \bar{\theta}_{t^*}, \varepsilon_{t^*}\right]}{\partial\varepsilon_{t^*}} \\
&\leq 1\left(m_{t^*} > 1\right) \frac{1-\beta^{t^{**}+1-t^*}}{1-\beta} = 1\left(m_{t^*} > 1\right) B_{t^*}.
\end{aligned}$$

Also,

$$\frac{\partial V_{t^*}[m_{t^*}, m_{t^*-1}, c_{t^*}, d_{t^*}, \bar{\theta}_{t^*}, \varepsilon_{t^*}, X_t]}{\partial\bar{\theta}_{t^*}} = 1\left(m_{t^*} > 1\right) \frac{\partial E\left[\sum_{t=t^*+1}^{t^{**}} \beta^{t-t^*} \varepsilon_t \mid \bar{\theta}_{t^*}, \varepsilon_{t^*}\right]}{\partial\bar{\theta}_{t^*}}. \tag{3.2}$$

If $m_{t^*} = 1$, then equation (3.2) is identically zero. If $m_{t^*} > 1$, then, since the mean of ε_t (for $t > t^{*)}$ is increasing in $\bar{\theta}_{t^*}$, equation (3.2) is positive. But it is bounded by B_t because $\partial E\left[\varepsilon_t \mid \bar{\theta}_{t^*}, \varepsilon_{t^*}\right] / \partial\bar{\theta}_{t^*} \leq 1$. Finally,

$$\begin{aligned}
V_{t^*}[m_{t^*}, m_{t^*-1}, c_{t^*}, d_{t^*}, 0, 0, X_t] &= \sum_{t=t^*}^{t^{**}} \beta^{t-t^*} f_t\left(m_t, c_t, d_t, X_t\right) + \\
&\quad 1\left(m_{t^*} > 1\right) \sum_{t=t^*+1}^{t^{**}} \beta^{t-t^*} E\left[\varepsilon_t \mid 0, 0\right]
\end{aligned}$$

which is finite because $E\left[\varepsilon_t \mid 0, 0\right]$ is finite.

Now assume that there exists a t' such that

$$\begin{aligned}
\partial V_t[m_t, m_{t-1}, c_t, d_t, \bar{\theta}_t, \varepsilon_t, X_t]/\partial\varepsilon_t &= 0 \text{ and} \\
\partial V_t[m_t, m_{t-1}, c_t, d_t, \bar{\theta}_t, \varepsilon_t, X_t]/\partial\bar{\theta}_t &= 0 \quad \text{if } m_t = 1;
\end{aligned}$$

$$\begin{aligned}
B_t &\geq \partial V_t[m_t, m_{t-1}, c_t, d_t, \bar{\theta}_t, \varepsilon_t, X_t]/\partial\varepsilon_t \geq 1 \text{ and} \\
B_t &\geq \partial V_t[m_t, m_{t-1}, c_t, d_t, \bar{\theta}_t, \varepsilon_t, X_t]/\partial\bar{\theta}_t \geq 0 \quad \text{if } m_t > 1
\end{aligned}$$

for all $t > t'$; $t^* - 1$ is such a time. Then

$$\begin{aligned}
&\frac{\partial V_{t'}[m_{t'}, m_{t'-1}, c_{t'}, d_{t'}, \bar{\theta}_{t'}, \varepsilon_{t'}, X_t]}{\partial\varepsilon_{t'}} \qquad (3.3) \\
&= 1\left(m_{t'} > 1\right) + \beta \frac{\partial E_{c_{t'+1}, \varepsilon_{t'+1}} \left\{\max_{m_{t'+1} \in F(m_{t'})} \left(V_{t'+1}\left[S_{t'+1}, X_{t'+1}\right]\right) \mid \bar{\theta}_{t'}, \varepsilon_{t'}, c_{t'}\right\}}{\partial\varepsilon_{t'}}
\end{aligned}$$

from equation (2.1). The first term on the right is positive it $m_{t'} > 1$, and it is zero if $m_{t'} = 1$. Consider the second term:

$$\partial V_{t'+1}\left[S_{t'+1}, X_{t'+1}\right]/\partial\varepsilon_{t'+1} = 0 \quad \text{if } m_t = 1;$$

$$B_{t'+1} \geq \partial V_{t'+1}\left[S_{t'+1}, X_{t'+1}\right]/\partial\varepsilon_{t'+1} \geq 1 \quad \text{if } m_t > 1$$

by assumption. Therefore

$$B_{t'+1} \geq \partial \max_{m_{t'+1}\in F(m_{t'})}\left(V_{t'+1}\left[S_{t'+1}, X_{t'+1}\right]\right)/\partial\varepsilon_{t'+1} \geq 0$$

everywhere the derivative exists. Therefore, since $c_{t'+1}$ and $\varepsilon_{t'+1}$ conditional on $\varepsilon_{t'}$ are independent, ε_t is positively serially correlated,

$$B_{t'+1} \geq \partial E_{\varepsilon_{t'+1}}\left\{\max_{m_{t'+1}\in F(m_{t'})}\left(V_{t'+1}\left[S_{t'+1}, X_{t'+1}\right]\right) \mid \bar{\theta}_{t'}, \varepsilon_{t'}, c_{t'+1}\right\}/\partial\varepsilon_{t'} \geq 0$$

and

$$B_{t'+1} \geq \partial E_{\varepsilon_{t'+1}}\left\{\max_{m_{t'+1}\in F(m_{t'})}\left(V_{t'+1}\left[S_{t'+1}, X_{t'+1}\right]\right) \mid \bar{\theta}_{t'}, \varepsilon_{t'}, c_{t'+1}\right\}/\partial\bar{\theta}_{t'} \geq 0$$

for all $c_{t'+1}$ (because the mean of $\varepsilon_{t'+1}$ is increasing in $\bar{\theta}_{t'}$) which implies that

$$\begin{aligned}
B_{t'+1} &\geq \partial E_{c_{t'+1},\varepsilon_{t'+1}}\left\{\max_{m_{t'+1}\in F(m_{t'})}\left(V_{t'+1}\left[S_{t'+1}, X_{t'+1}\right]\right) \mid \bar{\theta}_{t'}, \varepsilon_{t'}, c_{t'}\right\}/\partial\varepsilon_{t'} \geq 0; \\
B_{t'+1} &\geq \partial E_{c_{t'+1},\varepsilon_{t'+1}}\left\{\max_{m_{t'+1}\in F(m_{t'})}\left(V_{t'+1}\left[S_{t'+1}, X_{t'+1}\right]\right) \mid \bar{\theta}_{t'}, \varepsilon_{t'}, c_{t'}\right\}/\partial\bar{\theta}_{t'} \geq 0.
\end{aligned}$$

Therefore, since the first term in equation (3.3) is always unity if $m_t > 1$ and $B_t = 1 + \beta B_{t+1}$,

$$\begin{aligned}
\partial V_{t'}\left[S_{t'}, X_{t'}\right]/\partial\varepsilon_{t'} &= 0 \text{ and} \\
\partial V_{t'}\left[S_{t'}, X_{t'}\right]/\partial\bar{\theta}_{t'} &= 0 \quad \text{if } m_t = 1;
\end{aligned}$$

$$\begin{aligned}
B_{t'} &\geq \partial V_{t'}\left[S_{t'}, X_{t'}\right]/\partial\varepsilon_{t'} \geq 1 \text{ and} \\
B_{t'} &\geq \partial V_{t'}\left[S_{t'}, X_{t'}\right]/\partial\bar{\theta}_{t'} \geq 0 \quad \text{if } m_t > 1.
\end{aligned}$$

Also,

$$
\begin{aligned}
& V_{t'}[m_{t'}, m_{t'-1}, c_{t'}, d_{t'}, 0, 0, X_{t'}] \\
= \; & f_{t'}\left(m_{t'}, c_{t'}, d_{t'}, X_{t'}\right) - D_{m_{t'-1}} 1\left(m_{t'} = 1\right) + \beta E_{c_{t'+1}, \varepsilon_{t'+1}} \left\{ \max_{m_{t'+1} \in F(m_{t'})} \left(V_{t'+1}\left[S_{t'+1}, X_{t'+1}\right]\right) \mid 0, 0, c_t \right\}
\end{aligned}
$$

The first term is finite. Consider

$$
\begin{aligned}
& E_{\varepsilon_{t'+1}} \left\{ \max_{m_{t'+1} \in F(m_{t'})} \left(V_{t'+1}\left[S_{t'+1}, X_{t'+1}\right]\right) \mid 0, 0, c_{t'+1} \right\} \\
= \; & \int_{-\infty}^{\infty} \left[\max_{m_{t'+1} \in F(m_{t'})} V_{t'+1}\left[S_{t'+1}, X_{t'+1}\right] \right] \phi\left(\varepsilon_{t'+1} \mid 0, 0\right) d\varepsilon_{t'+1}
\end{aligned} \tag{3.4}
$$

where $\phi\left(\varepsilon_{t'+1} \mid \bar{\theta}_{t'}, \varepsilon_{t'}\right)$ is the (normal) density of $\varepsilon_{t'+1}$ conditional on $\bar{\theta}_{t'}$ and $\varepsilon_{t'}$. Since $\max_{m_{t'+1} \in F(m_{t'})} V_{t'+1}\left[S_{t'+1}, X_{t'+1}\right]$ grows linearly and $\phi\left(\varepsilon_{t'+1} \mid \bar{\theta}_{t'}, \varepsilon_{t'}\right)$ declines faster than exponentially, equation (3.4) is finite implying that

$$
E_{c_{t'+1}, \varepsilon_{t'+1}} \left\{ \max_{m_{t'+1} \in F(m_{t'})} \left(V_{t'+1}\left[S_{t'+1}, X_{t'+1}\right]\right) \mid 0, 0, c_t \right\}
$$

is also finite. Thus $V_{t'}[m_{t'}, m_{t'-1}, c_{t'}, d_{t'}, 0, 0, X_{t'}]$ is finite.

■

The facts that $V_t[m_t, m_{t-1}, c_t, d_t, 0, 0, X_t]$ is finite, $\partial V_t[1, m_{t-1}, c_t, d_t, \bar{\theta}_t, \varepsilon_t, X_t]/\partial \varepsilon_t = 0$, and $\partial V_t[1, m_{t-1}, c_t, d_t, \bar{\theta}_t, \varepsilon_t, X_t]/\partial \bar{\theta}_t = 0$ imply that $V_t[1, m_{t-1}, c_t, d_t, \bar{\theta}_t, \varepsilon_t, X_t]$ is finite for any $\bar{\theta}_t$ and ε_t.

Theorem 2. *$\exists\, \varepsilon_t^*\left(m_t, 1\right) : V_t[m_t, 1, c_t, 1, \bar{\theta}_t\left(\varepsilon_t\right), \varepsilon_t, X_t] > V_t[1, 1, c_t, d_t, \bar{\theta}_t\left(\varepsilon_t\right), \varepsilon_t, X_t]$ for all $\varepsilon_t > \varepsilon_t^*\left(m_t, 1\right)$ and $V_t[m_t, 1, c_t, 1, \bar{\theta}_t\left(\varepsilon_t\right), \varepsilon_t, X_t] < V_t[1, 1, c_t, d_t, \bar{\theta}_t\left(\varepsilon_t\right), \varepsilon_t, X_t]$ for all $\varepsilon_t < \varepsilon_t^*\left(m_t, 1\right)$ for $m_t > 1$.*

Proof. $V_t[1, 1, c_t, d_t, \bar{\theta}_t\left(\varepsilon_t\right), \varepsilon_t, X_t]$ is finite. From Theorem 1, $\partial V_{t'}\left[S_{t'}, X_{t'}\right]/\partial \varepsilon_{t'} = 0$ if $m_t = 1$. Then, also by Theorem 1, since $\partial V_{t'}\left[S_{t'}, X_{t'}\right]/\partial \varepsilon_{t'} \geq 1$ if $m_t > 1$ also by Theorem 1, the result follows.

■

Theorem 3. *$\exists\, \varepsilon_t^*\left(1, m_{t-1}\right) : V_t[m_{t-1}, m_{t-1}, c_t, d_t, \bar{\theta}_t, \varepsilon_t, X_t] > V_t[1, m_{t-1}, c_t, 1, \bar{\theta}_t, \varepsilon_t, X_t]$ for all $\varepsilon_t > \varepsilon_t^*\left(1, m_{t-1}\right)$ and $V_t[m_{t-1}, m_{t-1}, c_t, d_t, \bar{\theta}_t, \varepsilon_t, X_t] < V_t[1, m_{t-1}, c_t, 1, \bar{\theta}_t, \varepsilon_t, X_t]$ for all $\varepsilon_t < \varepsilon_t^*\left(1, m_{t-1}\right)$ for $m_{t-1} > 1$.*

Proof. $V_t[1, m_{t-1}, c_t, 1, \bar{\theta}_t, \varepsilon_t, X_t]$ is finite for all $\bar{\theta}_t$ and ε_t, and $\partial V_t[m_{t-1}, m_{t-1}, c_t, d_t, \bar{\theta}_t, \varepsilon_t, X_t]/\partial \varepsilon_t \geq 1$. The result follows. ■

Assumption 1: $D_3 \geq D_2 > 0$ and $f_t(3, c_t, d_t, X_t) \geq f_t(2, c_t, d_t, X_t)$.

Theorem 4. *If* $D_3 > D_2 > 0$ *and* $f_t(3, c_t, d_t, X_t) = f_t(2, c_t, d_t, X_t)$*, then* $V_t\left[3, m_{t-1}, c_t, d_t, \bar{\theta}_t, \varepsilon_t, X_t\right] < V_t\left[2, m_{t-1}, c_t, d_t, \bar{\theta}_t, \varepsilon_t, X_t\right]$ *for* $m_{t-1} \neq 3$ *and* $t < t^*$.

Proof. Note that

$$\begin{aligned}
& V_{t^*}\left[3, m_{t^*-1}, c_{t^*}, d_{t^*}, \bar{\theta}_{t^*}, \varepsilon_{t^*}, X_{t^*}\right] - V_{t^*}\left[2, m_{t^*-1}, c_{t^*}, d_{t^*}, \bar{\theta}_{t^*}, \varepsilon_{t^*}, X_{t^*}\right] \\
= \; & f_{t^*}(3, c_{t^*}, d_{t^*}, X_{t^*}) - f_{t^*}(2, c_{t^*}, d_{t^*}, X_{t^*}) \\
& + \beta E_{c_{t^*+1}, \varepsilon_{t^*+1}}\left\{ \max_{m_{t^*+1} \in F(3)} \left(V_{t^*+1}\left[S_{t^*+1}, X_{t^*+1}\right]\right) \mid \bar{\theta}_{t^*}, \varepsilon_{t^*}, c_{t^*} \right\} \\
& - \beta E_{c_{t^*+1}, \varepsilon_{t^*+1}}\left\{ \max_{m_{t^*+1} \in F(2)} \left(V_{t^*+1}\left[S_{t^*+1}, X_{t^*+1}\right]\right) \mid \bar{\theta}_{t^*}, \varepsilon_{t^*}, c_{t^*} \right\} \\
= \; & 0
\end{aligned}$$

because $f_t(3, c_t, d_t, X_t) = f_t(2, c_t, d_t, X_t)$ and no choices can be made beyond t^*. Assume that there exists a t such that $V_{t'}\left[3, m_{t'-1}, c_{t'}, d_{t'}, \bar{\theta}_{t'}, \varepsilon_{t'}, X_{t'}\right] \leq V_{t'}\left[2, m_{t'-1}, c_{t'}, d_{t'}, \bar{\theta}_{t'}, \varepsilon_{t'}, X_{t'}\right]$ for all $t' > t$ (except when $.m_{t'-1} = 3$). Then

$$\begin{aligned}
& V_t\left[3, m_{t-1}, c_t, d_t, \bar{\theta}_t, \varepsilon_t, X_t\right] - V_t\left[2, m_{t-1}, c_t, d_t, \bar{\theta}_t, \varepsilon_t, X_t\right] \qquad (3.5) \\
= \; & f_t(3, c_t, d_t, X_t) - f_t(2, c_t, d_t, X_t) \\
& + \beta E_{c_{t+1}, \varepsilon_{t+1}}\left\{ \max_{m_{t+1} \in F(3)} \left(V_{t+1}\left[S_{t+1}, X_{t+1}\right]\right) \mid \bar{\theta}_t, \varepsilon_t, c_t \right\} \\
& - \beta E_{c_{t+1}, \varepsilon_{t+1}}\left\{ \max_{m_{t+1} \in F(2)} \left(V_{t+1}\left[S_{t+1}, X_{t+1}\right]\right) \mid \bar{\theta}_t, \varepsilon_t, c_t \right\} \\
= \; & \beta E_{c_{t+1}, \varepsilon_{t+1}}\left\{ \max_{m_{t+1} \in F(3)} \left(V_{t+1}\left[S_{t+1}, X_{t+1}\right]\right) \mid \bar{\theta}_t, \varepsilon_t, c_t \right\} \\
& - \beta E_{c_{t+1}, \varepsilon_{t+1}}\left\{ \max_{m_{t+1} \in F(2)} \left(V_{t+1}\left[S_{t+1}, X_{t+1}\right]\right) \mid \bar{\theta}_t, \varepsilon_t, c_t \right\}.
\end{aligned}$$

Now divide the range of ε_{t+1}: (a) For ε_{t+1} where $V_{t+1}\left[2, 2, c_{t+1}, d_{t+1}, \bar{\theta}_{t+1}, \varepsilon_{t+1}, X_{t+1}\right] \geq V_{t+1}\left[3, m_t, c_{t+1}, d_{t+1}, \bar{\theta}_{t+1}, \varepsilon_{t+1}, X_{t+1}\right] \geq V_{t+1}\left[1, m_t, c_{t+1}, d_{t+1}, \bar{\theta}_{t+1}, \varepsilon_{t+1}, X_{t+1}\right]$ for

$m_t > 1$, then $m_{t+1} = m_t$ (because $2 \notin F(3)$); (b) for ε_{t+1} where $V_{t+1}\left[2, 2, c_{t+1}, d_{t+1}, \bar{\theta}_{t+1}, \varepsilon_{t+1}, X_{t+1}\right] \geq$ $V_{t+1}\left[1, m_t, c_{t+1}, d_{t+1}, \bar{\theta}_{t+1}, \varepsilon_{t+1}, X_{t+1}\right] \geq V_{t+1}\left[3, m_t, c_{t+1}, d_{t+1}, \bar{\theta}_{t+1}, \varepsilon_{t+1}, X_{t+1}\right]$ for $m_t > 1$, then if $m_t = 2$, then $m_{t+1} = 2$ and if $m_t = 3$, then $m_{t+1} = 1$; (c) For ε_{t+1} where $V_{t+1}\left[1, m_t, c_{t+1}, d_{t+1}, \bar{\theta}_{t+1}, \varepsilon_{t+1}, X_{t+1}\right] \geq V_{t+1}\left[2, 2, c_{t+1}, d_{t+1}, \bar{\theta}_{t+1}, \varepsilon_{t+1}, X_{t+1}\right] \geq$ $V_{t+1}\left[3, m_t, c_{t+1}, d_{t+1}, \bar{\theta}_{t+1}, \varepsilon_{t+1}, X_{t+1}\right]$ for $m_t > 1$, then $m_{t+1} = 1$ (note that $V_{t+1}\left[1, 2, c_{t+1}, d_{t+1}, \bar{\theta}_{t+1}, \varepsilon_{t-}\right.$ $V_{t+1}\left[1, 3, c_{t+1}, d_{t+1}, \bar{\theta}_{t+1}, \varepsilon_{t+1}, X_{t+1}\right]$ because $D_3 > D_2$); (d) For ε_{t+1} where $V_{t+1}\left[1, 2, c_{t+1}, d_{t+1}, \bar{\theta}_{t+1}, \varepsilon_{t+}\right.$ $V_{t+1}\left[2, 2, c_{t+1}, d_{t+1}, \bar{\theta}_{t+1}, \varepsilon_{t+1}, X_{t+1}\right] \geq V_{t+1}\left[3, 3, c_{t+1}, d_{t+1}, \bar{\theta}_{t+1}, \varepsilon_{t+1}, X_{t+1}\right] \geq V_{t+1}\left[1, 3, c_{t+1}, d_{t+1}, \bar{\theta}_{t+1}\right.$ These are all the possible cases subject to the condition that $V_t\left[3, m_{t-1}, c_t, d_t, \bar{\theta}_t, \varepsilon_t, X_t\right] \leq$ $V_t\left[2, m_{t-1}, c_t, d_t, \bar{\theta}_t, \varepsilon_t, X_t\right]$. Note that for each case the relevant part of equation (3.5) is less than or equal to zero and for some parts it is less than zero. Thus equation (3.5) is negative. ■

Theorem 5. *If $D_3 = D_2 > 0$ and $f_t(3, c_t, d_t, X_t) > f_t(2, c_t, d_t, X_t)$, then $V_t\left[3, m_{t-1}, c_t, d_t, \bar{\theta}_t, \varepsilon_t, X_t\right] >$ $V_t\left[2, m_{t-1}, c_t, d_t, \bar{\theta}_t, \varepsilon_t, X_t\right]$ for $m_{t-1} \neq 3$.*

