Technical Appendices for “Cohabitation, Marriage, and Divorce in a Model of Match Quality”

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A Bayesian Updating Rule for $\widehat{\theta}_{d_t}$

Let $i'_{d_t} = \begin{pmatrix} 1 & 1 & .. & 1 \end{pmatrix}$. Then

$$\begin{aligned} \hat{\theta}_{d_t} &= \frac{\sigma_\theta^{-2} E\theta + i'_{d_t}\Omega^{-1}\underline{\varepsilon}_{d_t}}{\sigma_\theta^{-2} + i'_{d_t}\Omega^{-1} i_{d_t}} \\ &= \begin{cases} \frac{\sigma_\theta^{-2}E\theta + \left(1-\rho^2\right)\sigma_\eta^{-2}\varepsilon_t}{\sigma_\theta^{-2}+(1-\rho^2)\sigma_\eta^{-2}} & \text{if } d_t = 1 \\ \frac{\sigma_\theta^{-2}E\theta+(1-\rho)\sigma_\eta^{-2}(\varepsilon_t+\varepsilon_{t-1})}{\sigma_\theta^{-2}+2(1-\rho)\sigma_\eta^{-2}} & \text{if } d_t = 2 \\ \frac{\sigma_\theta^{-2}E\theta+\frac{1}{\sigma_\eta^2}\left[(1-\rho)\left(\varepsilon_{t+1-d_t}+\varepsilon_t\right)+(1-\rho)^2\sum_{i=t+2-d_t}^{t-1}\varepsilon_i\right]}{\sigma_\theta^{-2}+\frac{1}{\sigma_\eta^2}\left[2(1-\rho)+(d_t-2)(1-\rho)^2\right]} & \text{if } d_t > 2 \end{cases} \end{aligned} \tag{1}$$

In the Section 3.3, it will be necessary to write $\hat{\theta}_{d_t}$ recursively in terms of ε_t and $\hat{\theta}_{d_t-1}$. Equation (1) is difficult to use for this purpose because the weights for ε_i, $t \geq i \geq t+1-d_t$, vary over i. However, one can write a simple, recursive formula for $\hat{\theta}_{d_t}$ in terms of $\hat{\theta}_{d_t-1}$ and ε_t as

$$\hat{\theta}_{d_t} = \begin{cases} \frac{\sigma_\theta^{-2}E\theta+\sigma_\eta^{-2}\varepsilon_t}{\sigma_\theta^{-2}+\sigma_\eta^{-2}} & \text{if } d_t = 1 \\ \frac{\sigma_\theta^{-2}E\theta+\sigma_\eta^{-2}[(1-\rho)\varepsilon_{t-1}+(1-\rho)\varepsilon_t]}{\sigma_\theta^{-2}+2(1-\rho)\sigma_\eta^{-2}} & \text{if } d_t = 2 \end{cases} \tag{2}$$

and

$$\begin{aligned}\hat{\theta}_{d_t} &= \frac{\sigma_\theta^{-2} E\theta + \frac{1}{\sigma_\eta^2}\left[\varsigma_{d_t-1} + (1-\rho)^2 \varepsilon_{t-1} + (1-\rho)\varepsilon_t\right]}{A_2} \qquad (3)\\ &= \frac{\sigma_\theta^{-2} E\theta + \frac{1}{\sigma_\eta^2}\left\{\sigma_\eta^2 \hat{\theta}_{d_t-1} A_3 - \sigma_\eta^2 \sigma_\theta^{-2} E\theta - (1-\rho)\varepsilon_{t-1} + (1-\rho)^2 \varepsilon_{t-1} + (1-\rho)\varepsilon_t\right\}}{A_2}\\ &= \frac{\sigma_\theta^{-2} E\theta + \frac{1}{\sigma_\eta^2}\left\{\sigma_\eta^2 \hat{\theta}_{d_t-1} A_3 - \sigma_\eta^2 \sigma_\theta^{-2} E\theta - (1-\rho)\varepsilon_{t-1} + (1-\rho)^2 \varepsilon_{t-1} + (1-\rho)\varepsilon_t\right\}}{A_2}\\ &= \frac{\hat{\theta}_{d_t-1} A_3 - \frac{1}{\sigma_\eta^2}\rho(1-\rho)\varepsilon_{t-1} + \frac{1}{\sigma_\eta^2}(1-\rho)\varepsilon_t}{A_2} \text{ if } d_t > 2\end{aligned}$$

where A_2 and A_3 are constants defined as

$$A_\tau = \sigma_\theta^{-2} + \frac{1}{\sigma_\eta^2}\left[2(1-\rho) + (d_t - \tau)(1-\rho)^2\right].$$