Proof. Note that

$$\begin{aligned}
& V_{t^*}\left[3, m_{t^*-1}, c_{t^*}, d_{t^*}, \bar{\theta}_{t^*}, \varepsilon_{t^*}, X_{t^*}\right] - V_{t^*}\left[2, m_{t^*-1}, c_{t^*}, d_{t^*}, \bar{\theta}_{t^*}, \varepsilon_{t^*}, X_{t^*}\right] \\
= \; & f_{t^*}(3, c_{t^*}, d_{t^*}, X_{t^*}) - f_{t^*}(2, c_{t^*}, d_{t^*}, X_{t^*}) \\
& +\beta E_{c_{t^*+1}, \varepsilon_{t^*+1}}\left\{\max_{m_{t^*+1} \in F(3)}\left(V_{t^*+1}\left[S_{t^*+1}, X_{t^*+1}\right]\right) \mid \bar{\theta}_{t^*}, \varepsilon_{t^*}, c_{t^*}\right\} \\
& -\beta E_{c_{t^*+1}, \varepsilon_{t^*+1}}\left\{\max_{m_{t^*+1} \in F(2)}\left(V_{t^*+1}\left[S_{t^*+1}, X_{t^*+1}\right]\right) \mid \bar{\theta}_{t^*}, \varepsilon_{t^*}, c_{t^*}\right\} \\
> \; & 0
\end{aligned}$$

because $f_t(3, c_t, d_t, X_t) > f_t(2, c_t, d_t, X_t)$ and no choices can be made beyond t^*. Assume that there exists a t such that $V_{t'}\left[3, m_{t'-1}, c_{t'}, d_{t'}, \bar{\theta}_{t'}, \varepsilon_{t'}, X_{t'}\right] >$ $V_{t'}\left[2, m_{t'-1}, c_{t'}, d_{t'}, \bar{\theta}_{t'}, \varepsilon_{t'}, X_{t'}\right]$ for all $t' > t$ (except when $.m_{t'-1} = 3$). Then

$$\begin{aligned}
& V_t\left[3, m_{t-1}, c_t, d_t, \bar{\theta}_t, \varepsilon_t, X_t\right] - V_t\left[2, m_{t-1}, c_t, d_t, \bar{\theta}_t, \varepsilon_t, X_t\right] \qquad (3.6) \\
= \; & f_t(3, c_t, d_t, X_t) - f_t(2, c_t, d_t, X_t) \\
& +\beta E_{c_{t+1}, \varepsilon_{t+1}}\left\{\max_{m_{t+1} \in F(3)}\left(V_{t+1}\left[S_{t+1}, X_{t+1}\right]\right) \mid \bar{\theta}_t, \varepsilon_t, c_t\right\}
\end{aligned}$$

$$
\begin{aligned}
&-\beta E_{c_{t+1},\varepsilon_{t+1}}\left\{\max_{m_{t+1}\in F(2)}\left(V_{t+1}\left[S_{t+1},X_{t+1}\right]\right)\mid\bar{\theta}_t,\varepsilon_t,c_t\right\}\\
= \; &\beta E_{c_{t+1},\varepsilon_{t+1}}\left\{\max_{m_{t+1}\in F(3)}\left(V_{t+1}\left[S_{t+1},X_{t+1}\right]\right)\mid\bar{\theta}_t,\varepsilon_t,c_t\right\}\\
&-\beta E_{c_{t+1},\varepsilon_{t+1}}\left\{\max_{m_{t+1}\in F(2)}\left(V_{t+1}\left[S_{t+1},X_{t+1}\right]\right)\mid\bar{\theta}_t,\varepsilon_t,c_t\right\}.
\end{aligned}
$$

Now divide the range of ε_{t+1}: (a) For ε_{t+1} where $V_{t+1}\left[3,m_t,c_{t+1},d_{t+1},\bar{\theta}_{t+1},\varepsilon_{t+1},X_{t+1}\right] > V_{t+1}\left[2,2,c_{t+1},d_{t+1},\bar{\theta}_{t+1},\varepsilon_{t+1},X_{t+1}\right], V_{t+1}\left[1,m_t,c_{t+1},d_{t+1},\bar{\theta}_{t+1},\varepsilon_{t+1},X_{t+1}\right]$ for $m_t > 1$, then $m_{t+1}=3$; (b) For ε_{t+1} where $V_{t+1}\left[1,m_t,c_{t+1},d_{t+1},\bar{\theta}_{t+1},\varepsilon_{t+1},X_{t+1}\right] \geq V_{t+1}\left[3,m_t,c_{t+1},d_{t+1},\bar{\theta}_{t+1},\varepsilon_{t+1},X_{t+1}\right], V_{t+1}\left[2,2,c_{t+1},d_{t+1},\bar{\theta}_{t+1},\varepsilon_{t+1},X_{t+1}\right]$ for $m_t > 1$, then $m_{t+1}=1$ (note that $V_{t+1}\left[1,2,c_{t+1},d_{t+1},\bar{\theta}_{t+1},\varepsilon_{t+1},X_{t+1}\right] = V_{t+1}\left[1,3,c_{t+1},d_{t+1},\bar{\theta}_{t+1},\varepsilon_{t+1},X_{t+1}\right]$ because $D_3 = D_2$); (c) For ε_{t+1} where $V_{t+1}\left[1,3,c_{t+1},d_{t+1},\bar{\theta}_{t+1},\varepsilon_{t+1},X_{t+1}\right] \geq V_{t+1}\left[3,m_t,c_{t+1},d_{t+1},\bar{\theta}_{t+1},\varepsilon_{t+1},X_{t+1}\right] > V_{t+1}\left[2,2,c_{t+1},d_{t+1},\bar{\theta}_{t+1},\varepsilon_{t+1},X_{t+1}\right] \geq V_{t+1}\left[1,2,c_{t+1},d_{t+1},\bar{\theta}_{t+}\right.$ These are all the possible cases subject to the condition that $V_t\left[3,m_{t-1},c_t,d_t,\bar{\theta}_t,\varepsilon_t,X_t\right] > V_t\left[2,m_{t-1},c_t,d_t,\bar{\theta}_t,\varepsilon_t,X_t\right]$. Note that for each case the relevant part of equation (3.6) is greater than or equal to zero and for some parts it is greater than zero. Thus equation (3.6) is positive. ■

Assumption 2: $D_3 > D_2$, $f_t\left(3,c_t,d_t,X_t\right) > f_t\left(2,c_t,d_t,X_t\right)$, and $D_3 - D_2$ does not grow over t or d_t.[3]

Theorem 6. $\varepsilon_t^*(1,3) < \varepsilon_t^*(1,2)$.

Proof. At t^*,

$$V_{t^*}\left[3,3,c_{t^*},d_{t^*},\bar{\theta}_{t^*},\varepsilon_{t^*},X_{t^*}\right] - V_{t^*}\left[2,2,c_{t^*},d_{t^*},\bar{\theta}_{t^*},\varepsilon_{t^*},X_{t^*}\right] = \sum_{t=t^*}^{t^{**}}\beta^{(t-t^*)}\left[f_t\left(3,c_t,d_t,X_t\right) - f_t\left(2,c_t,d_t,X\right.\right.$$

At $\varepsilon_{t^*}^*(1,2)$,

$$
\begin{aligned}
&\left\{V_{t^*}\left[3,3,c_{t^*},d_{t^*},\bar{\theta}_{t^*},\varepsilon_{t^*},X_{t^*}\right] - V_{t^*}\left[1,3,c_{t^*},d_{t^*},\bar{\theta}_{t^*},\varepsilon_{t^*},X_{t^*}\right]\right\} - \left\{V_{t^*}\left[2,2,c_{t^*},d_{t^*},\bar{\theta}_{t^*},\varepsilon_{t^*},X_{t^*}\right] - \right.\\
= \; &V_{t^*}\left[3,3,c_{t^*},d_{t^*},\bar{\theta}_{t^*},\varepsilon_{t^*},X_{t^*}\right] - V_{t^*}\left[2,2,c_{t^*},d_{t^*},\bar{\theta}_{t^*},\varepsilon_{t^*},X_{t^*}\right] + D_3 - D_2 > 0
\end{aligned}
$$

[3] The last part of this assumption is necessary to rule out the possibility of divorce costs rising so rapidly that agents divorce early to avoid being trapped later in the relationship. We can relax this part of the assumption to allow $D_3 - D_2$ to grow at a rate no greater than β^{-1}.

$\Rightarrow$at $\varepsilon^*_{t^*}(1,2)$, $V_{t^*}\left[3,3,c_{t^*},d_{t^*},\bar{\theta}_{t^*},\varepsilon_{t^*},X_{t^*}\right]-V_{t^*}\left[1,3,c_{t^*},d_{t^*},\bar{\theta}_{t^*},\varepsilon_{t^*},X_{t^*}\right]>0\Rightarrow$ $\varepsilon^*_{t^*}(1,3)<\varepsilon^*_{t^*}(1,2)$ (because $\partial V_{t^*}\left[3,3,c_{t^*},d_{t^*},\bar{\theta}_{t^*},\varepsilon_{t^*},X_{t^*}\right]/\partial\varepsilon_{t^*}>0$). At t^*-1,

$$\begin{aligned}
& V_{t^*-1}\left[3,3,c_{t^*-1},d_{t^*-1},\bar{\theta}_{t^*-1},\varepsilon_{t^*-1},X_{t^*-1}\right]-V_{t^*-1}\left[2,2,c_{t^*-1},d_{t^*-1},\bar{\theta}_{t^*-1},\varepsilon_{t^*-1},X_{t^*-1}\right] \\
= \; & f_{t^*-1}\left(3,c_{t^*-1},d_{t^*-1},X_{t^*-1}\right)-f_{t^*-1}\left(2,c_{t^*-1},d_{t^*-1},X_{t^*-1}\right)+ \\
& \beta E\max_{m\neq 2}V_{t^*}\left[m,3,c_{t^*},d_{t^*},\bar{\theta}_{t^*},\varepsilon_{t^*},X_{t^*}\right]-\beta E\max_{m}V_{t^*}\left[m,2,c_{t^*},d_{t^*},\bar{\theta}_{t^*},\varepsilon_{t^*},X_{t^*}\right] \\
> \; & f_{t^*-1}\left(3,c_{t^*-1},d_{t^*-1},X_{t^*-1}\right)-f_{t^*-1}\left(2,c_{t^*-1},d_{t^*-1},X_{t^*-1}\right)+ \\
& \beta\left(D_2-D_3\right)\int_{-\infty}^{\varepsilon^*_{t^*}(1,2)}d\Phi\left(\varepsilon_{t^*}\right) \\
& +\beta\int_{\varepsilon^*_{t^*}(1,2)}^{\infty}\left\{V_{t^*}\left[3,3,c_{t^*},d_{t^*},\bar{\theta}_{t^*},\varepsilon_{t^*},X_{t^*}\right]-V_{t^*}\left[2,2,c_{t^*},d_{t^*},\bar{\theta}_{t^*},\varepsilon_{t^*},X_{t^*}\right]\right\}d\Phi\left(\varepsilon_{t^*}\right) \\
> \; & \beta\left(D_2-D_3\right),
\end{aligned}$$

and

$$\begin{aligned}
& \left\{V_{t^*-1}\left[3,3,c_{t^*-1},d_{t^*-1},\bar{\theta}_{t^*-1},\varepsilon_{t^*-1},X_{t^*-1}\right]-V_{t^*-1}\left[1,3,c_{t^*-1},d_{t^*-1},\bar{\theta}_{t^*-1},\varepsilon_{t^*-1},X_{t^*-1}\right]\right\} \\
& -\left\{V_{t^*-1}\left[2,2,c_{t^*-1},d_{t^*-1},\bar{\theta}_{t^*-1},\varepsilon_{t^*-1},X_{t^*-1}\right]-V_{t^*-1}\left[1,2,c_{t^*-1},d_{t^*-1},\bar{\theta}_{t^*-1},\varepsilon_{t^*-1},X_{t^*-1}\right]\right\} \\
= \; & V_{t^*-1}\left[3,3,c_{t^*-1},d_{t^*-1},\bar{\theta}_{t^*-1},\varepsilon_{t^*-1},X_{t^*-1}\right]-V_{t^*-1}\left[2,2,c_{t^*-1},d_{t^*-1},\bar{\theta}_{t^*-1},\varepsilon_{t^*-1},X_{t^*-1}\right]+D_3-D_2 \\
> \; & \beta\left(D_2-D_3\right)+D_3-D_2 \\
= \; & \left(D_3-D_2\right)\left(1-\beta\right)>0
\end{aligned}$$

$\Rightarrow$at $\varepsilon^*_{t^*-1}(1,2)$, $V_{t^*-1}\left[3,3,c_{t^*-1},d_{t^*-1},\bar{\theta}_{t^*-1},\varepsilon_{t^*-1},X_{t^*-1}\right]-V_{t^*-1}\left[1,3,c_{t^*-1},d_{t^*-1},\bar{\theta}_{t^*-1},\varepsilon_{t^*-1},X_{t^*-1}\right]>$ $0\Rightarrow\varepsilon^*_{t^*-1}(1,3)<\varepsilon^*_{t^*-1}(1,2)$ (because $\partial V_{t^*-1}\left[3,3,c_{t^*-1},d_{t^*-1},\bar{\theta}_{t^*-1},\varepsilon_{t^*-1},X_{t^*-1}\right]/\partial\varepsilon_{t^*-1}>$ 0).

Now assume that $\exists\, t: V_{t'}\left[3,3,c_{t'},d_{t'},\bar{\theta}_{t'},\varepsilon_{t'},X_{t'}\right]-V_{t'}\left[2,2,c_{t'},d_{t'},\bar{\theta}_{t'},\varepsilon_{t'},X_{t'}\right]>$ $\beta\left(D_2-D_3\right)$ and $\varepsilon^*_{t'}(1,3)<\varepsilon^*_{t'}(1,2)\quad\forall t'>t$. Then

$$\begin{aligned}
& V_t\left[3,3,c_t,d_t,\bar{\theta}_t,\varepsilon_t,X_t\right]-V_t\left[2,2,c_t,d_t,\bar{\theta}_t,\varepsilon_t,X_t\right] \\
= \; & f_t\left(3,c_t,d_t,X_t\right)-f_t\left(2,c_t,d_t,X_t\right) \\
& +\beta E\max_{m\neq 2}V_{t+1}\left[m,3,c_{t+1},d_{t+1},\bar{\theta}_{t+1},\varepsilon_{t+1},X_{t+1}\right]-\beta E\max_{m}V_{t+1}\left[m,2,c_{t+1},d_{t+1},\bar{\theta}_{t+1},\varepsilon_{t+1},X_{t+1}\right] \\
> \; & f_t\left(3,c_t,d_t,X_t\right)-f_t\left(2,c_t,d_t,X_t\right)+ \\
& \beta\left[\left(D_2-D_3\right)\int_{-\infty}^{\varepsilon^*_{t+1}(1,2)}d\Phi\left(\varepsilon_t\right)+\int_{\varepsilon^*_{t+1}(1,2)}^{\infty}\left\{V_{t+1}\left[3,3,c_{t+1},d_{t+1},\bar{\theta}_{t+1},\varepsilon_{t+1},X_{t+1}\right]-V_{t+1}\left[2,2,c_{t+1},\right.\right.\right.
\end{aligned}$$

$$
\begin{aligned}
&> \beta\left[\left(D_2 - D_3\right)\int_{-\infty}^{\varepsilon^*_{t+1}(1,2)} d\Phi\left(\varepsilon_t\right) + \beta\left(D_2 - D_3\right)\int_{\varepsilon^*_{t+1}(1,2)}^{\infty} d\Phi\left(\varepsilon_{t+1}\right)\right] \\
&> \beta\left(D_2 - D_3\right),
\end{aligned}
$$

and

$$
\begin{aligned}
& \left\{V_t\left[3,3,c_t,d_t,\bar{\theta}_t,\varepsilon_t,X_t\right] - V_t\left[1,3,c_t,d_t,\bar{\theta}_t,\varepsilon_t,X_t\right]\right\} - \left\{V_t\left[2,2,c_t,d_t,\bar{\theta}_t,\varepsilon_t,X_t\right] - V_t\left[1,2,c_t,d_t,\bar{\theta}_t\right.\right. \\
&= V_t\left[3,3,c_t,d_t,\bar{\theta}_t,\varepsilon_t,X_t\right] - V_t\left[2,2,c_t,d_t,\bar{\theta}_t,\varepsilon_t,X_t\right] + D_3 - D_2 \\
&> \beta\left(D_2 - D_3\right) + D_3 - D_2 \\
&= \left(D_3 - D_2\right)\left(1 - \beta\right) > 0
\end{aligned}
$$

$\Rightarrow$at $\varepsilon^*_t(1,2)$, $V_t\left[3,3,c_t,d_t,\bar{\theta}_t,\varepsilon_t,X_t\right] - V_t\left[1,3,c_t,d_t,\bar{\theta}_t,\varepsilon_t,X_t\right] > 0 \Rightarrow \varepsilon^*_t(1,3) < \varepsilon^*_t(1,2)$ (because $\partial V_t\left[3,3,c_t,d_t,\bar{\theta}_t,\varepsilon_t,X_t\right]/\partial\varepsilon_t > 0$). ■

Theorem 7. $\partial V_t[3,1,c_t,1,\bar{\theta}_t(\varepsilon_t),\varepsilon_t,X_t]/\partial\varepsilon_t > \partial V_t[2,1,c_t,1,\bar{\theta}_t(\varepsilon_t),\varepsilon_t,X_t]/\partial\varepsilon_t$.

Proof. Let $m_t > 1$. Then, by equation (2.1),

$$
\begin{aligned}
& V_t[m_t,m_{t-1},c_t,d_t,\bar{\theta}_t,\varepsilon_t,X_t] \\
&= f_t\left(m_t,c_t,d_t,X_t\right) + 1\left(m_t > 1\right)\varepsilon_t + \\
& \beta E_{c_{t+1}}[\int_{\varepsilon^*_{t+1}(1,m_t)}^{\infty}\left\{\max_{\substack{m_{t+1}\in F(m_t)\\ m_{t+1}\neq 1}} V_{t+1}\left[S_{t+1},X_{t+1}\right]\right\}\phi\left(\varepsilon_{t+1}\mid\bar{\theta}_t,\varepsilon_t\right)d\varepsilon_{t+1} + \\
& \Phi\left(\varepsilon^*_{t+1}\left(1,m_t\right)\mid\bar{\theta}_t,\varepsilon_t\right)V_t[1,m_t,c_{t+1},1,0,0,X_{t+1}]\mid c_t],
\end{aligned}
$$

and its partial derivative is

$$
\begin{aligned}
\partial V_t[m_t,m_{t-1},c_t,d_t,\bar{\theta}_t,\varepsilon_t,X_t]/\partial\varepsilon_t &= 1 + \\
\beta E_{c_{t+1}}[\int_{\varepsilon^*_{t+1}(1,m_t)}^{\infty}\left\{\max_{\substack{m_{t+1}\in F(m_t)\\ m_{t+1}\neq 1}} V_{t+1}\left[S_{t+1},X_{t+1}\right]\right\}\frac{\partial\phi\left(\varepsilon_{t+1}\mid\bar{\theta}_t,\varepsilon_t\right)}{\partial\varepsilon_t}d\varepsilon_{t+1} &+ \qquad (3.7) \\
\frac{\partial\Phi\left(\varepsilon^*_{t+1}\left(1,m_t\right)\mid\bar{\theta}_t,\varepsilon_t\right)}{\partial\varepsilon_t}V_{t+1}[1,m_t,c_{t+1},1,0,0,X_{t+1}]\mid c_t].
\end{aligned}
$$

Note:

$$
\frac{\partial\phi\left(\varepsilon_{t+1}\mid\bar{\theta}_t,\varepsilon_t\right)}{\partial\varepsilon_t} = -\rho\frac{\partial\phi\left(\varepsilon_{t+1}\mid\bar{\theta}_t,\varepsilon_t\right)}{\partial\varepsilon_{t+1}}.
$$

Thus, the integral in the second line of equation (3.7) can be written as

$$-\rho \int_{\varepsilon^*_{t+1}(1,m_t)}^{\infty} \left\{ \max_{\substack{m_{t+1} \in F(m_t) \\ m_{t+1} \neq 1}} V_{t+1}\left[S_{t+1}, X_{t+1}\right] \right\} \frac{\partial \phi\left(\varepsilon_{t+1} \mid \bar{\theta}_t, \varepsilon_t\right)}{\partial \varepsilon_{t+1}} d\varepsilon_{t+1}$$

which, through integration by parts, can be written as

$$\rho \int_{\varepsilon^*_{t+1}(1,m_t)}^{\infty} \frac{\partial}{\partial \varepsilon_{t+1}} \left\{ \max_{\substack{m_{t+1} \in F(m_t) \\ m_{t+1} \neq 1}} V_{t+1}\left[m_{t+1}, m_t, c_{t+1}, d_{t+1}, \bar{\theta}_{t+1}, \varepsilon_{t+1}, X_{t+1}\right] \right\} \phi\left(\varepsilon_{t+1} \mid \bar{\theta}_t, \varepsilon_t\right) d\varepsilon_{t+1} \tag{3.8}$$

$$\rho \left\{ \max_{\substack{m_{t+1} \in F(m_t) \\ m_{t+1} \neq 1}} V_{t+1}\left[m_{t+1}, m_t, c_{t+1}, d_{t+1}, \bar{\theta}_{t+1}, \varepsilon^*_{t+1}(1, m_t), X_{t+1}\right] \right\} \phi\left(\varepsilon^*_{t+1}(1, m_t) \mid \bar{\theta}_t, \varepsilon_t\right).$$

The last term in equation (3.7) can be written as

$$-\rho\phi\left(\varepsilon^*_{t+1}(1, m_t) \mid \bar{\theta}_t, \varepsilon_t\right) V_{t+1}[1, m_t, c_{t+1}, 1, 0, 0, X_{t+1}]. \tag{3.9}$$

Combining equations (3.8) and (3.9) leads to

$$\rho \int_{\varepsilon^*_{t+1}(1,m_t)}^{\infty} \frac{\partial}{\partial \varepsilon_{t+1}} \left\{ \max_{\substack{m_{t+1} \in F(m_t) \\ m_{t+1} \neq 1}} V_{t+1}\left[m_{t+1}, m_t, c_{t+1}, d_{t+1}, \bar{\theta}_{t+1}, \varepsilon_{t+1}, X_{t+1}\right] \right\} \phi\left(\varepsilon_{t+1} \mid \bar{\theta}_t, \varepsilon_t\right) d\varepsilon_{t+1} \tag{3.10}$$

(the second term in equation (3.8) cancels with equation (3.9) by the definition of $\varepsilon^*_{t+1}(1, m_t)$). The integrand in equation (3.10) is the same for $m_t = 2$ or $m_t = 3$, but, since $\varepsilon^*_{t+1}(1, 3) < \varepsilon^*_{t+1}(1, 2)$,

$$\frac{\partial V_t[3, m_{t-1}, c_t, d_t, \bar{\theta}_t, \varepsilon_t, X_t]}{\partial \varepsilon_t} - \frac{\partial V_t[2, m_{t-1}, c_t, d_t, \bar{\theta}_t, \varepsilon_t, X_t]}{\partial \varepsilon_t} =$$

$$\rho \int_{\varepsilon^*_{t+1}(1,3)}^{\varepsilon^*_{t+1}(1,2)} \frac{\partial}{\partial \varepsilon_{t+1}} \left\{ \max_{\substack{m_{t+1} \in F(m_t) \\ m_{t+1} \neq 1}} V_{t+1}\left[m_{t+1}, m_t, c_{t+1}, d_{t+1}, \bar{\theta}_{t+1}, \varepsilon_{t+1}, X_{t+1}\right] \right\} \phi\left(\varepsilon_{t+1} \mid \bar{\theta}_t, \varepsilon_t\right) d\varepsilon_{t+1} > 0.$$

By a similar argument, we can show that

$$\frac{\partial V_t[3, m_{t-1}, c_t, d_t, \bar{\theta}_t, \varepsilon_t, X_t]}{\partial \bar{\theta}_t} - \frac{\partial V_t[2, m_{t-1}, c_t, d_t, \bar{\theta}_t, \varepsilon_t, X_t]}{\partial \bar{\theta}_t} > 0.$$

Therefore, since $\partial\bar{\theta}_t/\partial\varepsilon_t > 0$ when $d_t = 1$, the result follows. ■

Theorem 8. *Consider the case where* $m_{t-1} < 3$. *Then*

a) $\exists\, \varepsilon_t^*(1,2) : V_t\left[2,2,c_t,d_t,\bar{\theta}_t,\varepsilon_t,X_t\right] > V_t\left[1,2,c_t,d_t,\bar{\theta}_t,\varepsilon_t,X_t\right]$ $\forall \varepsilon_t > \varepsilon_t^*(1,2)$ and $V_t\left[2,2,c_t,d_t,\bar{\theta}_t,\varepsilon_t,X_t\right] < V_t\left[1,2,c_t,d_t,\bar{\theta}_t,\varepsilon_t,X_t\right]$ $\forall \varepsilon_t < \varepsilon_t^*(1,2)$.

b) $\exists\, \varepsilon_t^*(2,1) : V_t\left[2,1,c_t,d_t,\bar{\theta}_t,\varepsilon_t,X_t\right] > V_t\left[1,1,c_t,d_t,\bar{\theta}_t,\varepsilon_t,X_t\right]$ $\forall \varepsilon_t > \varepsilon_t^*(2,1)$ and $V_t\left[2,1,c_t,d_t,\bar{\theta}_t,\varepsilon_t,X_t\right] < V_t\left[1,1,c_t,d_t,\bar{\theta}_t,\varepsilon_t,X_t\right]$ $\forall \varepsilon_t < \varepsilon_t^*(2,1)$.

c) $\exists\, \varepsilon_t^{**}(3,m_{t-1}) : V_t\left[3,m_{t-1},c_t,d_t,\bar{\theta}_t,\varepsilon_t,X_t\right] > \max_{m=1}^{2} V_t\left[m,m_{t-1},c_t,d_t,\bar{\theta}_t,\varepsilon_t,X_t\right]$ $\forall \varepsilon_t > \varepsilon_t^{**}(3,m_{t-1})$ and $V_t\left[3,m_{t-1},c_t,d_t,\bar{\theta}_t,\varepsilon_t,X_t\right] < \max_{m=1}^{2} V_t\left[m,m_{t-1},c_t,d_t,\bar{\theta}_t,\varepsilon_t,X_t\right]$ $\forall \varepsilon_t < \varepsilon_t^{**}(3,m_{t-1})$.

Proof. a) This follows from the fact that $\partial V_t[2,m_{t-1},c_t,d_t,\bar{\theta}_t,\varepsilon_t,X_t]/\partial\varepsilon_t$ is bounded from below and above and that $\partial V_t[1,m_{t-1},c_t,d_t,\bar{\theta}_t,\varepsilon_t,X_t]/\partial\varepsilon_t = 0$.

b) This follows from the same argument.

c) Consider

$$
\begin{aligned}
& V_{t^*}\left[3,m_{t^*-1},c_{t^*},d_{t^*},\bar{\theta}_{t^*},\varepsilon_{t^*},X_{t^*}\right] - V_{t^*}\left[2,m_{t^*-1},c_{t^*},d_{t^*},\bar{\theta}_{t^*},\varepsilon_{t^*},X_{t^*}\right] \\
= & \sum_{t=t^*}^{t^{**}} \beta^{t-t^*}\left(f_t'(3,c_t,d_t,X_t) - f_t'(2,c_t,d_t,X_t)\right) > 0
\end{aligned}
$$

by assumption. There exists a $\varepsilon_{t^*}^{**}(3,m_{t^*-1})$ such that $V_{t^*}\left[3,m_{t^*-1},c_{t^*},d_{t^*},\bar{\theta}_{t^*},\varepsilon_{t^*},X_{t^*}\right] >$ $V_{t^*}\left[1,m_{t^*-1},c_{t^*},d_{t^*},\bar{\theta}_{t^*},\varepsilon_{t^*},X_{t^*}\right]$ $\forall \varepsilon_{t^*} > \varepsilon_{t^*}^{**}(3,m_{t^*-1})$ and $V_{t^*}\left[3,m_{t^*-1},c_{t^*},d_{t^*},\bar{\theta}_{t^*},\varepsilon_{t^*},X_{t^*}\right] <$ $V_{t^*}\left[1,m_{t^*-1},c_{t^*},d_{t^*},\bar{\theta}_{t^*},\varepsilon_{t^*},X_{t^*}\right]$ $\forall \varepsilon_{t^*} < \varepsilon_{t^*}^{**}(3,m_{t^*-1})$ by the previous arguments about bounded slopes.

Now assume that there is some t' and $\varepsilon_{t'+1}^{**}(3,m_{t'})$ for $m_{t'} = 1,2$ such that $V_{t'+1}\left[3,2,c_{t'+1},d_{t'+1},\bar{\theta}_{t'+1},\varepsilon_{t'+1},X_{t'+1}\right] > \max_{m=1}^{2} V_{t'+1}\left[m,2,c_{t'+1},d_{t'+1},\bar{\theta}_{t'+1},\varepsilon_{t'+1},X_{t'+1}\right]$ $\forall \varepsilon_{t'+1} > \varepsilon_{t'+1}^{**}(3,2)$ and $V_{t'+1}\left[3,2,c_{t'+1},d_{t'+1},\bar{\theta}_{t'+1},\varepsilon_{t'+1},X_{t'+1}\right] < \max_{m=1}^{2} V_{t'+1}\left[m,2,c_{t'+1},d_{t'+1},\bar{\theta}_{t'+1},\right.$ $\forall \varepsilon_{t'+1} < \varepsilon_{t'+1}^{**}(3,2)$. Then

$$
V_{t'}\left[3,m_{t'-1},c_{t'},d_{t'},\bar{\theta}_{t'},\varepsilon_{t'},X_{t'}\right] - V_{t'}\left[2,m_{t'-1},c_{t'},d_{t'},\bar{\theta}_{t'},\varepsilon_{t'},X_{t'}\right] =
$$

$$
f_{t'}(3,c_{t'},d_{t'},X_{t'}) - f_{t'}(2,c_{t'},d_{t'},X_{t'}) + \beta\int_{-\infty}^{\varepsilon_{t'+1}^{**}(3,2)} [\max_{m_{t'+1}\in F(3)}\left(V_{t'+1}\left[m_{t'+1},3,c_{t'+1},d_{t'+1},\bar{\theta}_{t'+1},\varepsilon_{t'+1},X_{t'+1}\right.\right.
$$

$$
\max_{m_{t'+1}\in F(2)}\left(V_{t'+1}\left[m_{t'+1},2,c_{t'+1},d_{t'+1},\bar{\theta}_{t'+1},\varepsilon_{t'+1},X_{t'+1}\right]\right) \mid \bar{\theta}_{t'},\varepsilon_{t'},c_{t'}]\phi\left(\varepsilon_{t'+1} \mid \bar{\theta}_{t'},\varepsilon_{t'}\right)d\varepsilon_{t'+1} + \qquad (3.11)
$$

$$\beta \int_{\varepsilon^{**}_{t'+1}(3,2)}^{\infty} [\max_{m_{t'+1} \in F(3)} \left(V_{t'+1} \left[m_{t'+1}, 3, c_{t'+1}, d_{t'+1}, \bar{\theta}_{t'+1}, \varepsilon_{t'+1}, X_{t'+1} \right] \right) -$$

$$\max_{m_{t+1} \in F(2)} \left(V_{t'+1} \left[m_{t'+1}, 2, c_{t'+1}, d_{t'+1}, \bar{\theta}_{t'+1}, \varepsilon_{t'+1}, X_{t'+1} \right] \right) \mid \bar{\theta}_{t'}, \varepsilon_{t'}, c_{t'}] \phi \left(\varepsilon_{t'+1} \mid \bar{\theta}_{t'}, \varepsilon_{t'} \right) d\varepsilon_{t'+1}.$$

Note that $\varepsilon^{**}_{t'+1}(3,2) \geq \varepsilon^{*}_{t'+1}(1,2)$ because $\partial V_{t'+1} \left[m_{t'+1}, 2, c_{t'+1}, d_{t'+1}, \bar{\theta}_{t'+1}, \varepsilon_{t'+1}, X_{t'+1} \right] / \partial \varepsilon_{t'+1}$ ≥ 1. Therefore, for any $\varepsilon_{t'+1} > \varepsilon^{**}_{t'+1}(3,2)$, the agent will choose marriage whether $m_{t'-1}$ was 2 or 3. Therefore, the last integral in equation (3.11) is equal to zero. The integrand of the first integral is bounded from above because $V_{t'+1} \left[m_{t'+1}, 3, c_{t'+1}, d_{t'+1}, 0, 0, X_{t'+1} \right]$ is finite, the partial derivatives of $V_{t'+1} \left[m_{t'+1}, 3, c_{t'+1}, d_{t'+1}, \bar{\theta}_{t'+1}, \varepsilon \right.$ with respect to $\bar{\theta}_{t'+1}$ and $\varepsilon_{t'+1}$ are bounded from below and above and $\phi \left(\varepsilon_{t'+1} \mid \bar{\theta}_{t'}, \varepsilon_{t'} \right)$ is declining faster than exponentially as $\varepsilon_{t'+1} \to -\infty$. Let the bound be Γ. Then the first integral can be written as

$$\beta \Gamma \Phi \left(\varepsilon^{**}_{t'+1}(3,2) \mid \bar{\theta}_{t'}, \varepsilon_{t'} \right) \to 0 \text{ as } \varepsilon_{t'} \to \infty.$$

Thus, since the first term in equation (3.11) is positive, there exists a $\varepsilon_{t'}$ where $V_{t'} \left[3, m_{t'-1}, c_{t'}, d_{t'}, \bar{\theta}_{t'}, \varepsilon, X_{t'} \right] - V_{t'} \left[2, m_{t'-1}, c_{t'}, d_{t'}, \bar{\theta}_{t'}, \varepsilon, X_{t'} \right] > 0$ for all $\varepsilon > \varepsilon_{t'}$. But there is an $\varepsilon_{t'}$ small enough such that $V_{t'} \left[3, m_{t'-1}, c_{t'}, d_{t'}, \bar{\theta}_{t'}, \varepsilon, X_{t'} \right] - V_{t'} \left[2, m_{t'-1}, c_{t'}, d_{t'}, \bar{\theta}_{t'}, \varepsilon, X_{t'} \right]$ 0 for all $\varepsilon < \varepsilon_{t'}$ because of the bounded partial derivative of $V_{t'} \left[3, m_{t'-1}, c_{t'}, d_{t'}, \bar{\theta}_{t'}, \varepsilon, X_{t'} \right]$. Thus, because $\partial V_{t'} \left[3, m_{t'-1}, c_{t'}, d_{t'}, \bar{\theta}_{t'}, \varepsilon, X_{t'} \right] / \partial \varepsilon > \partial V_{t'} \left[2, m_{t'-1}, c_{t'}, d_{t'}, \bar{\theta}_{t'}, \varepsilon, X_{t'} \right] / \partial \varepsilon$, there exists a $\varepsilon^{**}_{t'}(3, m_{t'-1})$ for $m_{t'-1} = 1, 2$. ■

Theorem 9. *Let $m_{t-1} < 3$. There exists $D_3 > D_2$ and $f_t(3, c_t, d_t, X_t) > f_t(2, c_t, d_t, X_t)$ where there exist $\underline{\varepsilon}^{**}_t(2, m_{t-1}) < \overline{\varepsilon}^{**}_t(2, m_{t-1})$ such that*

$$V_t \left[2, m_{t-1}, c_t, d_t, \bar{\theta}_t, \varepsilon_t, X_t \right] > \max_{m=1,3} V_t \left[m, m_{t-1}, c_t, d_t, \bar{\theta}_t, \varepsilon_t, X_t \right]$$

for all $\underline{\varepsilon}^{**}_t(2, m_{t-1}) < \varepsilon_t < \overline{\varepsilon}^{**}_t(2, m_{t-1})$.

Proof. From a previous, theorem, we know that if $f_t(3, c_t, d_t, X_t) = f_t(2, c_t, d_t, X_t)$, then $V_t \left[2, m_{t-1}, c_t, d_t, \bar{\theta}_t, \varepsilon_t, X_t \right] > V_t \left[3, m_{t-1}, c_t, d_t, \bar{\theta}_t, \varepsilon_t, X_t \right]$ and that there exists $\underline{\varepsilon}^{**}_t(2, m_{t-1})$ such that $V_t \left[2, m_{t-1}, c_t, d_t, \bar{\theta}_t, \varepsilon_t, X_t \right] > V_t \left[1, m_{t-1}, c_t, d_t, \bar{\theta}_t, \varepsilon_t, X_t \right]$ for all $\varepsilon_t > \underline{\varepsilon}^{**}_t(2, m_{t-1})$. We also know that if $f_t(3, c_t, d_t, X_t) > f_t(2, c_t, d_t, X_t)$, there is some $\overline{\varepsilon}^{**}_t(2, m_{t-1})$ such that $V_t \left[2, m_{t-1}, c_t, d_t, \bar{\theta}_t, \varepsilon_t, X_t \right] < V_t \left[3, m_{t-1}, c_t, d_t, \bar{\theta}_t, \varepsilon_t, X_t \right]$ for all $\varepsilon_t > \overline{\varepsilon}^{**}_t(2, m_{t-1})$.

The arguments in part (b) of the previous proof imply that $\varepsilon_t^{**}(3, m_{t-1})$ is a continuous function of $\Delta f_t(c_t, d_t, X_t) = f_t(3, c_t, d_t, X_t) - f_t(2, c_t, d_t, X_t)$, and we already showed that $\varepsilon_t^{**}(3, m_{t-1}) = \infty$ when $\Delta f_t(c_t, d_t, X_t) = 0$. Thus, for any $D_3 - D_2 > 0$ and any $\varepsilon_t^{**}(3, m_{t-1})$, there will be a positive $\Delta f_t(c_t, d_t, X_t)$ small enough. Set $\overline{\varepsilon}_t^{**}(2, m_{t-1}) = \varepsilon_t^{**}(3, m_{t-1})$ and $\underline{\varepsilon}_t^{**}(2, m_{t-1})$ equal to the value of ε_t where $V_t\left[2, m_{t-1}, c_t, d_t, \bar{\theta}_t, \varepsilon_t, X_t\right] = V_t\left[1, m_{t-1}, c_t, d_t, \bar{\theta}_t, \varepsilon_t, X_t\right]$. ■

Lemma 10. *Let* $(\varsigma_1, \varsigma_2, \varsigma_3)' \sim N[0, \Omega]$ *where* $\Omega_{jk} > 0 \ \forall \ j, k$. *Then*

$$\Pr[\varsigma_3 < \alpha_{31} \mid \alpha_{11} < \varsigma_1 < \alpha_{12}, \alpha_{22} < \varsigma_2] > \Pr[\varsigma_3 < \alpha_{31} \mid \alpha_{12} < \varsigma_1, \alpha_{22} < \varsigma_2]$$

and

$$\Pr[\varsigma_3 < \alpha_{31} \mid \alpha_{11} < \varsigma_1 < \alpha_{12}] > \Pr[\varsigma_3 < \alpha_{31} \mid \alpha_{12} < \varsigma_1]$$

Proof.

$$\begin{aligned}
&\Pr[\varsigma_3 < \alpha_{31} \mid \varsigma_1, \alpha_{22} < \varsigma_2] \\
= \;& \frac{\int_{\alpha_{22}}^{\infty}\int_{-\infty}^{\alpha_{31}} \phi_{3|12}(\varsigma_3 \mid \varsigma_1, \varsigma_2)\, \phi_{2|1}(\varsigma_2 \mid \varsigma_1)\, \phi_1(\varsigma_1)\, d\varsigma_3 d\varsigma_2}{\int_{\alpha_{22}}^{\infty}\int_{-\infty}^{\infty} \phi_{3|12}(\varsigma_3 \mid \varsigma_1, \varsigma_2)\, \phi_{2|1}(\varsigma_2 \mid \varsigma_1)\, \phi_1(\varsigma_1)\, d\varsigma_3 d\varsigma_2} \\
= \;& \frac{\int_{\alpha_{22}}^{\infty} \Phi_{3|12}(\alpha_{31} \mid \varsigma_1, \varsigma_2)\, \phi_{2|1}(\varsigma_2 \mid \varsigma_1)\, d\varsigma_2}{\int_{\alpha_{22}}^{\infty} \phi_{2|1}(\varsigma_2 \mid \varsigma_1)\, d\varsigma_2}
\end{aligned}$$

and

$$\begin{aligned}
&\frac{\partial \Pr[\varsigma_3 < \alpha_{31} \mid \varsigma_1, \alpha_{22} < \varsigma_2]}{\partial \varsigma_1} \\
= \;& \int_{\alpha_{22}}^{\infty} \left[\frac{\phi_{2|1}(\varsigma_2 \mid \varsigma_1)}{\int_{\alpha_{22}}^{\infty} \phi_{2|1}(\varsigma_2 \mid \varsigma_1)\, d\varsigma_2} \frac{\partial}{\partial \varsigma_1} \Phi_{3|12}(\alpha_{31} \mid \varsigma_1, \varsigma_2) + \right. \\
& \left. \Phi_{3|12}(\alpha_{31} \mid \varsigma_1, \varsigma_2) \frac{\partial}{\partial \varsigma_1} \frac{\phi_{2|1}(\varsigma_2 \mid \varsigma_1)}{\int_{\alpha_{22}}^{\infty} \phi_{2|1}(\varsigma_2 \mid \varsigma_1)\, d\varsigma_2} \right] d\varsigma_2.
\end{aligned}$$

We know that

$$\frac{\partial}{\partial \varsigma_1} \Phi_{3|12}(\alpha_{31} \mid \varsigma_1, \varsigma_2) < 0,$$

so

$$\int_{\alpha_{22}}^{\infty} \frac{\phi_{2|1}(\varsigma_2 \mid \varsigma_1)}{\int_{\alpha_{22}}^{\infty} \phi_{2|1}(\varsigma_2 \mid \varsigma_1)\, d\varsigma_2} \frac{\partial}{\partial \varsigma_1} \Phi_{3|12}(\alpha_{31} \mid \varsigma_1, \varsigma_2)\, d\varsigma_2 < 0.$$

We can write the second term as[4]

$$\begin{aligned}
&\int_{\alpha_{22}}^{\infty} \Phi_{3|12}\left(\alpha_{31} \mid \varsigma_1, \varsigma_2\right) \frac{\partial}{\partial \varsigma_1} \frac{\phi_{2|1}\left(\varsigma_2 \mid \varsigma_1\right)}{\int_{\alpha_{22}}^{\infty} \phi_{2|1}\left(\varsigma_2 \mid \varsigma_1\right) d\varsigma_2} d\varsigma_2 \\
= \; & -\Phi_{3|12}\left(\alpha_{31} \mid \varsigma_1, \varsigma_2\right) \frac{\partial}{\partial \varsigma_1} \frac{\left[1-\Phi_{2|1}\left(\varsigma_2 \mid \varsigma_1\right)\right]}{\left[1-\Phi_{2|1}\left(\alpha_{22} \mid \varsigma_1\right)\right]} \Big|_{\alpha_{22}}^{\infty} \\
& + \int_{\alpha_{22}}^{\infty} \frac{\partial}{\partial \varsigma_1} \frac{\left[1-\Phi_{2|1}\left(\varsigma_2 \mid \varsigma_1\right)\right]}{\left[1-\Phi_{2|1}\left(\alpha_{22} \mid \varsigma_1\right)\right]} \frac{\partial}{\partial \varsigma_2} \Phi_{3|12}\left(\alpha_{31} \mid \varsigma_1, \varsigma_2\right) d\varsigma_2 \\
= \; & \int_{\alpha_{22}}^{\infty} \frac{\partial}{\partial \varsigma_1} \frac{\left[1-\Phi_{2|1}\left(\varsigma_2 \mid \varsigma_1\right)\right]}{\left[1-\Phi_{2|1}\left(\alpha_{22} \mid \varsigma_1\right)\right]} \frac{\partial}{\partial \varsigma_2} \Phi_{3|12}\left(\alpha_{31} \mid \varsigma_1, \varsigma_2\right) d\varsigma_2 .
\end{aligned}$$

We know that

$$\frac{\partial}{\partial \varsigma_2} \Phi_{3|12}\left(\alpha_{31} \mid \varsigma_1, \varsigma_2\right) < 0$$

and

$$\frac{\partial}{\partial \varsigma_1} \frac{\left[1-\Phi_{2|1}\left(\varsigma_2 \mid \varsigma_1\right)\right]}{\left[1-\Phi_{2|1}\left(\alpha_{22} \mid \varsigma_1\right)\right]} > 0.$$

So

$$\frac{\partial \Pr\left[\varsigma_3 < \alpha_{31} \mid \varsigma_1, \alpha_{22} < \varsigma_2\right]}{\partial \varsigma_1} < 0.$$

Now define

$$H\left(\varsigma_1\right) = \Pr\left[\varsigma_3 < \alpha_{31} \mid \varsigma_1, \alpha_{22} < \varsigma_2\right]$$

as a function of ς_1, and write $\Pr\left[\varsigma_3 < \alpha_{31} \mid \alpha_{11} < \varsigma_1 < \alpha_{12}, \alpha_{22} < \varsigma_2\right]$ as

$$\int H\left(\varsigma_1\right) g^*\left(\varsigma_1\right) d\varsigma_1$$

where

$$g^*\left(\varsigma_1\right) = \frac{\phi\left(\varsigma_1\right)}{\Phi\left(\alpha_{12}\right) - \Phi\left(\alpha_{11}\right)} 1\left[\alpha_{11} < \varsigma_1 < \alpha_{12}\right]$$

4

$$u = \Phi_{3|12}\left(\alpha_{31} \mid \varepsilon_1, \varepsilon_2\right) \Rightarrow du = \frac{\partial}{\partial \varepsilon_2} \Phi_{3|12}\left(\alpha_{31} \mid \varepsilon_1, \varepsilon_2\right) d\varepsilon_2$$

$$dv = \frac{\partial}{\partial \varepsilon_1} \frac{\phi_{2|1}\left(\varepsilon_2 \mid \varepsilon_1\right)}{\int_{\alpha_{22}}^{\infty} \phi_{2|1}\left(\varepsilon_2 \mid \varepsilon_1\right) d\varepsilon_2} d\varepsilon_2 \Rightarrow v = \frac{\partial}{\partial \varepsilon_1} \frac{-\left[1-\Phi_{2|1}\left(\varepsilon_2 \mid \varepsilon_1\right)\right]}{\left[1-\Phi_{2|1}\left(\alpha_{22} \mid \varepsilon_1\right)\right]}$$

and $\Pr\left[\varsigma_3 < \alpha_{31} \mid \alpha_{12} < \varsigma_1, \alpha_{22} < \varsigma_2\right]$ as

$$\int H(\varsigma_1)\, g^{**}(\varsigma_1)\, d\varsigma_1$$

where

$$g^{**}(\varsigma_1) = \frac{\phi(\varsigma_1)}{1 - \Phi(\alpha_{12})} 1\left[\alpha_{12} < \varsigma_1\right].$$

It is clear that $G^*(\varsigma_1) > G^{**}(\varsigma_1)$, and we have already shown that $H'(\varsigma_1) < 0$. Thus, the result follows by properties of stochastic dominance. The second condition follows by setting $\alpha_{22} = -\infty$.

Definition 1. *Let* $\tilde{P}_t(k, \tau) = \Pr[m_t = 1 \mid m_s = 3 \; \forall \; t-1 \geq s \geq \tau + k, m_s = 2 \forall \tau + k > s \geq \tau, m_{\tau-1} = 1]$.

Theorem 11. $\tilde{P}_t(1, \tau) > \tilde{P}_t(0, \tau)$.

Proof. Using the notation from the above lemma, let $\varsigma_{3-s} = \varepsilon_{t-s}, s = 0, 1, 2$. Let $\alpha_{31} = \varepsilon_t^*(1, 3)$, $\alpha_{11} = \varepsilon_{t-2}^*(2, 1)$, $\alpha_{12} = \overline{\varepsilon}_{t-2}^{**}(2, 1)$, $\alpha_{22} = \overline{\varepsilon}_{t-1}^{**}(2, 1)$. The condition that all the covariances are positive is satisfied because the two ways in which early errors affect later errors is through positive serial correlation and through updates of θ. Thus the result follows. ■

Theorem 12. $\tilde{P}_t(k, \tau) > \tilde{P}_t(k-1, \tau) \; \forall \; k > 1$.

Proof. Using the notation from the above lemma, let $\varsigma_{3-s} = \varepsilon_{t-s}, s = k-1, k, k+1$. Let $\alpha_{31} = \varepsilon_t^*(1, 3)$, $\alpha_{11} = \varepsilon_{t-2}^*(2, 1)$, $\alpha_{12} = \overline{\varepsilon}_{t-2}^{**}(2, 1)$, $\alpha_{22} = \overline{\varepsilon}_{t-1}^{**}(2, 1)$. The condition that all the covariances are positive is satisfied because the two ways in which early errors affect later errors is through positive serial correlation and through updates of θ. Thus the result follows. ■

Theorem 13. $\tilde{P}_t(k, \tau) > \tilde{P}_t(l, \tau) \; \forall \; k > l$.

Proof. The result follows from the previous theorem.

Assume $f_t(m_t, c_t, d_t + 1, X_t) \geq f_t(m_t, c_t, d_t, X_t)$ for all t, m_t, c_t, d_t, and X_t. Then $V_t\left[m_t, m_t, c_t, d_t, \bar{\theta}_t, \varepsilon_t, X_t\right] \geq V_t\left[m_t, m_{t-1}, c_t, d_t, \bar{\theta}_t, \varepsilon_t, X_t\right]$ for all m_t and m_{t-1}.

Proof. When $m_t = 1$, $m_{t-1} \neq 1$,

$$V_t\left[1, 1, c_t, d_t, \bar{\theta}_t, \varepsilon_t, X_t\right] - V_t\left[1, m_{t-1}, c_t, d_t, \bar{\theta}_t, \varepsilon_t, X_t\right] = -D_{m_{t-1}} > 0.$$

When $m_t = 2$, $m_{t-1} = 1$ $(2 \notin F(3))$,

$V_t\left[2, 2, c_t, d_t, \bar{\theta}_t, \varepsilon_t, X_t\right] - V_t\left[2, 1, c_t, d_t, \bar{\theta}_t, \varepsilon_t, X_t\right] = f_t(2, c_t, d_t, X_t) - f_t(2, c_t, 1, X_t) \geq 0.$

When $m_t = 3$, $m_{t-1} = 1$,

$V_t\left[3, 3, c_t, d_t, \bar{\theta}_t, \varepsilon_t, X_t\right] - V_t\left[3, 1, c_t, d_t, \bar{\theta}_t, \varepsilon_t, X_t\right] = f_t(3, c_t, d_t, X_t) - f_t(3, c_t, 1, X_t) \geq 0.$

When $m_t = 3$, $m_{t-1} = 2$,

$V_t\left[3, 3, c_t, d_t, \bar{\theta}_t, \varepsilon_t, X_t\right] - V_t\left[3, 2, c_t, d_t, \bar{\theta}_t, \varepsilon_t, X_t\right] = f_t(3, c_t, d_t, X_t) - f_t(3, c_t, d_t, X_t) = 0.$

■

Definition 2. *Let*

$$\begin{aligned} P_t[m_{t+1} \mid m_t, c_t, d_t] &= \Pr[m_{t+1} \mid t, m_t, c_t, d_t] \\ &= \frac{\int\int \Pr\left[m_{t+1} \mid t, m_t, c_t, d_t, \bar{\theta}_t, \varepsilon_t\right] h_t\left(\bar{\theta}_t, \varepsilon_t \mid m_t, c_t, d_t\right) d\bar{\theta}_t d\varepsilon_t}{\int\int h_t\left(\bar{\theta}_t, \varepsilon_t \mid m_t, c_t, d_t\right) d\bar{\theta}_t d\varepsilon_t} \end{aligned}$$

where $h_t\left(\bar{\theta}_t, \varepsilon_t \mid m_t, c_t, d_t\right)$ is the joint conditional density of $\bar{\theta}_t$ and ε_t.

Theorem 14. $P_t[1 \mid 2, c_t, d_t] \geq P_t[1 \mid 3, c_t, d_t]$.

Proof. This follows directly from the lemma above and the result that $\varepsilon_t^*(1, 3) < \varepsilon_t^*(1, 2)$.

Conjecture 15. $\partial P_t[1 \mid 3, c_t, d_t] / \partial D_3 < 0$

Proof.

$P_{t^*-1}[1 \mid 3, c_{t^*-1}, d_{t^*-1}] = \Pr\left\{V_{t^*}\left[1, 3, c_{t^*}, 1, \bar{\theta}_{t^*}, \varepsilon_{t^*}, X_{t^*}\right] > V_{t^*}\left[3, 3, c_{t^*}, d_{t^*}, \bar{\theta}_{t^*}, \varepsilon_{t^*}, X_{t^*}\right] \mid \bar{\theta}_{t^*-1}, \varepsilon_{t^*-1}\right\}$

$$\frac{\partial V_{t^*}\left[1, 3, c_{t^*}, 1, \bar{\theta}_{t^*}, \varepsilon_{t^*}, X_{t^*}\right]}{\partial D_3} = -1$$

and

$$\frac{\partial V_{t^*}\left[3, 3, c_{t^*}, d_{t^*}, \bar{\theta}_{t^*}, \varepsilon_{t^*}, X_{t^*}\right]}{\partial D_3} = 0.$$

So $\partial P_t\left[1 \mid 3, c_t, d_t\right] / \partial D_3 < 0$ at $t^* - 1$. At $t^* - 2$,

$$\frac{\partial V_{t^*-1}\left[1, 3, c_{t^*-1}, 1, \bar{\theta}_{t^*-1}, \varepsilon_{t^*-1}, X_{t^*-1}\right]}{\partial D_3} = -1$$

and

$$\frac{\partial V_{t^*-1}\left[3, 3, c_{t^*-1}, d_{t^*-1}, \bar{\theta}_{t^*-1}, \varepsilon_{t^*-1}, X_{t^*-1}\right]}{\partial D_3} = -\beta P_{t^*-1}\left[1 \mid 3, c_{t^*-1}, d_{t^*-1}\right] > -1.$$

So $\partial P_t\left[1 \mid 3, c_t, d_t\right] / \partial D_3 < 0$ at $t^* - 2$.

At $t^* - 3$,

$$\frac{\partial V_{t^*-2}\left[1, 3, c_{t^*-2}, 1, \bar{\theta}_{t^*-2}, \varepsilon_{t^*-2}, X_{t^*-2}\right]}{\partial D_3} = -1 - \beta^2 P_{t^*-2}\left[3 \mid 1, c_{t^*-2}, 1\right] P_{t^*-1}\left[1 \mid 3, c_{t^*-1}, 1\right]$$

and

$$\begin{aligned}
& \frac{\partial V_{t^*-2}\left[3, 3, c_{t^*-2}, d_{t^*-2}, \bar{\theta}_{t^*-2}, \varepsilon_{t^*-2}, X_{t^*-2}\right]}{\partial D_3} \\
= \;& -\beta P_{t^*-2}\left[1 \mid 3, c_{t^*-2}, d_{t^*-2}\right] - \beta^2 P_{t^*-2}\left[3 \mid 3, c_{t^*-2}, d_{t^*-2}\right] P_{t^*-1}\left[1 \mid 3, c_{t^*-1}, d_{t^*-1}\right] \\
= \;& -\beta\left\{P_{t^*-2}\left[1 \mid 3, c_{t^*-2}, d_{t^*-2}\right] + \beta P_{t^*-2}\left[3 \mid 3, c_{t^*-2}, d_{t^*-2}\right] P_{t^*-1}\left[1 \mid 3, c_{t^*-1}, d_{t^*-1}\right]\right\} \\
> \;& -\beta.
\end{aligned}$$

Therefore,

$$\frac{\partial V_{t^*-2}\left[1, 3, c_{t^*-2}, 1, \bar{\theta}_{t^*-2}, \varepsilon_{t^*-2}, X_{t^*-2}\right]}{\partial D_3} - \frac{\partial V_{t^*-2}\left[3, 3, c_{t^*-2}, d_{t^*-2}, \bar{\theta}_{t^*-2}, \varepsilon_{t^*-2}, X_{t^*-2}\right]}{\partial D_3} < 0.$$

So $\partial P_t\left[1 \mid 3, c_t, d_t\right] / \partial D_3 < 0$ at $t^* - 3$.

At t, let $P_1^*(t+s, t)$ be the probability that a first divorce occurs at $t+s$ conditional on being married at t:

$$P_1^*(t+s, t) = P_{t+s}\left[1 \mid 3, c_{t+s}, d_{t+s}\right] \Pi_{r=0}^{s-1} P_{t+r}\left[3 \mid 3, c_{t+s}, d_{t+s}\right],$$

and let $P_k^{**}(t+s, t)$ be the probability that the *kth* remarriage occurs at $t+s$ conditional on being single at t. Then

$$\frac{\partial V_t\left[1, 3, c_t, 1, \bar{\theta}_t, \varepsilon_t, X_t\right]}{\partial D_3} - \frac{\partial V_t\left[3, 3, c_t, d_t, \bar{\theta}_t, \varepsilon_t, X_t\right]}{\partial D_3}$$

$$
\begin{aligned}
= \quad & -1+\left\{\sum_{s=0}^{\infty} \beta^{s+1} P_1^*(t+s, t)\right\} \\
& -\sum_{k=1}^{\infty}\{\sum_{j=2+k}^{\infty} \beta^j \sum_{s=1}^{j-1} P_k^{**}(t+s, t) P_1^*(j+t, t+s)- \\
& \sum_{s=0}^{\infty} \beta^{s+j+1} P_1^*(t+s, t) \sum_{r=1}^{j-1} P_k^{**}(t+s+r, t+s) P_1^*(j+t+s, t+s+r)\}.
\end{aligned}
$$

So we need to be able to say something about

$$
\sum_{r=1}^{j-1} P_k^{**}(t+r, t) P_1^*(j+t, t+r)-\beta^{s+1} P_1^*(t+s, t) \sum_{r=1}^{j-1} P_k^{**}(t+s+r, t+s) P_1^*(j+t+s, t+s+r)
$$

for all s and j.

Theorem 16. *Assume $f_t(m_t, c_t, d_t, X_t)$ does not depend upon t, c_t, d_t, or X_t. Let $f_m = f_t(m, c_t, d_t, X_t)$. Then a) $\partial P_t[1 \mid 3, c_t, d_t] / \partial D_3 < 0$ and b) for $t^* - t$ large enough, $\partial P_t[m \mid m_{t-1}, c_t, d_t] / \partial f_m \geq 0$.*

Proof. a) Let $P_{31}^*(t+s, t)$ be the probability that a first divorce occurs at $t+s$ conditional on being married at t:

$$
P_{31}^*(t+s, t)=P_{t+s}[1 \mid 3, c_{t+s}, d_{t+s}] \Pi_{r=0}^{s-1} P_{t+r}[3 \mid 3, c_{t+s}, d_{t+s}],
$$

and let $P_k^{**}(t+s, t)$ be the probability that the kth remarriage occurs at $t+s$ conditional on being single at t. Then

$$
\begin{aligned}
& \frac{\partial V_t\left[1,3, c_t, 1, \bar{\theta}_t, \varepsilon_t, X_t\right]}{\partial D_3}-\frac{\partial V_t\left[3,3, c_t, d_t, \bar{\theta}_t, \varepsilon_t, X_t\right]}{\partial D_3} \qquad (3.12) \\
= \quad & -1+\left\{\sum_{s=0}^{\infty} \beta^{s+1} P_{31}^*(t+s, t)\right\} \\
& -\sum_{k=1}^{\infty}\{\sum_{j=2+k}^{\infty} \beta^j \sum_{r=1}^{j-1} P_k^{**}(t+r, t) P_{31}^*(j+t, t+r)- \\
& \sum_{s=0}^{\infty} \beta^{s+j+1} P_{31}^*(t+s, t) \sum_{r=1}^{j-1} P_k^{**}(t+s+r, t+s) P_{31}^*(j+t+s, t+s+r)\}.
\end{aligned}
$$

So we need to be able to say something about

$$
\sum_{r=1}^{j-1} P_k^{**}(t+r, t) P_{31}^*(j+t, t+r)-\sum_{s=0}^{\infty} \beta^{s+1} P_{31}^*(t+s, t) \sum_{r=1}^{j-1} P_k^{**}(t+s+r, t+s) P_1^*(j+t+s, t+s+r) \qquad (3.13)
$$

for all s and j. But given our assumption about no age and children effects, $P_k^{**}(t+r,t) = P_k^{**}(t+s+r,t+s)$ and $P_{31}^{*}(j+t,t+r) = P_{31}^{*}(j+t+s,t+s+r)$ for all s. Thus equation (3.13) becomes

$$\sum_{r=1}^{j-1} P_k^{**}(t+r,t)\, P_{31}^{*}(j+t,t+r)\left[1-\sum_{s=0}^{\infty}\beta^{s+1}P_{31}^{*}(t+s,t)\right] < 1-\sum_{s=0}^{\infty}\beta^{s+1}P_{31}^{*}(t+s,t).$$

Thus, equation (3.12) is less than

$$\begin{aligned}
& -1+\left\{\sum_{s=0}^{\infty}\beta^{s+1}P_{31}^{*}(t+s,t)\right\}-\sum_{k=1}^{\infty}\{\sum_{j=2+k}^{\infty}\beta^{j}\sum_{s=1}^{j-1}\left\{1-\sum_{s=0}^{\infty}\beta^{s+1}P_{31}^{*}(t+s,t)\right\} \\
= & -\left\{1-\sum_{s=0}^{\infty}\beta^{s+1}P_{31}^{*}(t+s,t)\right\}\left\{1+\sum_{k=1}^{\infty}\sum_{j=2+k}^{\infty}\beta^{j}\right\}<0.
\end{aligned}$$

b) If $m \notin F(m_{t-1})$, then $P_t[m \mid m_{t-1}, c_t, d_t] = 0$ independent of f_m, so $\partial P_t[m \mid m_{t-1}, c_t, d_t]/\partial f_m = 0$. Consider the case where $m \in F(m_{t-1})$, and let $m' \in F(m_{t-1})$. Let $P_m^{***}(t+s,t)$ be the probability that one is in state m at time $t+s$ conditional on having been in state m at time t (possibly with transitions in between), and $P_{m'm}^{*}(t+s,t)$ be the probability that the first transition into state m occurs at time $t+s$ conditional on being in state m' at time t. Then

$$\begin{aligned}
& \frac{\partial V_t\left[m, m_{t-1}, c_t, 1, \bar{\theta}_t, \varepsilon_t, X_t\right]}{\partial f_m}-\frac{\partial V_t\left[m', m_{t-1}, c_t, d_t, \bar{\theta}_t, \varepsilon_t, X_t\right]}{\partial f_m} \qquad (3.14)\\
= & \left\{1+\sum_{s=1}^{t^*-1-t}\beta^{s}P_m^{***}(t+s,t)+P_m^{***}(t^*,t)\sum_{s=t^*}^{t^{**}}\beta^{s-t}\right\}- \\
& \sum_{r=1}^{t^*-1-t}P_{m'm}^{*}(t+r,t)\,\beta^{r}\left\{1+\sum_{s=1}^{t^*-1-(t+r)}\beta^{s}P_m^{***}(t+r+s,t+r)+P_m^{***}(t^*,t+r)\sum_{s=t^*}^{t^{**}}\beta^{s-t-r}\right\}.
\end{aligned}$$

Since $P_m^{***}(t+s,t) = P_m^{***}(t+r+s,t+r)$ for all r, equation (3.14) becomes

$$\begin{aligned}
& \left\{1+\sum_{s=1}^{t^*-1-t}\beta^{s}P_m^{***}(t+s,t)+P_m^{***}(t^*,t)\sum_{s=t^*}^{t^{**}}\beta^{s-t}\right\}- \\
& \sum_{r=1}^{t^*-1-t}P_{m'm}^{*}(t+r,t)\,\beta^{r}\left\{1+\sum_{s=1}^{t^*-1-(t+r)}\beta^{s}P_m^{***}(t+s,t)+P_m^{***}(t^*,t+r)\sum_{s=t^*}^{t^{**}}\beta^{s-t-r}\right\} \\
= & \left\{1+\sum_{s=1}^{t^*-1-t}\beta^{s}P_m^{***}(t+s,t)\right\}+P_m^{***}(t^*,t)\sum_{s=t^*}^{t^{**}}\beta^{s-t}-
\end{aligned}$$

$$\sum_{r=1}^{t^*-1-t} P^*_{m'm}(t+r,t)\,\beta^r \left\{1+\sum_{s=1}^{t^*-1-(t+r)} \beta^s P^{***}_m(t+s,t)\right\} -$$
$$\sum_{r=1}^{t^*-1-t} P^*_{m'm}(t+r,t)\,\beta^r P^{***}_m(t^*,t+r)\sum_{s=t^*}^{t^{**}}\beta^{s-t-r}$$
$$= \left\{1+\sum_{s=1}^{t^*-1-t}\beta^s P^{***}_m(t+s,t)\right\} + P^{***}_m(t^*,t)\sum_{s=t^*}^{t^{**}}\beta^{s-t} -$$
$$\sum_{r=1}^{t^*-1-t} P^*_{m'm}(t+r,t)\,\beta^r \left\{1+\sum_{s=1}^{t^*-1-t}\beta^s P^{***}_m(t+s,t)\right\} +$$
$$\sum_{r=1}^{t^*-1-t} P^*_{m'm}(t+r,t)\,\beta^r \sum_{s=t^*-(t+r)+1}^{t^*-1-t}\beta^s P^{***}_m(t+s,t+r) -$$
$$\sum_{r=1}^{t^*-1-t} P^*_{m'm}(t+r,t)\,\beta^r P^{***}_m(t^*,t+r)\sum_{s=t^*}^{t^{**}}\beta^{s-t-r}$$
$$= \left\{1+\sum_{s=1}^{t^*-1-t}\beta^s P^{***}_m(t+s,t)\right\}\left\{1-\sum_{r=1}^{t^*-1-t}P^*_{m'm}(t+r,t)\,\beta^r\right\} +$$
$$\left\{P^{***}_m(t^*,t)-\sum_{r=1}^{t^*-1-t}P^*_{m'm}(t+r,t)\,P^{***}_m(t^*,t+r)\right\}\sum_{s=t^*}^{t^{**}}\beta^{s-t} +$$
$$\sum_{r=1}^{t^*-1-t} P^*_{m'm}(t+r,t)\,\beta^r \left\{\sum_{s=t^*-(t+r)+1}^{t^*-1-t}\beta^s P^{***}_m(t+s,t+r)\right\}.$$

The first term is always positive because $1-\sum_{r=1}^{t^*-1-t}P^*_{m'm}(t+r,t)\,\beta^r > 1-\sum_{r=1}^{t^*-1-t}P^*_{m'm}(t+r,t) > 0$. The last term is positive. The middle term is proportional to the difference between $P^{***}_m(t^*,t)$ and the probability that the agent will be in state m at t^* conditional on being in state m' at t which can not be signed. But the proportionality constant, $\sum_{s=t^*}^{t^{**}}\beta^{s-t} = \beta^{t^*-t}\sum_{s=0}^{t^{**}-t^*}\beta^s \to 0$ uniformly as $t^*-t\to\infty$. Note: this qualification occurs only because $t^{**}-t^*>0$ which is the particular way we end lives in this model. ■

Conjecture 17. *Assume $f_t(m_t,c_t,d_t,X_t)$ does not depend upon t, c_t, d_t, or X_t. Then $\varepsilon^*_t(1,3)\geq\varepsilon^*_{t-1}(1,3)$.*

Proof. $\varepsilon^*_t(1,3)$ is the value of ε_t where $V_t\left[3,3,c_t,d_t,\bar\theta_t,\varepsilon_t,X_t\right]=V_t\left[1,3,c,d_t,\bar\theta_t,\varepsilon_t,X_t\right]$. Given the assumption that $f_t(m_t,c_t,d_t,X_t)$ does not depend upon t, c_t, d_t, or X_t, $V_t\left[1,3,c_t,d_t,\bar\theta_t,\varepsilon_t,X_t\right]$ does not depend on t or d_t. Thus, the change in $\varepsilon^*_t(1,3)$

with t can be measured by comparing $V_{t+1}\left[3,3,c,d_t+1,\bar{\theta},\varepsilon,X\right]-V_t\left[3,3,c,d_t,\bar{\theta},\varepsilon,X\right]$ for given values of c and X and for $\varepsilon=\varepsilon_t^*(1,3)$. First consider t^*-2. Since flows are the same,

$$\begin{aligned}
\Delta\left[\varepsilon_{t^*-2}^*(1,3)\right] &= V_{t^*-1}\left[3,3,c,d+1,\bar{\theta},\varepsilon_{t^*-1}^*(1,3),X\right]-V_{t^*-2}\left[3,3,c,d,\bar{\theta},\varepsilon_{t^*-2}^*(1,3),X\right] \\
&= \varepsilon_{t^*-1}^*(1,3)+\beta E\left[\sum_{t=t^*}^{t^{**}}\beta^{t-t^*}(f_3+\varepsilon_t)\mid\bar{\theta},\varepsilon_{t^*-1}=\varepsilon_{t^*-1}^*(1,3)\right]-\varepsilon_{t^*-2}^*(1,3)- \\
&\quad \beta E\left[f_3+\varepsilon_{t^*-1}+\beta\sum_{t=t^*}^{t^{**}}\beta^{t-t^*}(f_3+\varepsilon_t)\mid\bar{\theta},\varepsilon_{t^*-2}=\varepsilon_{t^*-2}^*(1,3),\varepsilon_{t^*-1}>\varepsilon_{t^*-1}^*(1,3)\right]\bullet \\
&\quad \Pr\left[\varepsilon_{t^*-1}>\varepsilon_{t^*-1}^*(1,3)\mid\bar{\theta},\varepsilon_{t^*-2}=\varepsilon_{t^*-2}^*(1,3)\right]- \\
&\quad \beta E\left[f_1+\beta\sum_{t=t^*}^{t^{**}}\beta^{t-t^*}f_1\right]\Pr\left[\varepsilon_{t^*-1}<\varepsilon_{t^*-1}^*(1,3)\mid\bar{\theta},\varepsilon_{t^*-2}=\varepsilon_{t^*-2}^*(1,3)\right],
\end{aligned}$$

and $\partial\Delta\left[\varepsilon_{t^*-2}^*(1,3)\right]/\partial\varepsilon_{t^*-2}^*(1,3)<0$. Evaluating $\Delta\left[\varepsilon_{t^*-2}^*(1,3)\right]$ at $\varepsilon_{t^*-1}^*(1,3)$ leads to

$$\begin{aligned}
\Delta\left[\varepsilon_{t^*-1}^*(1,3)\right] &= \beta E\left[\sum_{t=t^*}^{t^{**}}\beta^{t-t^*}(f_3+\varepsilon_t)\mid\bar{\theta},\varepsilon_{t^*-1}=\varepsilon_{t^*-1}^*(1,3)\right]- \\
&\quad \beta E\left[f_3+\varepsilon_{t^*-1}+\beta\sum_{t=t^*}^{t^{**}}\beta^{t-t^*}(f_3+\varepsilon_t)\mid\bar{\theta},\varepsilon_{t^*-2}=\varepsilon_{t^*-1}^*(1,3),\varepsilon_{t^*-1}>\varepsilon_{t^*-1}^*(1,3)\right]\bullet \\
&\quad \Pr\left[\varepsilon_{t^*-1}>\varepsilon_{t^*-1}^*(1,3)\mid\bar{\theta},\varepsilon_{t^*-2}=\varepsilon_{t^*-1}^*(1,3)\right]- \\
&\quad \beta E\left[f_1+\beta\sum_{t=t^*}^{t^{**}}\beta^{t-t^*}f_1\right]\Pr\left[\varepsilon_{t^*-1}<\varepsilon_{t^*-1}^*(1,3)\mid\bar{\theta},\varepsilon_{t^*-2}=\varepsilon_{t^*-1}^*(1,3)\right] \\
&= \beta E\left[\sum_{t=t^*}^{t^{**}}\beta^{t-t^*}(f_3+\varepsilon_t)\mid\bar{\theta},\varepsilon_{t^*-1}=\varepsilon_{t^*-1}^*(1,3)\right]- \\
&\quad \beta E\left[\sum_{t=t^*-1}^{t^{**}}\beta^{t-(t^*-1)}(f_3+\varepsilon_t)\mid\bar{\theta},\varepsilon_{t^*-2}=\varepsilon_{t^*-1}^*(1,3),\varepsilon_{t^*-1}>\varepsilon_{t^*-1}^*(1,3)\right]\bullet \\
&\quad \Pr\left[\varepsilon_{t^*-1}>\varepsilon_{t^*-1}^*(1,3)\mid\bar{\theta},\varepsilon_{t^*-2}=\varepsilon_{t^*-1}^*(1,3)\right]- \\
&\quad \beta E\left[\sum_{t=t^*-1}^{t^{**}}\beta^{t-(t^*-1)}f_1\right]\Pr\left[\varepsilon_{t^*-1}<\varepsilon_{t^*-1}^*(1,3)\mid\bar{\theta},\varepsilon_{t^*-2}=\varepsilon_{t^*-1}^*(1,3)\right] \\
&= \beta E\left[\sum_{t=t^*}^{t^{**}}\beta^{t-t^*}(f_3+\varepsilon_t)\mid\bar{\theta},\varepsilon_{t^*-1}=\varepsilon_{t^*-1}^*(1,3)\right]-
\end{aligned}$$

$$
\begin{aligned}
&\beta E\left[\sum_{t=t^*}^{t^{**}} \beta^{t-t^*}\left(f_3+\varepsilon_t\right) \mid \bar{\theta}, \varepsilon_{t^*-1}=\varepsilon_{t^*-1}^*(1,3), \varepsilon_{t^*}>\varepsilon_{t^*-1}^*(1,3)\right] \bullet \\
&\Pr\left[\varepsilon_{t^*}>\varepsilon_{t^*-1}^*(1,3) \mid \bar{\theta}, \varepsilon_{t^*-1}=\varepsilon_{t^*-1}^*(1,3)\right]- \\
&\beta E\left[\sum_{t=t^*}^{t^{**}} \beta^{t-t^*} f_1\right] \Pr\left[\varepsilon_{t^*}<\varepsilon_{t^*-1}^*(1,3) \mid \bar{\theta}, \varepsilon_{t^*-1}=\varepsilon_{t^*-1}^*(1,3)\right] \\
< \;& 0.
\end{aligned}
$$

Thus, $\varepsilon_{t^*-2}^*(1,3)<\varepsilon_{t^*-1}^*(1,3)$.

Now assume that there is a t such that $\varepsilon_{t'}^*(1,3)<\varepsilon_{t'+1}^*(1,3)$ for all $t' \geq t$. Then

$$
\begin{aligned}
\Delta\left[\varepsilon_{t-1}^*(1,3)\right] &= V_t\left[3,3,c,d+1,\bar{\theta},\varepsilon_t^*(1,3),X\right]-V_{t-1}\left[3,3,c,d,\bar{\theta},\varepsilon_{t-1}^*(1,3),X\right] \\
&= \varepsilon_t^*(1,3)+\beta E\left\{V_{t+1}\left[3,3,c,d+2,\bar{\theta}\left(d+2,\varepsilon_{t+1},\bar{\theta}\right),\varepsilon_{t+1},X\right] \mid \varepsilon_{t+1}>\varepsilon_{t+1}^*(1,3),\bar{\theta},\varepsilon\right. \\
&\quad \Pr\left[\varepsilon_{t+1}>\varepsilon_{t+1}^*(1,3) \mid \bar{\theta},\varepsilon_t=\varepsilon_t^*(1,3)\right]+ \\
&\quad \beta V_{t+1}\left[1,3,c,1,\bar{\theta},\varepsilon_{t+1},X\right] \Pr\left[\varepsilon_{t+1}<\varepsilon_{t+1}^*(1,3) \mid \bar{\theta},\varepsilon_t=\varepsilon_t^*(1,3)\right]- \\
&\quad \varepsilon_{t-1}^*(1,3)-\beta E\left\{V_t\left[3,3,c,d+1,\bar{\theta}\left(d+1,\varepsilon_t,\bar{\theta}\right),\varepsilon_t,X\right] \mid \varepsilon_t>\varepsilon_t^*(1,3),\bar{\theta},\varepsilon_{t-1}=\varepsilon_{t-}^*\right. \\
&\quad \Pr\left[\varepsilon_t>\varepsilon_t^*(1,3) \mid \bar{\theta},\varepsilon_{t-1}=\varepsilon_{t-1}^*(1,3)\right]- \\
&\quad \beta V_t\left[1,3,c,1,\bar{\theta},\varepsilon_t,X\right] \Pr\left[\varepsilon_t<\varepsilon_t^*(1,3) \mid \bar{\theta},\varepsilon_{t-1}=\varepsilon_{t-1}^*(1,3)\right],
\end{aligned}
$$

and $\partial\Delta\left[\varepsilon_{t-1}^*(1,3)\right]/\partial\varepsilon_{t-1}^*(1,3)<0$. Evaluating $\Delta\left[\varepsilon_{t-1}^*(1,3)\right]$ at $\varepsilon_t^*(1,3)$ leads to

$$
\begin{aligned}
&\beta E\left\{V_{t+1}\left[3,3,c,d+2,\bar{\theta}\left(d+2,\varepsilon_{t+1},\bar{\theta}\right),\varepsilon_{t+1},X\right] \mid \varepsilon_{t+1}>\varepsilon_{t+1}^*(1,3),\bar{\theta},\varepsilon_t=\varepsilon_t^*(1,3)\right\} \bullet \\
&\Pr\left[\varepsilon_{t+1}>\varepsilon_{t+1}^*(1,3) \mid \bar{\theta},\varepsilon_t=\varepsilon_t^*(1,3)\right]+ \\
&\beta V_{t+1}\left[1,3,c,1,\bar{\theta},\varepsilon_{t+1},X\right] \Pr\left[\varepsilon_{t+1}<\varepsilon_{t+1}^*(1,3) \mid \bar{\theta},\varepsilon_t=\varepsilon_t^*(1,3)\right]- \\
&\beta E\left\{V_t\left[3,3,c,d+1,\bar{\theta}\left(d+1,\varepsilon_t,\bar{\theta}\right),\varepsilon_t,X\right] \mid \varepsilon_t>\varepsilon_t^*(1,3),\bar{\theta},\varepsilon_{t-1}=\varepsilon_t^*(1,3)\right\} \bullet \\
&\Pr\left[\varepsilon_t>\varepsilon_t^*(1,3) \mid \bar{\theta},\varepsilon_{t-1}=\varepsilon_t^*(1,3)\right]- \\
&\beta V_t\left[1,3,c,1,\bar{\theta},\varepsilon_t,X\right] \Pr\left[\varepsilon_t<\varepsilon_t^*(1,3) \mid \bar{\theta},\varepsilon_{t-1}=\varepsilon_t^*(1,3)\right]
\end{aligned}
$$

[what next? worry about $\bar{\theta}\left(d+2,\varepsilon_{t+1},\bar{\theta}\right) \neq \bar{\theta}\left(d+1,\varepsilon_{t+1},\bar{\theta}\right)$ with the difference not signable]. I don't think we can prove this. ■

Theorem 18. *Assume $f_t(m_t, c_t, d_t, X_t)$ does not depend upon t, c_t, d_t, or X_t. Then $P_t[1 \mid 1, c_t, d_t]$ is nondecreasing in d_t.*

Proof. Since $V_{t+1}\left[1, 1, c_{t+1}, d_{t+1}, \bar{\theta}_{t+1}, \varepsilon_{t+1}, X_{t+1}\right]$ does not depend upon $\bar{\theta}_{t+1}$ and ε_{t+1}, $\Pr\left[m_{t+1} \mid m_t, c_t, d_t, \bar{\theta}_t, \varepsilon_t\right]$ does not depend upon $\bar{\theta}_t$ or ε_t. Therefore, $P_t[1 \mid 1, c_t, d_t] = \Pr\left[1 \mid t, 1, c_t, d_t, \bar{\theta}_t, \varepsilon_t\right]$ for any value of $\bar{\theta}_t$ and ε_t.

$$
\begin{aligned}
P_t[1 \mid 1, c_t, d_t] &= \Pr\left\{V_{t+1}\left[1, 1, c_{t+1}, d_{t+1}, \bar{\theta}_{t+1}, \varepsilon_{t+1}, X_{t+1}\right] > \max_{m>1} V_{t+1}\left[m, 1, c_{t+1}, d_{t+1}, \bar{\theta}_{t+1}, \varepsilon_{t+1}, X_{t+}\right.\right. \\
&= \Pr\left\{V_{t+1}\left[1, 1, c_{t+1}, d_{t+1}, \bar{\theta}_{t+1}, \varepsilon_{t+1}, X_{t+1}\right] > \max_{m>1} V_{t+1}\left[m, 1, c_{t+1}, d_{t+1}, \bar{\theta}_{t+1}, \varepsilon_{t+1}, X_{t+}\right.\right.
\end{aligned}
$$

because none of the $\bar{\theta}_{t+1}$ or ε_{t+1} terms depend upon $\bar{\theta}_t$ or ε_t. But when $m_t = 1$, $d_{t+1} = 1$ for $m_{t+1} > 1$, and $d_{t+1} = \min[d_t + 1, \tau_d]$ for $m_{t+1} = 1$. Thus $\max_{m>1} V_{t+1}\left[m, 1, c_{t+1}, d_{t+1}, \bar{\theta}_{t+1}, \varepsilon_{t+1}, X_{t+1}\right]$ does not depend on d_t, and $V_{t+1}\left[1, 1, c_{t+1}, d_{t+1}, \bar{\theta}_{t+1}, \varepsilon_{t+1},\right.$ is nondecreasing in d_t because $V_{t+1}\left[1, 1, c_{t+1}, d_{t+1}, \bar{\theta}_{t+1}, \varepsilon_{t+1}, X_{t+1}\right]$ depends on d_t only through $f_{t+1}(1, c_{t+1}, d_t + 1, X_{t+1})$ and $E_{c_{t+2}, \varepsilon_{t+2}}\left\{\max_{m_{t+2} \in F(1)}(V_{t+2}[S_{t+2}, X_{t+2}]) \mid c_{t+1}\right\}$ both of which are nondecreasing in d_t.

■

Theorem 19. *Assume $f_t(m_t, c_t, d_t, X_t)$ does not depend upon t, c_t, d_t, or X_t. Assume $t^* \to \infty$. Then $P_t[3 \mid 3, c_t, d_t] > P_{t-1}[3 \mid 3, c_t, d_t - 1]$ as $t, d_t \to \infty$ (with $t^* - t \to \infty$).*

Proof. Consider the case where $d_t \geq \tau_d$; at such a point, there is no more learning about θ. Therefore, $\varepsilon_t^*(1, 3)$ is not changing with d_t or t (call this level $\varepsilon_\infty^*(1, 3)$). Consider the distribution of ε_t conditional on other (fixed) state variables (and explicitly on $\bar{\theta}$):

$$
\begin{aligned}
\Psi_t\left[x \mid \bar{\theta}\right] &= \Pr\left[\varepsilon_t < x \mid \bar{\theta}\right] \qquad (3.15) \\
&= \Pr\left[\bar{\theta} + \rho\left(\varepsilon_{t-1} - \bar{\theta}\right) + \eta_t < x\right] \\
&= \frac{\int_{\varepsilon_\infty^*(1,3)}^{\infty} \Pr\left[\bar{\theta} + \rho\left(\varepsilon_{t-1} - \bar{\theta}\right) + \eta_t < x \mid \varepsilon_{t-1}\right] d\Psi_{t-1}\left(\varepsilon_{t-1} \mid \bar{\theta}\right)}{1 - \Psi_{t-1}\left[\varepsilon_\infty^{**}(3) \mid \bar{\theta}\right]} \\
&= \frac{\int_{\varepsilon_\infty^*(1,3)}^{\infty} \Phi\left[\frac{x - \bar{\theta} - \rho\left(\varepsilon_{t-1} - \bar{\theta}\right)}{\sigma_\eta}\right] d\Psi_{t-1}\left(\varepsilon_{t-1} \mid \bar{\theta}\right)}{1 - \Psi_{t-1}\left[\varepsilon_\infty^*(1, 3) \mid \bar{\theta}\right]}
\end{aligned}
$$

(the distribution is truncated at $\varepsilon^*_\infty(1,3)$ because anyone with $\varepsilon_{t-1} < \varepsilon^*_\infty(1,3)$ divorces). Using integration by parts (and defining $\Psi^*_t\left[x \mid \bar{\theta}\right] = 1- \Psi_t\left[x \mid \bar{\theta}\right]$, equation (3.15) can be written as

$$\Psi^*_t\left[x \mid \bar{\theta}\right] = \Phi\left[\frac{\bar{\theta} - x + \rho\left(\varepsilon^*_\infty(1,3) - \bar{\theta}\right)}{\sigma_\eta}\right] - \rho\int_{\varepsilon^*_\infty(1,3)}^{\infty} \frac{\Psi^*_{t-1}\left[\varepsilon_{t-1} \mid \bar{\theta}\right]}{\Psi^*_{t-1}\left[\varepsilon^*_\infty(1,3) \mid \bar{\theta}\right]}\frac{1}{\sigma_\eta}\phi\left[\frac{x - \bar{\theta} - \rho\left(\varepsilon_{t-1} - \bar{\theta}\right)}{\sigma_\eta}\right] d \tag{3.16}$$

Let $\Upsilon\left\{\Psi^*_{t-1}\right\}$ be defined by equation (3.16); i.e., $\Psi^*_t = \Upsilon\left\{\Psi^*_{t-1}\right\}$. Consider two potential distribution functions, Ψ^* and Ψ^{**}. Then it is straightforward to show that

$$\sup_x\left|\Upsilon\left\{\Psi^*\left[x \mid \bar{\theta}\right]\right\} - \Upsilon\left\{\Psi^{**}\left[x \mid \bar{\theta}\right]\right\}\right| = \rho\sup_x\left|\Psi^*\left[x \mid \bar{\theta}\right] - \Psi^{**}\left[x \mid \bar{\theta}\right]\right|$$

which is the (contraction mapping) condition necessary for the existence of a unique asymptotic conditional distribution, $\Psi^*\left[x \mid \bar{\theta}\right]$ and for the asymptotic convergence of $\Psi^*_t\left[x \mid \bar{\theta}\right]$ to $\Psi^*\left[x \mid \bar{\theta}\right]$. Note that

$$\frac{d\Psi^*\left[x \mid \bar{\theta}\right]}{d\bar{\theta}} = \frac{\partial\Psi^*\left[x \mid \bar{\theta}\right]}{\partial\bar{\theta}} + \frac{\partial\Psi^*\left[x \mid \bar{\theta}\right]}{\partial\varepsilon^*_\infty(1,3)}\frac{\partial\varepsilon^*_\infty(1,3)}{\partial\bar{\theta}}.$$

Consider first

$$\begin{aligned}
&\frac{\partial\Psi^*\left[x \mid \bar{\theta}\right]}{\partial\bar{\theta}} \\
= \;& \frac{\partial}{\partial\bar{\theta}}\Phi\left[\frac{\bar{\theta} - x + \rho\left(\varepsilon^*_\infty(1,3) - \bar{\theta}\right)}{\sigma_\eta}\right] \\
&+ \frac{\partial}{\partial\bar{\theta}}\rho\int_{\varepsilon^*_\infty(1,3)}^{\infty} \frac{\Psi^*\left[\varepsilon_{t-1} \mid \bar{\theta}\right]}{\Psi^*\left[\varepsilon^*_\infty(1,3) \mid \bar{\theta}\right]}\frac{1}{\sigma_\eta}\phi\left[\frac{x - \bar{\theta} - \rho\left(\varepsilon_{t-1} - \bar{\theta}\right)}{\sigma_\eta}\right] d\varepsilon_{t-1} \\
= \;& \frac{\partial}{\partial\bar{\theta}}\Phi\left[\frac{\bar{\theta} - x + \rho\left(\varepsilon^*_\infty(1,3) - \bar{\theta}\right)}{\sigma_\eta}\right] \\
&+ \rho\int_{\varepsilon^*_\infty(1,3)}^{\infty} \frac{\partial}{\partial\bar{\theta}}\left\{\frac{\Psi^*\left[\varepsilon_{t-1} \mid \bar{\theta}\right]}{\Psi^*\left[\varepsilon^*_\infty(1,3) \mid \bar{\theta}\right]}\right\}\frac{1}{\sigma_\eta}\phi\left[\frac{x - \bar{\theta} - \rho\left(\varepsilon_{t-1} - \bar{\theta}\right)}{\sigma_\eta}\right] d\varepsilon_{t-1} \\
&+ \rho\int_{\varepsilon^*_\infty(1,3)}^{\infty} \frac{\Psi^*\left[\varepsilon_{t-1} \mid \bar{\theta}\right]}{\Psi^*\left[\varepsilon^*_\infty(1,3) \mid \bar{\theta}\right]}\frac{1}{\sigma_\eta}\frac{\partial}{\partial\bar{\theta}}\phi\left[\frac{x - \bar{\theta} - \rho\left(\varepsilon_{t-1} - \bar{\theta}\right)}{\sigma_\eta}\right] d\varepsilon_{t-1}
\end{aligned}$$

$$\begin{aligned}
&= \frac{\partial}{\partial\bar{\theta}}\Phi\left[\frac{\bar{\theta}-x+\rho\left(\varepsilon_{\infty}^{*}(1,3)-\bar{\theta}\right)}{\sigma_{\eta}}\right] \\
&\quad +\rho\int_{\varepsilon_{\infty}^{*}(1,3)}^{\infty}\frac{\partial}{\partial\bar{\theta}}\left\{\frac{\Psi^{*}\left[\varepsilon_{t-1}\mid\bar{\theta}\right]}{\Psi^{*}\left[\varepsilon_{\infty}^{*}(1,3)\mid\bar{\theta}\right]}\right\}\frac{1}{\sigma_{\eta}}\phi\left[\frac{x-\bar{\theta}-\rho\left(\varepsilon_{t-1}-\bar{\theta}\right)}{\sigma_{\eta}}\right]d\varepsilon_{t-1} \\
&\quad -(1-\rho)\int_{\varepsilon_{\infty}^{*}(1,3)}^{\infty}\frac{\Psi^{*}\left[\varepsilon_{t-1}\mid\bar{\theta}\right]}{\Psi^{*}\left[\varepsilon_{\infty}^{*}(1,3)\mid\bar{\theta}\right]}\frac{1}{\sigma_{\eta}}\frac{\partial}{\partial\varepsilon_{t-1}}\phi\left[\frac{x-\bar{\theta}-\rho\left(\varepsilon_{t-1}-\bar{\theta}\right)}{\sigma_{\eta}}\right]d\varepsilon_{t-1}
\end{aligned}$$

$$\begin{aligned}
&= \frac{(1-\rho)}{\sigma_{\eta}}\phi\left[\frac{\bar{\theta}-x+\rho\left(\varepsilon_{\infty}^{*}(1,3)-\bar{\theta}\right)}{\sigma_{\eta}}\right] \\
&\quad +\rho\int_{\varepsilon_{\infty}^{*}(1,3)}^{\infty}\frac{\partial}{\partial\bar{\theta}}\left\{\frac{\Psi^{*}\left[\varepsilon_{t-1}\mid\bar{\theta}\right]}{\Psi^{*}\left[\varepsilon_{\infty}^{*}(1,3)\mid\bar{\theta}\right]}\right\}\frac{1}{\sigma_{\eta}}\phi\left[\frac{x-\bar{\theta}-\rho\left(\varepsilon_{t-1}-\bar{\theta}\right)}{\sigma_{\eta}}\right]d\varepsilon_{t-1} \\
&\quad -(1-\rho)\frac{1}{\sigma_{\eta}}\phi\left[\frac{x-\bar{\theta}-\rho\left(\varepsilon_{\infty}^{*}(1,3)-\bar{\theta}\right)}{\sigma_{\eta}}\right] \\
&\quad -(1-\rho)\int_{\varepsilon_{\infty}^{*}(1,3)}^{\infty}\frac{\partial}{\partial\varepsilon_{t-1}}\frac{\Psi^{*}\left[\varepsilon_{t-1}\mid\bar{\theta}\right]}{\Psi^{*}\left[\varepsilon_{\infty}^{*}(1,3)\mid\bar{\theta}\right]}\frac{1}{\sigma_{\eta}}\phi\left[\frac{x-\bar{\theta}-\rho\left(\varepsilon_{t-1}-\bar{\theta}\right)}{\sigma_{\eta}}\right]d\varepsilon_{t-1} \\
&= +\rho\int_{\varepsilon_{\infty}^{*}(1,3)}^{\infty}\frac{\partial}{\partial\bar{\theta}}\left\{\frac{\Psi^{*}\left[\varepsilon_{t-1}\mid\bar{\theta}\right]}{\Psi^{*}\left[\varepsilon_{\infty}^{*}(1,3)\mid\bar{\theta}\right]}\right\}\frac{1}{\sigma_{\eta}}\phi\left[\frac{x-\bar{\theta}-\rho\left(\varepsilon_{t-1}-\bar{\theta}\right)}{\sigma_{\eta}}\right]d\varepsilon_{t-1} \\
&\quad -(1-\rho)\int_{\varepsilon_{\infty}^{*}(1,3)}^{\infty}\frac{\partial}{\partial\varepsilon_{t-1}}\frac{\Psi^{*}\left[\varepsilon_{t-1}\mid\bar{\theta}\right]}{\Psi^{*}\left[\varepsilon_{\infty}^{*}(1,3)\mid\bar{\theta}\right]}\frac{1}{\sigma_{\eta}}\phi\left[\frac{x-\bar{\theta}-\rho\left(\varepsilon_{t-1}-\bar{\theta}\right)}{\sigma_{\eta}}\right]d\varepsilon_{t-1}
\end{aligned}$$

The second term is a weighted average of $\partial\Psi^{*}\left[\varepsilon_{t-1}\mid\bar{\theta}\right]/\partial\varepsilon_{t-1}$ over ε_{t-1} which is negative, so it is negative (times a negative). If $\inf\frac{\partial\Psi^{*}\left[\varepsilon_{t-1}\mid\bar{\theta}\right]}{\partial\bar{\theta}}\geq 0$, then $\inf\frac{\partial\Psi^{*}\left[x\mid\bar{\theta}\right]}{\partial\bar{\theta}}>0$. Since there is a unique solution to $\Psi^{*}\left[x\mid\bar{\theta}\right]$, there is a unique solution to $\frac{\partial\Psi^{*}\left[x\mid\bar{\theta}\right]}{\partial\bar{\theta}}$. Thus $\partial\Psi^{*}\left[x\mid\bar{\theta}\right]/\partial\bar{\theta}>0$. Also,

$$\frac{\partial\Psi_{t}^{*}\left[x\mid\bar{\theta}\right]}{\partial\varepsilon_{\infty}^{*}(1,3)}$$

$$
\begin{aligned}
&= \frac{\partial}{\partial \varepsilon_{\infty}^{*}(1,3)} \Phi\left[\frac{\bar{\theta}-x+\rho\left(\varepsilon_{\infty}^{*}(1,3)-\bar{\theta}\right)}{\sigma_{\eta}}\right] \\
&\quad -\rho \frac{\partial}{\partial \varepsilon_{\infty}^{*}(1,3)} \int_{\varepsilon_{\infty}^{*}(1,3)}^{\infty} \frac{\Psi_{t-1}^{*}\left[\varepsilon_{t-1} \mid \bar{\theta}\right]}{\Psi_{t-1}^{*}\left[\varepsilon_{\infty}^{*}(1,3) \mid \bar{\theta}\right]} \frac{1}{\sigma_{\eta}} \phi\left[\frac{x-\bar{\theta}-\rho\left(\varepsilon_{t-1}-\bar{\theta}\right)}{\sigma_{\eta}}\right] d\varepsilon_{t-1} \\
&= \frac{\rho}{\sigma_{\eta}} \phi\left[\frac{\bar{\theta}-x+\rho\left(\varepsilon_{\infty}^{*}(1,3)-\bar{\theta}\right)}{\sigma_{\eta}}\right] \\
&\quad +\frac{\rho}{\sigma_{\eta}} \phi\left[\frac{x-\bar{\theta}-\rho\left(\varepsilon_{\infty}^{*}(1,3)-\bar{\theta}\right)}{\sigma_{\eta}}\right]-\rho \int_{\varepsilon_{\infty}^{*}(1,3)}^{\infty} \frac{\partial}{\partial \varepsilon_{\infty}^{*}(1,3)}\left\{\frac{\Psi_{t-1}^{*}\left[\varepsilon_{t-1} \mid \bar{\theta}\right]}{\Psi_{t-1}^{*}\left[\varepsilon_{\infty}^{*}(1,3) \mid \bar{\theta}\right]}\right\} \frac{1}{\sigma_{\eta}} \phi\left[\frac{x-\bar{\theta}-\rho}{\sigma}\right. \\
&= -\rho \int_{\varepsilon_{\infty}^{*}(1,3)}^{\infty} \frac{\partial}{\partial \varepsilon_{\infty}^{*}(1,3)}\left\{\frac{\Psi_{t-1}^{*}\left[\varepsilon_{t-1} \mid \bar{\theta}\right]}{\Psi_{t-1}^{*}\left[\varepsilon_{\infty}^{*}(1,3) \mid \bar{\theta}\right]}\right\} \frac{1}{\sigma_{\eta}} \phi\left[\frac{x-\bar{\theta}-\rho\left(\varepsilon_{t-1}-\bar{\theta}\right)}{\sigma_{\eta}}\right] d\varepsilon_{t-1} \\
&= \rho \int_{\varepsilon_{\infty}^{*}(1,3)}^{\infty} \frac{\Psi_{t-1}^{*}\left[\varepsilon_{t-1} \mid \bar{\theta}\right]}{\left\{\Psi_{t-1}^{*}\left[\varepsilon_{\infty}^{*}(1,3) \mid \bar{\theta}\right]\right\}^{2}} \frac{\partial \Psi_{t-1}^{*}\left[\varepsilon_{\infty}^{*}(1,3) \mid \bar{\theta}\right]}{\partial \varepsilon_{\infty}^{*}(1,3)} \frac{1}{\sigma_{\eta}} \phi\left[\frac{x-\bar{\theta}-\rho\left(\varepsilon_{t-1}-\bar{\theta}\right)}{\sigma_{\eta}}\right] d\varepsilon_{t-1}<0 .
\end{aligned}
$$

and $\partial \varepsilon_{\infty}^{*}(1,3) / \partial \bar{\theta} < 0$ (because $\partial V_t[m_t, 1, c_t, 1, \bar{\theta}_t(\varepsilon_t), \varepsilon_t, X_t]/\partial \bar{\theta}_t > 0$, an agent is willing to tolerate worse ε's as $\bar{\theta}$ increases) Thus, $d\Psi^*\left[x \mid \bar{\theta}\right]/d\bar{\theta} > 0$ which implies that $d\Psi^*\left[\varepsilon_{\infty}^{*}(1,3) \mid \bar{\theta}\right]/d\bar{\theta} > 0$.

The unconditional distribution is

$$\Psi_t^*[x] = \int \Psi_t^*\left[x \mid \bar{\theta}\right] d\Im_t\left(\bar{\theta}\right)$$

where $\Im_t\left(\bar{\theta}\right)$ is the distribution of $\bar{\theta}$ at t,

$$\Im_t\left(\bar{\theta}\right) = \frac{\int_{-\infty}^{\bar{\theta}} \Psi_t^*\left[\varepsilon_{\infty}^{*}(1,3) \mid x\right] d\Im_{t-1}(x)}{\int_{-\infty}^{\infty} \Psi_t^*\left[\varepsilon_{\infty}^{*}(1,3) \mid x\right] d\Im_{t-1}(x)}.$$

The reciprocal of $\Im_t\left(\bar{\theta}\right)$ is

$$
\begin{aligned}
&\frac{\int_{-\infty}^{\infty} \Psi_t^*\left[\varepsilon_{\infty}^{*}(1,3) \mid x\right] d\Im_{t-1}(x)}{\int_{-\infty}^{\bar{\theta}} \Psi_t^*\left[\varepsilon_{\infty}^{*}(1,3) \mid x\right] d\Im_{t-1}(x)} \\
&= 1+\frac{\int_{\bar{\theta}}^{\infty} \Psi_t^*\left[\varepsilon_{\infty}^{*}(1,3) \mid x\right] d\Im_{t-1}(x)}{\int_{-\infty}^{\bar{\theta}} \Psi_t^*\left[\varepsilon_{\infty}^{*}(1,3) \mid x\right] d\Im_{t-1}(x)}
\end{aligned}
$$

$$
\begin{aligned}
&< 1+\frac{\int_{\bar{\theta}}^{\infty} \Psi_t^*\left[\varepsilon_\infty^*(1,3) \mid \bar{\theta}\right] d\Im_{t-1}(x)}{\int_{-\infty}^{\bar{\theta}} \Psi_t^*\left[\varepsilon_\infty^*(1,3) \mid \bar{\theta}\right] d\Im_{t-1}(x)} \\
&= 1+\frac{\Psi_t^*\left[\varepsilon_\infty^*(1,3) \mid \bar{\theta}\right]\left\{1-\Im_{t-1}\left(\bar{\theta}\right)\right\}}{\Psi_t^*\left[\varepsilon_\infty^*(1,3) \mid \bar{\theta}\right]\Im_{t-1}\left(\bar{\theta}\right)} \\
&= 1+\frac{\left\{1-\Im_{t-1}\left(\bar{\theta}\right)\right\}}{\Im_{t-1}\left(\bar{\theta}\right)}=\frac{1}{\Im_{t-1}\left(\bar{\theta}\right)}.
\end{aligned}
$$

Therefore $\Im_{t-1}\left(\bar{\theta}\right) > \Im_t\left(\bar{\theta}\right)$. Since $\Psi_t^*\left[x \mid \bar{\theta}\right]$ is converging to $\Psi^*\left[x \mid \bar{\theta}\right]$ with $d\Psi^*\left[x \mid \bar{\theta}\right]/d\bar{\theta} > 0$ and $\Im_t\left(\bar{\theta}\right)$ is decreasing in t (shifting toward larger values of $\bar{\theta}$), $\Psi_t^*[x]$ is increasing in t. The probability of divorce is then $\Psi_t\left[\varepsilon_\infty^*(1,3)\right]$ which is declining in t. ■

Conjecture 20. *Assume $f_t(m_t, c_t, d_t, X_t)$ does not depend upon t, c_t, d_t, or X_t. Assume $t^* \to \infty$. Then $P_t[3 \mid 3, c_t, d_t]$ is asymptoting to a probability greater than zero from below as $t, d_t \to \infty$ (with $t^* - t \to \infty$).*

Proof. ■

Conjecture 21. *Assume $f_t(m_t, c_t, d_t, X_t)$ does not depend upon t, c_t, d_t, or X_t. Assume $t^* \to \infty$. Then a) $P_{t+1}[3 \mid 3, c_t, d_t+1] > P_t[3 \mid 3, c_t, d_t]$ and b) $P_{t+1}[2 \mid 2, c_t, d_t+1] > P_t[2 \mid 2, c_t, d_t]$*

Proof. Next, consider the case when $d_t < \tau_d$.

$$
P_t[3 \mid 3, c_t, d_t] = \frac{\int\int \Pr\left[3 \mid t, 3, c_t, d_t, \bar{\theta}_t, \varepsilon_t\right] h_t\left(\bar{\theta}_t, \varepsilon_t \mid m_t, c_t, d_t\right) d\bar{\theta}_t d\varepsilon_t}{\int\int h_t\left(\bar{\theta}_t, \varepsilon_t \mid m_t, c_t, d_t\right) d\bar{\theta}_t d\varepsilon_t}
$$

where

$$
\begin{aligned}
\Pr\left[3 \mid t, 3, c, d_t, \bar{\theta}, \varepsilon\right] &= \Pr\left\{V_{t+1}\left[3, 3, c_{t+1}, d_{t+1}, \bar{\theta}_{t+1}, \varepsilon_{t+1}, X_{t+1}\right] > V_{t+1}\left[1, 3, c_{t+1}, 1, \bar{\theta}_{t+1}, \varepsilon_{t+1}, X_{t+1}\right]\right. \\
&= \Pr\left[\varepsilon_{t+1} > \varepsilon_{t+1}^*(1,3) \mid c, \bar{\theta}_t, \varepsilon_t\right].
\end{aligned}
$$

By the same argument,

$$
\Pr\left[3 \mid t+1, 3, c, d_t+1, \bar{\theta}, \varepsilon\right] = \Pr\left[\varepsilon_{t+2} > \varepsilon_{t+2}^*(1,3) \mid c, \bar{\theta}, \varepsilon\right].
$$

Since there are no children, age, or duration effects to flows, ... [Things to show: a) $\varepsilon_{t+2}^*(1,3) > \varepsilon_{t+1}^*(1,3)$; b) selection effect leads to decrease in exit probability if $\varepsilon_{t+2}^*(1,3)$ not changing in t; effect (b) dominates effect (a).] ■

4. Numerical Methods Issues

1) $\bar{\theta}_t$ and ε_t are continuous state variables. First consider the case when $m_t \neq 1$ and $d_t < \bar{t}_d$. When evaluating $V_t[S_t, X_t]$,
$E\left\{\max \ (V_{t+1}[S_{t+1}, X_{t+1}]) \mid \bar{\theta}_t, \varepsilon_t, c_t\right\}$ is

$$\frac{p_{t+1}}{\sigma_\eta}\int_{-\infty}^{\infty}\phi\left(\frac{\varepsilon_{t+1}-\left[(1-\rho)\,\bar{\theta}_t+\rho\varepsilon_t\right]}{\sigma_\eta}\right)\left\{\max_{m_{t+1}\in F(m_t)} V_{t+1}\left[S_{t+1}^1\right]\right\}d\varepsilon_{t+1}+ \tag{4.1}$$
$$\frac{1-p_{t+1}}{\sigma_\eta}\int_{-\infty}^{\infty}\phi\left(\frac{\varepsilon_{t+1}-\left[(1-\rho)\,\bar{\theta}_t+\rho\varepsilon_t\right]}{\sigma_\eta}\right)\left\{\max_{m_{t+1}\in F(m_t)} V_{t+1}\left[S_{t+1}^0\right]\right\}d\varepsilon_{t+1}$$

if $m_t \neq 1$ where $S_{t+1}^1 = \left(m_{t+1}, m_t, c_{1t+1}^*, 1, d_{t+1}, \bar{\theta}_{t+1}, \varepsilon_{t+1}\right)$ is the vector of state variables conditional on a birth, $S_{t+1}^0 = \left(m_{t+1}, m_t, c_t, c_{2t+1}^*, d_{t+1}, \bar{\theta}_{t+1}, \varepsilon_{t+1}\right)$is the vector of state variables conditional on no birth,

$$\begin{aligned}
c_{1t+1}^* &= \min\left(c_{1t}+1, \bar{n}_c\right), \\
c_{2t+1}^* &= \min\left(c_{2t}+1, \bar{n}_{ca}\right), \\
d_{t+1} &= \min\left(d_t+1, \bar{t}_d\right)1\left(m_{t+1}=m_t \cup \left(m_{t+1}=3 \cap m_t=2\right)\right)+ \\
&\quad 1\left(m_{t+1}\neq m_t \cap \left(m_{t+1}\neq 3 \cup m_t \neq 2\right)\right),
\end{aligned} \tag{4.2}$$

is relationship duration implied by m_{t+1} (note that this definition implies that duration of relationship does not start over if a couple moves from cohabitation to marriage), and $\bar{\theta}_{t+1}$ is given by equation (7.9). Note that, because $\bar{\theta}_{t+1}$ is determined once ε_{t+1} is given, there is essentially only one continuous state variable to integrate over. The integral in equation (4.1) can be written as

$$\int_{-\infty}^{\infty}\frac{1}{\sigma_\eta\sqrt{2\pi}}\exp\left\{\frac{-1}{2}\left[\frac{\varepsilon_{t+1}-\left[(1-\rho)\,\bar{\theta}_t+\rho\varepsilon_t\right]}{\sigma_\eta}\right]^2\right\}H\left[\varepsilon_{t+1}\right]d\varepsilon_{t+1} \tag{4.3}$$

where

$$H\left[\varepsilon_{t+1}\right]=\max_{m_{t+1}\in F(m_t)} V_{t+1}\left[m_{t+1}, \bullet, \bullet, \bullet, \bullet, \bar{\theta}_{t+1}\left(\varepsilon_{t+1}\right), \varepsilon_{t+1}\right]. \tag{4.4}$$

Equation (4.3) can be written as

$$\frac{1}{\sigma_\eta\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp\left\{\frac{-1}{2}\left[\frac{\eta_{t+1}}{\sigma_\eta}\right]^2\right\}H\left[(1-\rho)\,\bar{\theta}_t+\rho\varepsilon_t+\eta_{t+1}\right]d\eta_{t+1} \tag{4.5}$$

which, with a change of variables ($u = \eta_{t+1}/\sqrt{2}\sigma_\eta \Rightarrow du = d\eta_{t+1}/\sqrt{2}\sigma_\eta$), can be written as

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp\left\{-u^2\right\} H\left[(1-\rho)\,\bar{\theta}_t + \rho\varepsilon_t + \sqrt{2}\sigma_\eta u\right] du. \tag{4.6}$$

This can be approximated using Gaussian quadrature as

$$\frac{1}{\sqrt{\pi}} \sum_{j=1}^{K} a_j H\left[(1-\rho)\,\bar{\theta}_t + \rho\varepsilon_t + \sqrt{2}\sigma_\eta \tilde{u}_j\right] \tag{4.7}$$

where $(a_j, \tilde{u}_j)_{j=1}^K$ are K-point Gaussian quadrature weights and points. But, since $H[\bullet]$ was not evaluated at $(1-\rho)\,\bar{\theta}_t + \rho\varepsilon_t + \sqrt{2}\sigma_\eta\tilde{u}_j$, it must be interpolated. Let $\tilde{\theta} = \left(\tilde{\theta}_1, \tilde{\theta}_2, .., \tilde{\theta}_K\right)$ and $\tilde{\varepsilon} = (\tilde{\varepsilon}_1, \tilde{\varepsilon}_2, .., \tilde{\varepsilon}_K)$ be a grid of K^2 (Gaussian quadrature) points at which to evaluate $V_t[S_{t+1}, X_{t+1}]$. Let $y_j = (1-\rho)\,\bar{\theta}_t + \rho\varepsilon_t + \sqrt{2}\sigma_\eta\tilde{u}_j$ or $y_j = \mu + \sqrt{2\left(\sigma_\eta^2 + \sigma_\theta^2\right)}\tilde{u}_j$ (depending on the case) and $z_j = \bar{\theta}_{t+1}(y_j)$. Then each $H[\bullet]$ term in equation (4.7) can be interpolated as

$$h[y_j] = \frac{\sum_{k=[j]}^{[j]+1} \sum_{l=[j]}^{[j]+1} \max_{m_{t+1} \in F(m_t)} V_{t+1}\left[m_{t+1}, \bullet, \bullet, \bullet, \bullet, \tilde{\theta}_k, \tilde{\varepsilon}_l\right] R\left[\tilde{\theta}_k - z_j, \tilde{\varepsilon}_l - y_j\right]}{\sum_{k=1}^{K} \sum_{l=1}^{K} R\left[\tilde{\theta}_k - z_j, \tilde{\varepsilon}_l - y_j\right]} \tag{4.8}$$

where

$$R\left[\tilde{\theta}_k - z_j, \tilde{\varepsilon}_l - y_j\right] = \left[\left(\tilde{\theta}_k - z_j\right)^2 + (\tilde{\varepsilon}_l - y_j)^2\right]^{-1/2}. \tag{4.9}$$

and $[j]$ is that value of k (or l) such that $\tilde{\theta}_{[j]} < z_j < \tilde{\theta}_{[j]+1}$ (or $\tilde{\varepsilon}_{[j]} < y_j < \tilde{\varepsilon}_{[j]+1}$). Alternatively,

$$R\left[\tilde{\theta}_k - z_j, \tilde{\varepsilon}_l - y_j\right] = \left|\tilde{\theta}_{k'} - z_j\right|^p |\tilde{\varepsilon}_{l'} - y_j|^p, \tag{4.10}$$

where $k' = [j]$ if $k = [j]+1$ and $k' = [j]+1$ if $k = [j]$ (and l' has a similar definition), and $1 < p \leq 2$. The second definition is better because it is continuous and differentiable at seems (where $\tilde{\theta}_k = z_j$ or $\tilde{\varepsilon}_l = y_j$) while the first definition is neither. A problem with the second is that the derivative of $h[y_j]$ at seems is zero. A way to avoid this is to set $p = 1$. There is no way to have nonzero derivatives that are continuous at seems when only 4 points are used to evaluate $h[y_j]$.

Now consider the case when $m_t \neq 1$ and $d_t = \bar{t}_d$. When evaluating $V_t[S_t, X_t]$, $E\left\{\max \ (V_{t+1}[S_{t+1}, X_{t+1}]) \mid \bar{\theta}_t, \varepsilon_t, c_t\right\}$ is

$$\frac{p_{t+1}}{\sigma_\eta} \int_{-\infty}^{\infty} \phi\left(\frac{\varepsilon_{t+1} - \left[(1-\rho)\,\bar{\theta}_t + \rho\varepsilon_t\right]}{\sigma_\eta}\right) \left\{\max_{m_{t+1} \in F(m_t)} V_{t+1}\left[S_{t+1}^1\right]\right\} d\varepsilon_{t+1} + \tag{4.11}$$

$$\frac{1-p_{t+1}}{\sigma_\eta}\int_{-\infty}^{\infty}\phi\left(\frac{\varepsilon_{t+1}-\left[(1-\rho)\,\bar{\theta}_t+\rho\varepsilon_t\right]}{\sigma_\eta}\right)\left\{\max_{m_{t+1}\in F(m_t)} V_{t+1}\left[S_{t+1}^0\right]\right\}d\varepsilon_{t+1}$$

where $S_{t+1}^1=\left(m_{t+1},m_t,c_{1t+1}^*,1,d_{t+1},\bar{\theta}_t,\varepsilon_{t+1}\right)$ is the vector of state variables conditional on a birth, $S_{t+1}^0=\left(m_{t+1},m_t,c_t,c_{2t+1}^*,d_{t+1},\bar{\theta}_t,\varepsilon_{t+1}\right)$is the vector of state variables conditional on no birth (note that S_{t+1}^1 and S_{t+1}^0 depend on $\bar{\theta}_t$ instead of $\bar{\theta}_{t+1}$ because $\theta=\bar{\theta}_t$ is treated as known), and c_{1t+1}^*, c_{2t+1}^*, and d_{t+1} are defined in equation (4.2). Note that, because θ is treated as known, there is essentially only one continuous state variable to integrate over. The integral in equation (4.11) can be written as

$$\int_{-\infty}^{\infty}\frac{1}{\sigma_\eta\sqrt{2\pi}}\exp\left\{\frac{-1}{2}\left[\frac{\varepsilon_{t+1}-\left[(1-\rho)\,\bar{\theta}_t+\rho\varepsilon_t\right]}{\sigma_\eta}\right]^2\right\}H\left[\varepsilon_{t+1}\right]d\varepsilon_{t+1} \tag{4.12}$$

where

$$H\left[\varepsilon_{t+1}\right]=\max_{m_{t+1}\in F(m_t)} V_{t+1}\left[m_{t+1},\bullet,\bullet,\bullet,\bullet,\bar{\theta}_t,\varepsilon_{t+1}\right]. \tag{4.13}$$

Equation (4.12) can be written as

$$\frac{1}{\sigma_\eta\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp\left\{\frac{-1}{2}\left[\frac{\eta_{t+1}}{\sigma_\eta}\right]^2\right\}H\left[(1-\rho)\,\bar{\theta}_t+\rho\varepsilon_t+\eta_{t+1}\right]d\eta_{t+1} \tag{4.14}$$

which, with a change of variables ($u=\eta_{t+1}/\sqrt{2}\sigma_\eta \Rightarrow du=d\eta_{t+1}/\sqrt{2}\sigma_\eta$), can be written as

$$\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\exp\left\{-u^2\right\}H\left[(1-\rho)\,\bar{\theta}_t+\rho\varepsilon_t+\sqrt{2}\sigma_\eta u\right]du. \tag{4.15}$$

This can be approximated using Gaussian quadrature as

$$\frac{1}{\sqrt{\pi}}\sum_{j=1}^{K}a_j H\left[(1-\rho)\,\bar{\theta}_t+\rho\varepsilon_t+\sqrt{2}\sigma_\eta\tilde{u}_j\right] \tag{4.16}$$

where $(a_j,\tilde{u}_j)_{j=1}^K$ are K-point Gaussian quadrature weights and points. But, since $H[\bullet]$ was not evaluated at $(1-\rho)\,\bar{\theta}_t+\rho\varepsilon_t+\sqrt{2}\sigma_\eta\tilde{u}_j$, it must be interpolated. Let

$y_j = (1-\rho)\bar{\theta}_t + \rho\varepsilon_t + \sqrt{2}\sigma_\eta\tilde{u}_j$ and $z_j = \bar{\theta}_t$. Then each $H[\bullet]$ term in equation (4.16) can be interpolated as

$$h[y_j] = \frac{\sum_{k=1}^{K}\sum_{l=1}^{K}\max_{m_{t+1}\in F(m_t)} V_{t+1}\left[m_{t+1},\bullet,\bullet,\bullet,\bullet,\tilde{\theta}_k,\tilde{\varepsilon}_l\right] R\left[\tilde{\theta}_k - z_j, \tilde{\varepsilon}_l - y_j\right]}{\sum_{k=1}^{K}\sum_{l=1}^{K} R\left[\tilde{\theta}_k - z_j, \tilde{\varepsilon}_l - y_j\right]} \tag{4.17}$$

where $\tilde{\theta}_k$ and $\tilde{\varepsilon}_l$ were defined for the previous case and $R\left[\tilde{\theta}_k - z_j, \tilde{\varepsilon}_l - y_j\right]$ is defined in equation (4.9) or (4.10).

Now consider the case when $m_t = 1$.When evaluating $V_t[S_t, X_t]$, $E\left\{\max\ \ (V_{t+1}[S_{t+1}, X_{t+1}]) \mid \bar{\theta}_t, \varepsilon_t, c_t\right\}$ is

$$\begin{aligned} &\frac{p_{t+1}}{\sqrt{\sigma_\eta^2+\sigma_\theta^2}}\int_{-\infty}^{\infty}\phi\left(\frac{\varepsilon_{t+1}-\mu}{\sqrt{\sigma_\eta^2+\sigma_\theta^2}}\right)\left\{\max_{m_{t+1}\in F(m_t)} V_{t+1}\left[S_{t+1}^1\right]\right\}d\varepsilon_{t+1} + \\ &\frac{1-p_{t+1}}{\sqrt{\sigma_\eta^2+\sigma_\theta^2}}\int_{-\infty}^{\infty}\phi\left(\frac{\varepsilon_{t+1}-\mu}{\sqrt{\sigma_\eta^2+\sigma_\theta^2}}\right)\left\{\max_{m_{t+1}\in F(m_t)} V_{t+1}\left[S_{t+1}^0\right]\right\}d\varepsilon_{t+1} \end{aligned} \tag{4.18}$$

where $S_{t+1}^1 = \left(m_{t+1}, m_t, c_{1t+1}^*, 1, d_{t+1}, \bar{\theta}_{t+1}, \varepsilon_{t+1}\right)$ is the vector of state variables conditional on a birth, $S_{t+1}^0 = \left(m_{t+1}, m_t, c_t, c_{2t+1}^*, d_{t+1}, \bar{\theta}_{t+1}, \varepsilon_{t+1}\right)$is the vector of state variables conditional on no birth, c_{1t+1}^*, c_{2t+1}^*, d_{t+1} and are defined in equation (4.2), and $\bar{\theta}_{t+1}$ is given by equation (7.9) when $d_t = 1$. Note that, because $\bar{\theta}_{t+1}$ is determined once ε_{t+1} is given, there is essentially only one continuous state variable to integrate over. The integral in equation (4.18) can be written as

$$\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi\left(\sigma_\eta^2+\sigma_\theta^2\right)}}\exp\left\{\frac{-1}{2}\left[\frac{\varepsilon_{t+1}-\mu}{\sqrt{\sigma_\eta^2+\sigma_\theta^2}}\right]^2\right\}H[\varepsilon_{t+1}]\,d\varepsilon_{t+1} \tag{4.19}$$

where

$$H[\varepsilon_{t+1}] = \max_{m_{t+1}\in F(m_t)} V_{t+1}\left[m_{t+1},\bullet,\bullet,\bullet,\bullet,\bar{\theta}_{t+1}(\varepsilon_{t+1}),\varepsilon_{t+1}\right]. \tag{4.20}$$

Equation (4.19) can be written as

$$\frac{1}{\sqrt{2\pi\left(\sigma_\eta^2+\sigma_\theta^2\right)}}\int_{-\infty}^{\infty}\exp\left\{\frac{-1}{2}\left[\frac{\eta_{t+1}}{\sqrt{\sigma_\eta^2+\sigma_\theta^2}}\right]^2\right\}H\left[\mu+\eta_{t+1}\right]d\eta_{t+1} \tag{4.21}$$

which, with a change of variables ($u = \eta_{t+1}/\sqrt{2\left(\sigma_\eta^2 + \sigma_\theta^2\right)} \Rightarrow du = d\eta_{t+1}/\sqrt{2\left(\sigma_\eta^2 + \sigma_\theta^2\right)}$), can be written as

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp\left\{-u^2\right\} H\left[\mu + \sqrt{2\left(\sigma_\eta^2 + \sigma_\theta^2\right)} u\right] du. \tag{4.22}$$

This can be approximated using Gaussian quadrature as

$$\frac{1}{\sqrt{\pi}} \sum_{j=1}^{K} a_j H\left[(1-\rho)\,\bar{\theta}_t + \rho\varepsilon_t + \sqrt{2}\sigma_\eta \tilde{u}_j\right] \tag{4.23}$$

where $(a_j, \tilde{u}_j)_{j=1}^K$ are K-point Gaussian quadrature weights and points. But, since $H[\bullet]$ was not evaluated at $\mu + \sqrt{2\left(\sigma_\eta^2 + \sigma_\theta^2\right)}\tilde{u}_j$, it must be interpolated. Let $y_j = \mu + \sqrt{2\left(\sigma_\eta^2 + \sigma_\theta^2\right)}\tilde{u}_j$ and $z_j = \bar{\theta}_{t+1}(y_j)$. Then each $H[\bullet]$ term in equation (4.23) can be interpolated as

$$h[y_j] = \frac{\sum_{k=1}^K \sum_{l=1}^K \max_{m_{t+1} \in F(m_t)} V_{t+1}\left[m_{t+1}, \bullet, \bullet, \bullet, \bullet, \tilde{\theta}_k, \tilde{\varepsilon}_l\right] R\left[\tilde{\theta}_k - z_j, \tilde{\varepsilon}_l - y_j\right]}{\sum_{k=1}^K \sum_{l=1}^K R\left[\tilde{\theta}_k - z_j, \tilde{\varepsilon}_l - y_j\right]} \tag{4.24}$$

where $\tilde{\theta}_k$ and $\tilde{\varepsilon}_l$ were defined for the first case and $R\left[\tilde{\theta}_k - z_j, \tilde{\varepsilon}_l - y_j\right]$ is defined in equation (4.9) or (4.10).

2) We can save a lot of time by not evaluating each person's value function matrix for each draw of μ. Create N representative people with different values of X. For example, if X includes race (2 values), religion (2 values), education (3 values), and region (4 values), and we discretize μ with 5 values, then there are $N = 2 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 240$ representative people (instead of $I = 6000$ times number of draws of μ people). Once we have evaluated value function matrices for these N representative people, we can interpolate only those value functions actually needed for frequency evaluation for the I people-draws in our sample. For those characteristics in X that are discrete and where all relevant values were included (e.g., race, religion, region), no interpolation is necessary. In the other dimensions, we can use a kernel-based interpolation method. Let X^c be the continuous elements of X that need to be interpolated over, let X^d be the discrete elements of X that do not need to be interpolated over, and let $\tilde{X} = \left(\tilde{X}^c, \tilde{X}^d\right)$ be the values of $X = \left(X^c, X^d\right)$ corresponding to representative people.

Let $\phi\left[X^c - \tilde{X}^c; \Omega_x\right]$ be a normal kernel function with bandwidth matrix Ω_x. Let $e(X)$ be some function (such as frequencies or derivatives of frequencies) that we need to compute. Then the interpolated value of $e(X)$ is

$$\frac{\sum_{n=1}^{N} 1\left(\tilde{X}^d = X^d\right)\phi\left[X^c - \tilde{X}^c; \Omega_x\right] e\left(\tilde{X}\right)}{\sum_{n=1}^{N} 1\left(\tilde{X}^d = X^d\right)\phi\left[X^c - \tilde{X}^c; \Omega_x\right]}. \tag{4.25}$$

5. Path Probabilities

Let $\tilde{m} = (m_1, m_2, .., m_T)$ and $\tilde{b} = (b_1, b_2, .., b_T)$ where T is the last period of data. The path probability can be written as

$$\Pr\left[V_t\left(m_t, S_t^{b*}, \bar{\theta}_t, \varepsilon_t\right) \geq V_t\left(j, S_t^{b*}, \bar{\theta}_t, \varepsilon_t\right) \forall j \in F(m_{t-1}); b_t \quad \forall 2 \leq t \leq T\right] \tag{5.1}$$

where $S_t^{1*} = (m_{t-1}, c_{1t}^*, 1, d_t)$ is the set of state variables, excluding m_t, $\bar{\theta}_t$, and ε_t, conditional on a birth at t and $S_t^{0*} = (m_{t-1}, c_{t-1}, c_{2t}^*, d_t)$ is the set of state variables, excluding m_t, $\bar{\theta}_t$, and ε_t, conditional on no birth at t. [Note that $S_t^b = \left(m_t, S_t^{b*}, \bar{\theta}_t, \varepsilon_t\right)$.] Equation (5.1) can be written as

$$\begin{aligned} & \Pi_{t=3}^{T} \Pr[V_t\left(m_t, S_t^{b*}, \bar{\theta}\left(\varepsilon_t, \bar{\theta}_{t-1}\right), \varepsilon_t\right) \geq V_t\left(j, S_t^{b*}, \bar{\theta}\left(\varepsilon_t, \bar{\theta}_{t-1}\right), \varepsilon_t\right) \\ \forall j \in\ & F(m_{t-1}); b_t \mid \varepsilon_{t-1}, \bar{\theta}_{t-1}] \bullet \\ & \Pr\left[V_2\left(m_2, S_2^{b*}, \bar{\theta}(\varepsilon_2), \varepsilon_2\right) \geq V_2\left(j, S_2^{b*}, \bar{\theta}(\varepsilon_2), \varepsilon_2\right) \forall j \in F(m_1); b_2\right] \end{aligned} \tag{5.2}$$

where $\bar{\theta}\left(\varepsilon_t, \bar{\theta}_{t-1}\right)$ determines $\bar{\theta}_t$ as a function of ε_t and $\bar{\theta}_{t-1}$ according to:

Case 1: $m_{t-1} \neq 1$, $d_{t-1} < \bar{t}_d$: $x = \bar{\theta}_t$.

Case 2: $m_{t-1} \neq 1$, $d_{t-1} = \bar{t}_d$: $x = \bar{\theta}_{t-1}$because θ $(= \bar{\theta}_{t-1})$ is treated as known at $t-1$.

Case 3: $m_{t-1} = 1$: $x = \bar{\theta}_t$ where $\bar{\theta}_t$ is defined by equation (7.9) when $d_{t-1} = 1$.

Each of the transition probabilities in (5.2) can be written as

$$\begin{aligned} \Pr[m_t = k, b_t = b] &= p_t^b (1-p_t)^{1-b} \frac{1}{\sigma_\eta} \int_{-\infty}^{\infty} \phi\left(\frac{\varepsilon_t - \left[(1-\rho)\bar{\theta}_{t-1} + \rho\varepsilon_{t-1}\right]}{\sigma_\eta}\right) \\ & 1\left[V_t\left(k, S_t^{b*}, \bar{\theta}_t, \varepsilon_t\right) \geq V_t\left(j, S_t^{b*}, \bar{\theta}_t, \varepsilon_t\right) \forall j \in F(m_{t-1})\right] d\varepsilon_t. \end{aligned} \tag{5.3}$$

The path probabilities can be simulated using a modified GHK algorithm. The complete algorithm is described below, and then issues associated with each step are discussed:

a) Compute $\Pr[m_2, b_2]$ using equation (5.3);
b) Draw $\varepsilon_2 \mid m_2$;
c) Compute $\bar{\theta}_2 = \bar{\theta}(\varepsilon_2)$;
d) Set $t = 2$ and $P = \Pr[m_2, b_2]$;
e) Set $t = t + 1$;
f) Compute $\Pr\left[m_t, b_t \mid \varepsilon_{t-1}, \bar{\theta}_{t-1}\right]$;
g) Set $P = P * \Pr\left[m_t, b_t \mid \varepsilon_{t-1}, \bar{\theta}_{t-1}\right]$;
h) Stop if $t = T$;
i) Draw $\varepsilon_t \mid m_t, \varepsilon_{t-1}, \bar{\theta}_{t-1}$;
j) Compute $\bar{\theta}_t = \bar{\theta}\left(\varepsilon_t, \bar{\theta}_{t-1}\right)$;
k) Go to (e).

Issues:

Issue f) Evaluating equation (5.3) can be done in two parts, $p_t^b (1 - p_t)^{1-b}$ and everything else. The first part is completely straightforward. For the second, consider a two dimensional space with one axis for ε_t and the other for $\bar{\theta}_t$. Then, conditional on $\bar{\theta}_{t-1}$, $\bar{\theta}_t = \bar{\theta}\left(\varepsilon_t, \bar{\theta}_{t-1}\right)$ is a curve in that space. Consider dividing up that space into rectangles whose vertices are the grid points where $V_t\left(k, S_t^{b*}, \bar{\theta}_t, \varepsilon_t\right)$ is evaluated. Then any value function, $V_t\left(k, S_t^{b*}, \bar{\theta}_t, \varepsilon_t\right)$, is a weighted average of the value functions at the four adjacent grid points. We need to find all values ε^* where $V_t\left(k, S_t^{b*}, \bar{\theta}\left(\varepsilon^*, \bar{\theta}_{t-1}\right), \varepsilon^*\right) = V_t\left(k', S_t^{b*}, \bar{\theta}\left(\varepsilon^*, \bar{\theta}_{t-1}\right), \varepsilon^*\right)$ for some $k \neq k'$ so that we can determine at which values of ε_t each choice is the best. The most straightforward way to do this would be to consider all (or almost all) values of ε_t and which value function is the greatest. But that would be very expensive. Instead, consider the following algorithm for each rectangle in $\left(\varepsilon_t, \bar{\theta}_t\right)$ that $\left[\varepsilon_t, \bar{\theta}\left(\varepsilon_t, \bar{\theta}_{t-1}\right)\right]$ passes through:

a) Define the vertices of the rectangle as $\left(\check{\varepsilon}_1, \check{\theta}_1\right)$, $\left(\check{\varepsilon}_1, \check{\theta}_2\right)$, $\left(\check{\varepsilon}_2, \check{\theta}_1\right)$, and $\left(\check{\varepsilon}_2, \check{\theta}_2\right)$;
b) Compute $V_t\left(k, S_t^{b*}, \bar{\theta}_t, \varepsilon_t\right)$ $\forall k$ and for vertex;
c) If $V_t\left(k, S_t^{b*}, \bar{\theta}_t, \varepsilon_t\right) > V_t\left(k', S_t^{b*}, \bar{\theta}_t, \varepsilon_t\right)$ $\forall k' \neq k$ at each vertex, then assign

$$\frac{1}{\sigma_\eta} \int_{\check{\varepsilon}_1}^{\check{\varepsilon}_2} \phi\left(\frac{\varepsilon_t - \left[(1-\rho)\,\bar{\theta}_{t-1} + \rho\varepsilon_{t-1}\right]}{\sigma_\eta}\right) d\varepsilon_t \tag{5.4}$$

to choice k (see theorem 1 and proof below for why this is valid) and go to next rectangle;

d) Otherwise, consider new vertices where $\check{\varepsilon}_1$ and $\check{\varepsilon}_2$ do not change and $\check{\theta}_l = \bar{\theta}\left(\check{\varepsilon}_l, \bar{\theta}_{t-1}\right)$ for $l = 1, 2$.

e) Compute $V_t\left(k, S_t^{b*}, \bar{\theta}_t, \varepsilon_t\right)$ $\forall k$ and for vertex;

f) If $V_t\left(k, S_t^{b*}, \bar{\theta}_t, \varepsilon_t\right) > V_t\left(k', S_t^{b*}, \bar{\theta}_t, \varepsilon_t\right)$ $\forall k' \neq k$ at each vertex, then assign the probability in equation (5.4) to choice k (see theorem 2 and proof below for why this is valid) and go to next rectangle;

g) Otherwise, let $\check{\varepsilon}_2 = (\check{\varepsilon}_1 + \check{\varepsilon}_2)/2$ and consider four new vertices, $\left(\check{\varepsilon}_1, \check{\theta}_1\right)$, $\left(\check{\varepsilon}_1, \check{\theta}_2\right)$,$\left(\check{\varepsilon}_2, \check{\theta}_1\right)$, and $\left(\check{\varepsilon}_2, \check{\theta}_2\right)$;

h) Compute $V_t\left(k, S_t^{b*}, \bar{\theta}_t, \varepsilon_t\right)$ $\forall k$ and for vertex;

i) If $V_t\left(k, S_t^{b*}, \bar{\theta}_t, \varepsilon_t\right) > V_t\left(k', S_t^{b*}, \bar{\theta}_t, \varepsilon_t\right)$ $\forall k' \neq k$ at each vertex, then assign the probability in equation (5.4) to choice k (see theorem 2 and proof below for why this is valid), redefine $\check{\varepsilon}_1 = \check{\varepsilon}_2$ and $\check{\varepsilon}_2 =$ the ε value of the rectangle one was in, and go to (e);

Note: There should be a value ε_δ: if $\check{\varepsilon}_2 - \check{\varepsilon}_1 < \varepsilon_\delta$, one interpolates instead of returning to (g).

Theorem 1. *Let* $\left(\check{\varepsilon}_1, \check{\theta}_1\right)$, $\left(\check{\varepsilon}_1, \check{\theta}_2\right)$, $\left(\check{\varepsilon}_2, \check{\theta}_1\right)$, *and* $\left(\check{\varepsilon}_2, \check{\theta}_2\right)$ *be four vertices in* $\left(\varepsilon_t, \bar{\theta}_t\right)$*-space. If* $V_t\left(k, S_t^{b*}, \bar{\theta}_t, \varepsilon_t\right) > V_t\left(k', S_t^{b*}, \bar{\theta}_t, \varepsilon_t\right)$ $\forall k' \neq k$ *at each vertex, then* $V_t\left(k, S_t^{b*}, \bar{\theta}_t, \varepsilon_t\right) > V_t\left(k', S_t^{b*}, \bar{\theta}_t, \varepsilon_t\right)$ $\forall k' \neq k$ $\forall \left(\varepsilon_t, \bar{\theta}_t\right)$ *where*

$$\begin{aligned} \check{\varepsilon}_1 &\leq \varepsilon_t \leq \check{\varepsilon}_2, \\ \check{\theta}_1 &\leq \bar{\theta}_t \leq \check{\theta}_2. \end{aligned} \tag{5.5}$$

Proof. Define $\tilde{V}_t\left(k', S_t^{b*}, \bar{\theta}_t, \varepsilon_t\right) = V_t\left(k, S_t^{b*}, \bar{\theta}_t, \varepsilon_t\right) - V_t\left(k', S_t^{b*}, \bar{\theta}_t, \varepsilon_t\right)$. Then $\tilde{V}_t\left(k', S_t^{b*}, \bar{\theta}_t, \varepsilon_t\right) \geq 0$ at each of the vertices, and $\tilde{V}_t\left(k', S_t^{b*}, \bar{\theta}_t, \varepsilon_t\right)$ is a weighted average of the $\tilde{V}_t\left(k', S_t^{b*}, \bar{\theta}_t, \varepsilon_t\right)$'s at the vertices for any other value of $\left(\varepsilon_t, \bar{\theta}_t\right)$ satisfying equation (5.5). Since a weighted average of positive numbers is positive, the proof follows. ■

Theorem 2. *Let* $\left(\check{\varepsilon}_1, \check{\theta}_1\right)$, $\left(\check{\varepsilon}_1, \check{\theta}_2\right)$, $\left(\check{\varepsilon}_2, \check{\theta}_1\right)$, *and* $\left(\check{\varepsilon}_2, \check{\theta}_2\right)$ *be any four points in* $\left(\varepsilon_t, \bar{\theta}_t\right)$*-space within a rectangle where* $\check{\theta}_l = \bar{\theta}\left(\check{\varepsilon}_l, \bar{\theta}_{t-1}\right)$ *for* $l = 1, 2$*. If* $V_t\left(k, S_t^{b*}, \bar{\theta}_t, \varepsilon_t\right) >$

$V_t\left(k', S_t^{b*}, \bar{\theta}_t, \varepsilon_t\right)$ $\forall k' \neq k$ *at each vertex, then* $V_t\left(k, S_t^{b*}, \bar{\theta}_t, \varepsilon_t\right) > V_t\left(k', S_t^{b*}, \bar{\theta}_t, \varepsilon_t\right)$ $\forall k' \neq k$ $\forall\left(\varepsilon_t, \bar{\theta}_t\right)$ *satisfying equation (5.5).*

Proof. $\tilde{V}_t\left(k', S_t^{b*}, \bar{\theta}_t, \varepsilon_t\right)$ is monotone in both ε_t and $\bar{\theta}_t$ because we are just changing the weights associated with the function monotonely. Since $\tilde{V}_t\left(k', S_t^{b*}, \bar{\theta}_t, \varepsilon_t\right) > 0$ at the points, $\tilde{V}_t\left(k', S_t^{b*}, \bar{\theta}_t, \varepsilon_t\right) > 0$ at any point satisfying equation (5.5).

Issue i) From step (a), we know that m_t is the best choice in the intervals $(\breve{\varepsilon}_{11}, \breve{\varepsilon}_{12})$, $(\breve{\varepsilon}_{21}, \breve{\varepsilon}_{22})$, ..,$(\breve{\varepsilon}_{M1}, \breve{\varepsilon}_{M2})$ [Note: hopefully near the true value of the parameters, $M = 1$]. Since we know the distribution $F\left(\varepsilon_t \mid \varepsilon_{t-1}, \bar{\theta}_{t-1}\right)$ of $\varepsilon_t \mid \varepsilon_{t-1}, \bar{\theta}_{t-1}$, we know the distribution of $\varepsilon_t \mid \varepsilon_{t-1}, \bar{\theta}_{t-1}, m_t$: Let k: $\breve{\varepsilon}_{k1} \leq \varepsilon_t \leq \breve{\varepsilon}_{k2}$. Then

$$\begin{aligned} F\left(\varepsilon_t \mid \varepsilon_{t-1}, \bar{\theta}_{t-1}, m_t\right) &= \{\sum_{j=1}^{k-1}\left[F\left(\breve{\varepsilon}_{j2} \mid \varepsilon_{t-1}, \bar{\theta}_{t-1}\right) - F\left(\breve{\varepsilon}_{j1} \mid \varepsilon_{t-1}, \bar{\theta}_{t-1}\right)\right] + \\ &\quad \left[F\left(\varepsilon_t \mid \varepsilon_{t-1}, \bar{\theta}_{t-1}\right) - F\left(\breve{\varepsilon}_{k1} \mid \varepsilon_{t-1}, \bar{\theta}_{t-1}\right)\right]\}/ \\ &\quad \sum_{j=1}^{M}\left[F\left(\breve{\varepsilon}_{j2} \mid \varepsilon_{t-1}, \bar{\theta}_{t-1}\right) - F\left(\breve{\varepsilon}_{j1} \mid \varepsilon_{t-1}, \bar{\theta}_{t-1}\right)\right]. \end{aligned} \tag{5.6}$$

Then we can simulate $\varepsilon_t \mid m_t, \varepsilon_{t-1}, \bar{\theta}_{t-1}$ as

$$\begin{aligned} \varepsilon_t \mid \varepsilon_{t-1}, \bar{\theta}_{t-1}, m_t = F^{-1}\{&\sum_{j=1}^{M}\left[F\left(\breve{\varepsilon}_{j2} \mid \varepsilon_{t-1}, \bar{\theta}_{t-1}\right) - F\left(\breve{\varepsilon}_{j1} \mid \varepsilon_{t-1}, \bar{\theta}_{t-1}\right)\right] u + \\ F\left(\breve{\varepsilon}_{k1} \mid \varepsilon_{t-1}, \bar{\theta}_{t-1}\right) - &\sum_{j=1}^{k-1}\left[F\left(\breve{\varepsilon}_{j2} \mid \varepsilon_{t-1}, \bar{\theta}_{t-1}\right) - F\left(\breve{\varepsilon}_{j1} \mid \varepsilon_{t-1}, \bar{\theta}_{t-1}\right)\right]\}. \end{aligned} \tag{5.7}$$

■

Unfortunately, because of the ordered nature of the ε error, most (if not all) values of the parameter vector have the property that there are two observations, i and j, where observation i includes a period of cohabitation with zero probability of occuring and j includes a period of marriage with zero probability of occuring. The two conditions together suggest that it would be difficult to find a parameter vector where the likelihood function was positive. One way to fix this problem is to replace $\Pr\left[m_t = k, b_t = b\right]$ in equation (5.3) with

$$p_t^b\left(1 - p_t\right)^{1-b}\left[\left(1 - \varsigma\right)P_t + \varsigma Q_t\right] \tag{5.8}$$

where

$$
\begin{aligned}
P_t &= \frac{1}{\sigma_\eta}\int_{-\infty}^{\infty}\phi\left(\frac{\varepsilon_t-\left[(1-\rho)\bar{\theta}_{t-1}+\rho\varepsilon_{t-1}\right]}{\sigma_\eta}\right)\bullet \\
&\quad 1\left[V_t\left(k,S_t^{b*},\bar{\theta}_t,\varepsilon_t\right)\geq V_t\left(j,S_t^{b*},\bar{\theta}_t,\varepsilon_t\right)\forall j\in F\left(m_{t-1}\right)\right]d\varepsilon_t,
\end{aligned} \tag{5.9}
$$

Q_t is a probability function such that $0 < Q_t$, and ς is a small number. Note that if $\varsigma = 0$, equation (5.8) reduces to equation (5.3). Thus, equation (5.8) should be thought of as an approximation to equation (5.3) that avoids the problems associated with equation (5.3) equalling zero. In particular, Q_t should be chosen so that $0 < Q_t$ and so that the partial derivative of Q_t with respect to the parameter vector is informative about the direction one should change the parameter vector to increase P_t. One can choose Q_t tobe a multinomial logit-type probability to satisfy these conditions. In particular, let

$$
Q_t=\frac{1}{\sigma_\eta}\int_{-\infty}^{\infty}\frac{\exp\left\{V_t\left(j,S_t^{b*},\bar{\theta}_t,\varepsilon_t\right)\right\}}{\sum_{k\in F(m_{t-1})}\exp\left\{V_t\left(k,S_t^{b*},\bar{\theta}_t,\varepsilon_t\right)\right\}}\phi\left(\frac{\varepsilon_t-\left[(1-\rho)\bar{\theta}_{t-1}+\rho\varepsilon_{t-1}\right]}{\sigma_\eta}\right)d\varepsilon_t \tag{5.10}
$$

for an observation where choice j is chosen at time t. Note that $\partial Q_t/\partial V_t\left(j,S_t^{b*},\bar{\theta}_t,\varepsilon_t\right) > 0$ and $\partial Q_t/\partial V_t\left(k,S_t^{b*},\bar{\theta}_t,\varepsilon_t\right) < 0$ for all $k\in F\left(m_{t-1}\right)\neq j$ as desired. Equation (5.10) can be approximated without losing any of its nice properties using Gaussian quadrature over ε_t. Note also that one can count the number of observations where $P_t = 0$, so one can see how dependent the likelihood function is on the inclusion of Q_t. I would suggest fixing ς as a very small number (e.g., 10^{-4}) so that it can be argued that a) this is just a small deviation from the model or alternatively b) it can be argued that there really is a small extreme value error in the model but that its size is so small that it has no appreciable effect on value functions.

The other issue here is that one must draw ε_t conditional on m_t which becomes more complicated. Let the extreme value random variable associated with multinomial logit probability be called ξ. First, let's derive the distribution of ε_t conditional on m_t. The joint density of ε_t and m_t is

$$
\begin{aligned}
\Pr\left[m_t=j,\varepsilon_t\right] &= \frac{1}{\sigma_\eta}\phi\left(\frac{\varepsilon_t-\left[(1-\rho)\bar{\theta}_{t-1}+\rho\varepsilon_{t-1}\right]}{\sigma_\eta}\right) \\
&\quad \{(1-\varsigma)\,1\left[V_t\left(j,S_t^{b*},\bar{\theta}_t,\varepsilon_t\right)\geq V_t\left(k,S_t^{b*},\bar{\theta}_t,\varepsilon_t\right)\forall k\in F\left(m_{t-1}\right)\right]
\end{aligned} \tag{5.11}
$$

$$+\varsigma\frac{\exp\left\{V_t\left(j,S_t^{b*},\bar{\theta}_t,\varepsilon_t\right)\right\}}{\sum_{k\in F(m_{t-1})}\exp\left\{V_t\left(k,S_t^{b*},\bar{\theta}_t,\varepsilon_t\right)\right\}}\}.$$

Consider three cases for the conditional distribution:

1) When P_t in equation (5.9) is relatively large, the multinomial logit term in equation (5.11) becomes insignificant and we are back to the conditional normal distribution for $\varepsilon_t \mid m_t = j$.

2) When, conditional on $\xi = 0$, there exists no value of ε_t where $V_t\left(j,S_t^{b*},\bar{\theta}_t,\varepsilon_t\right) \geq V_t\left(k,S_t^{b*},\bar{\theta}_t,\varepsilon_t\right) \forall k \in F\left(m_{t-1}\right)$, then equation (5.11) simplifies to

$$\Pr\left[m_t = j, \varepsilon_t\right] = \varsigma\frac{1}{\sigma_\eta}\phi\left(\frac{\varepsilon_t - \left[\left(1-\rho\right)\bar{\theta}_{t-1} + \rho\varepsilon_{t-1}\right]}{\sigma_\eta}\right)\frac{\exp\left\{V_t\left(j,S_t^{b*},\bar{\theta}_t,\varepsilon_t\right)\right\}}{\sum_{k\in F(m_{t-1})}\exp\left\{V_t\left(k,S_t^{b*},\bar{\theta}_t,\varepsilon_t\right)\right\}} \tag{5.12}$$

and $\Pr\left[\varepsilon_t < x \mid m_t = j\right]$

$$\begin{aligned}
&= \int_{-\infty}^{x}\frac{1}{\sigma_\eta}\phi\left(\frac{\varepsilon_t - \left[\left(1-\rho\right)\bar{\theta}_{t-1} + \rho\varepsilon_{t-1}\right]}{\sigma_\eta}\right)\frac{\exp\left\{V_t\left(j,S_t^{b*},\bar{\theta}_t,\varepsilon_t\right)\right\}}{\sum_{k\in F(m_{t-1})}\exp\left\{V_t\left(k,S_t^{b*},\bar{\theta}_t,\varepsilon_t\right)\right\}}d\varepsilon/ \\
&\quad \int_{-\infty}^{\infty}\frac{1}{\sigma_\eta}\phi\left(\frac{\varepsilon - \left[\left(1-\rho\right)\bar{\theta}_{t-1} + \rho\varepsilon_{t-1}\right]}{\sigma_\eta}\right)\frac{\exp\left\{V_t\left(j,S_t^{b*},\bar{\theta}_t,\varepsilon\right)\right\}}{\sum_{k\in F(m_{t-1})}\exp\left\{V_t\left(k,S_t^{b*},\bar{\theta}_t,\varepsilon\right)\right\}}d\varepsilon \\
&= \int_{-\infty}^{x}\frac{1}{\sigma_\eta}\phi\left(\frac{\varepsilon - \left[\left(1-\rho\right)\bar{\theta}_{t-1} + \rho\varepsilon_{t-1}\right]}{\sigma_\eta}\right)\frac{\exp\left\{V_t\left(j,S_t^{b*},\bar{\theta}_t,\varepsilon\right)\right\}}{\sum_{k\in F(m_{t-1})}\exp\left\{V_t\left(k,S_t^{b*},\bar{\theta}_t,\varepsilon\right)\right\}}d\varepsilon/Q_t.
\end{aligned} \tag{5.13}$$

One can simulate $\varepsilon_t \mid m_t = j$ by inverting equation (5.13). The way to do this is to:

a) Draw a standard uniform u and multiply it by Q_t;

b) Define

$$\check{P}_k = \sum_{l=1}^{k}\frac{a_l}{\sigma_\eta}\phi\left(\frac{\check{\varepsilon}_l - \left[\left(1-\rho\right)\bar{\theta}_{t-1} + \rho\varepsilon_{t-1}\right]}{\sigma_\eta}\right)\frac{\exp\left\{V_t\left(j,S_t^{b*},\bar{\theta}_t,\check{\varepsilon}_l\right)\right\}}{\sum_{k\in F(m_{t-1})}\exp\left\{V_t\left(k,S_t^{b*},\bar{\theta}_t,\check{\varepsilon}_l\right)\right\}} \tag{5.14}$$

where a_l are the Gaussian quadrature weights;

c) Find the first Gaussian quadrature value $\check{\varepsilon}_k$ such that $\check{P}_k > Q_t u$;

d) Interpolate:

$$\varepsilon_t = \check{\varepsilon}_k + \frac{\left(Q_t u - \check{P}_k\right)\left(\check{\varepsilon}_k - \check{\varepsilon}_{k-1}\right)}{\check{P}_k - \check{P}_{k-1}}. \tag{5.15}$$

3) When P_t in equation (5.9) is positive but small enough so that the multinomial logit term is not insignificant, then a modification of method (2) can be used:

a) Draw a standard uniform u and compare it to

$$\frac{(1-\varsigma)\,P_t}{(1-\varsigma)\,P_t+\varsigma Q_t}. \tag{5.16}$$

b) If u is less than equation (5.16), let

$$u \leftarrow u/\left[\frac{(1-\varsigma)\,P_t}{(1-\varsigma)\,P_t+\varsigma Q_t}\right] \tag{5.17}$$

and then use u in a GHK step;

c) If u is greater than equation (5.16), let

$$u \leftarrow \left[u-\frac{(1-\varsigma)\,P_t}{(1-\varsigma)\,P_t+\varsigma Q_t}\right]/\left[\frac{\varsigma Q_t}{(1-\varsigma)\,P_t+\varsigma Q_t}\right] \tag{5.18}$$

and use u in the algorithm described for case (2) with equations (5.14) and (5.15).

6. Storage and Computation Requirements

$S_t = \left(m_t, m_{t-1}, c_t, d_t, \bar{\theta}_t, \varepsilon_t\right) \Rightarrow$storage requirements of $M = N \cdot 8 \cdot \bar{n}_c \cdot \bar{n}_{ca} \cdot \bar{t}_d \cdot K^2$. If, for example, $N = 240$, $\bar{n}_c = 4$, $\bar{n}_{ca} = 5$, $\bar{t}_d = 5$, and $K = 7$, $M = 9,408,000$. This may be feasible.

7. Old Updating Rule

Let

$$\bar{\varepsilon}_t = \frac{1}{d_t}\sum_{s=t+1-d_t}^{t} \varepsilon_s. \tag{7.1}$$

Then

$$\begin{aligned} E\left(\bar{\varepsilon}_t \mid \theta\right) &= \theta, \qquad (7.2)\\ Var\left(\bar{\varepsilon}_t \mid \theta\right) &= Var\left(\frac{1}{d_t}\sum_{s=t+1-d_t}^{t} \varepsilon_s \mid \theta\right) \end{aligned}$$

$$= \frac{1}{{d_t}^2} E\left[\left(\sum_{s=t+1-d_t}^{t} \varepsilon_s - \theta\right)^2 \mid \theta\right] \quad (7.3)$$

$$= \frac{1}{{d_t}^2} E\left[\left(\sum_{s=t+1-d_t}^{t} \sum_{r=t+1-d_t}^{t} (\varepsilon_s - \theta)(\varepsilon_r - \theta)\right) \mid \theta\right]. \quad (7.4)$$

Note that the

$$E\left((\varepsilon_s - \theta)(\varepsilon_r - \theta) \mid \theta\right)$$
$$= \sigma_\eta^2 \sum_{i=t+1-d_t+\max\{0,s-r\}}^{s-1} \rho^{r-s+2i}.$$

So equation (7.2) becomes

$$\frac{\sigma_\eta^2}{{d_t}^2}\left(\sum_{s=t+1-d_t}^{t} \sum_{r=t+1-d_t}^{t} \sum_{i=t+1-d_t+\max\{0,s-r\}}^{s-1} \rho^{r-s+2i}\right)$$
$$= \frac{\sigma_\eta^2}{{d_t}^2}\left(\sum_{s=t+1-d_t}^{t} \sum_{r=t+1-d_t}^{t} \rho^{r-s} \sum_{i=t+1-d_t+\max\{0,s-r\}}^{s-1} \rho^{2i}\right)$$
$$= \frac{\sigma_\eta^2}{{d_t}^2}\left(\sum_{s=t+1-d_t}^{t} \sum_{r=t+1-d_t}^{t} \rho^{r-s}\rho^{2[t+1-d_t+\max\{0,s-r\}]} \sum_{i=0}^{s-1-[t+1-d_t+\max\{0,s-r\}]} \rho^{2i}\right)$$
$$= \frac{\sigma_\eta^2}{{d_t}^2}\left(\sum_{s=t+1-d_t}^{t} \sum_{r=t+1-d_t}^{t} \rho^{2[t+1-d_t+\max\{0,r-s\}]} \frac{1-\rho^{2[s-t-1+d_t-\max\{0,s-r\}]}}{1-\rho^2}\right)$$
$$= \frac{\sigma_\eta^2}{{d_t}^2}\left(\sum_{s=t+1-d_t}^{t} \sum_{r=t+1-d_t}^{t} \rho^{\max\{0,r-s\}} \frac{1-\rho^{2[s-\max\{0,s-r\}]}}{1-\rho^2}\right)$$
$$= \frac{\sigma_\eta^2}{{d_t}^2}\left(\sum_{s=t+1-d_t}^{t} \sum_{r=t+1-d_t}^{t} \rho^{\max\{0,r-s\}} \frac{1-\rho^{2\min\{s,r\}}}{1-\rho^2}\right)$$

$$\frac{\sigma_\eta^2}{{d_t}^2}\left[\frac{d_t+1}{1-\rho^2} - \frac{1-\rho^{2(d_t+1)}}{(1-\rho^2)^2}\right]$$

Let $\tau_\theta^2 = (\sigma_\theta^2)^{-1}$, $\tau_\eta^2 = \left(\sigma_\eta^2\right)^{-1}$, and $\tau_{d_t}^2 = [Var(\bar{\varepsilon}_t \mid \theta)]^{-1}$. Then

$$\hat{\theta}_t \mid \bar{\varepsilon}_t \sim N\left[\frac{\tau_{d_t}^2 \bar{\varepsilon}_t + \tau_\theta^2 \mu}{\tau_{d_t}^2 + \tau_\theta^2}, \left(\tau_{d_t}^2 + \tau_\theta^2\right)^{-1}\right]. \quad (7.5)$$

Define

$$\bar{\theta}_t = \frac{\tau_{d_t}^2 \bar{\varepsilon}_t + \tau_\theta^2 \mu}{\tau_{d_t}^2 + \tau_\theta^2}, \tag{7.6}$$
$$\bar{\tau}_{d_t}^2 = \tau_{d_t}^2 + \tau_\theta^2.$$

We can now write an updating rule for $\bar{\theta}_t$ and $\bar{\tau}_{d_t}$ in terms of $\bar{\theta}_{t-1}$, $\bar{\tau}_{d_t-1}^2$, $\tau_{d_t}^2$, d_t, and ε_t. Note that

$$\bar{\varepsilon}_t = \frac{(d_t - 1)\,\bar{\varepsilon}_{t-1} + \varepsilon_t}{d_t} \qquad \text{if } d_t > 1, \tag{7.7}$$
$$\bar{\varepsilon}_t = \varepsilon_t \qquad \text{if } d_t = 1,$$

and

$$\bar{\varepsilon}_{t-1} = \left[\bar{\theta}_{t-1}\bar{\tau}_{d_t-1}^2 - \tau_\theta^2 \mu\right] / \tau_{d_t}^2. \tag{7.8}$$

Plugging in gives

$$\bar{\theta}_t = \frac{(d_t - 1)\,\bar{\theta}_{t-1}\bar{\tau}_{d_t-1}^2 + \tau_\theta^2\mu + \tau_{d_t}^2\varepsilon_t}{d_t \bar{\tau}_{d_t}^2} \qquad \text{if } \bar{t}_d > d_t > 1, \tag{7.9}$$
$$\bar{\theta}_t = \frac{\tau_{d_t}^2 \varepsilon_t + \tau_\theta^2 \mu}{\tau_{d_t}^2 + \tau_\theta^2} \qquad \text{if } d_t = 1,$$

and $\bar{\theta}_t = \bar{\theta}_{t-1}$ if $d_t \geq \bar{t}_d$. Note: the derivation for equation (7.9) is

$$\begin{aligned}
\bar{\theta}_t &= \frac{\tau_{d_t}^2 \left[\frac{(d_t-1)\bar{\varepsilon}_{t-1}+\varepsilon_t}{d_t}\right] + \tau_\theta^2\mu}{\tau_{d_t}^2 + \tau_\theta^2} \qquad (7.10)\\
&= \frac{\tau_{d_t}^2 \left[\frac{(d_t-1)\frac{\bar{\theta}_{t-1}\bar{\tau}_{d_t-1}^2 - \tau_\theta^2\mu}{\tau_{d_t}^2} + \varepsilon_t}{d_t}\right] + \tau_\theta^2\mu}{\tau_{d_t}^2 + \tau_\theta^2}\\
&= \frac{\tau_{d_t}^2 \left[(d_t-1)\,\frac{\bar{\theta}_{t-1}\bar{\tau}_{d_t-1}^2 - \tau_\theta^2\mu}{\tau_{d_t}^2} + \varepsilon_t\right] + d_t\tau_\theta^2\mu}{d_t\left(\tau_{d_t}^2 + \tau_\theta^2\right)}\\
&= \frac{(d_t-1)\left(\bar{\theta}_{t-1}\bar{\tau}_{d_t-1}^2 - \tau_\theta^2\mu\right) + \tau_{d_t}^2\varepsilon_t + d_t\tau_\theta^2\mu}{d_t\left(\tau_{d_t}^2 + \tau_\theta^2\right)}\\
&= \frac{(d_t-1)\,\bar{\theta}_{t-1}\bar{\tau}_{d_t-1}^2 + \tau_\theta^2\mu + \tau_{d_t}^2\varepsilon_t}{d_t\bar{\tau}_{d_t}^2}.
\end{aligned}$$