It is necessary to write $\hat{\theta}_{d_t}$ recursively in terms of ε_t and $\hat{\theta}_{d_t-1}$. Equation (1) is difficult to use for this purpose because the weights for ε_i, $t \geq i \geq t+1-d_t$, vary over i. However, one can write a simple, recursive formula in terms of a newly defined variable that, together with ε_t, determines $\hat{\theta}_{d_t}$. In particular, define $\varsigma_1 = 0$,

$$\begin{aligned}\varsigma_2 &= (1-\rho)\varepsilon_{t+1-d_t},\\ \varsigma_{d_t} &= (1-\rho)\varepsilon_{t+1-d_t} + (1-\rho)^2 \sum_{i=t+2-d_t}^{t-1} \varepsilon_i \text{ if } d_t > 2.\end{aligned}$$

Then

$$\hat{\theta}_{d_t} = \begin{cases} \frac{\sigma_\theta^{-2} E\theta + \sigma_\eta^{-2}\varepsilon_t}{\sigma_\theta^{-2} + \sigma_\eta^{-2}} & \text{if } d_t = 1\\ \frac{\sigma_\theta^{-2} E\theta + \frac{1}{\sigma_\eta^2}\left[\varsigma_{d_t} + (1-\rho)\varepsilon_t\right]}{\sigma_\theta^{-2} + \frac{1}{\sigma_\eta^2}\left[2(1-\rho) + (d_t-2)(1-\rho)^2\right]} & \text{if } d_t > 1\end{cases} \qquad (4)$$

and

$$\begin{aligned}\varsigma_2 &= \varsigma_1 + (1-\rho)\varepsilon_1, \qquad (5)\\ \varsigma_{d_t} &= \varsigma_{d_t-1} + (1-\rho)^2 \varepsilon_{t-1} \text{ if } d_t > 2.\end{aligned}$$

Thus, at age t, we can define the state variables as ς_{d_t} and ε_t, we can write $\hat{\theta}_{d_t}$ in terms of the state variables (and $E\theta$), and we can define a recursive formula for ς_{d_t}.

Now we can define a recursive rule for $\hat{\theta}_{d_t}$ in terms of the state variables from the period before, $\hat{\theta}_{d_t-1}$ and ε_{t-1}, and ε_t. Define

$$A_\tau = \sigma_\theta^{-2} + \frac{1}{\sigma_\eta^2}\left[2(1-\rho) + (d_t - \tau)(1-\rho)^2\right].$$

From equation (4), we get

$$\varsigma_{d_t-1} = \sigma_\eta^2 \hat{\theta}_{d_t-1} A_3 - \sigma_\eta^2 \sigma_\theta^{-2} E\theta - (1-\rho)\,\varepsilon_{t-1}$$

and, from equations (4) and (5), we get equations (2) and (3).

B Existence and Boundedness Theorem Proof

Proof. (Theorem 4.1)

For this proof, we suppress the following arguments of the value and utility flow functions for brevity of notation: c_{t-1}, d_t, and X_t. Let

$$\begin{aligned}
& V_{t^*}[m_{t^*}, m_{t^*-1}, \widehat{\theta}_{d_{t^*}}, \varepsilon_{t^*}] \\
= \; & E\left[\sum_{t=t^*}^{t^{**}} \beta^{t-t^*}\left(f_t\left(m_t\right) + 1\left(m_t > 1\right)\varepsilon_t\right) \mid \widehat{\theta}_{d_{t^*}}, \varepsilon_{t^*}\right] \\
= \; & \sum_{t=t^*}^{t^{**}} \beta^{t-t^*} f_t\left(m_t\right) + 1\left(m_{t^*} > 1\right)\varepsilon_{t^*} \\
& +1\left(m_{t^*} > 1\right) E\left[\sum_{t=t^*+1}^{t^{**}} \beta^{t-t^*}\varepsilon_t \mid \widehat{\theta}_{d_{t^*}}, \varepsilon_{t^*}\right]
\end{aligned}$$

which implies that

$$\begin{aligned}
& \frac{\partial V_{t^*}[m_{t^*}, m_{t^*-1}, \widehat{\theta}_{d_{t^*}}, \varepsilon_{t^*}]}{\partial \varepsilon_{t^*}} \\
= \; & 1\left(m_{t^*} > 1\right) + 1\left(m_{t^*} > 1\right)\frac{\partial E\left[\sum_{t=t^*+1}^{t^{**}} \beta^{t-t^*}\varepsilon_t \mid \widehat{\theta}_{d_{t^*}}, \varepsilon_{t^*}\right]}{\partial \varepsilon_{t^*}}.
\end{aligned} \tag{6}$$

Note that $m_t = m_{t^*}$ for all $t \geq t^*$ by the definition of t^*. If $m_{t^*} = 1$, then equation (6) is identically zero. If $m_{t^*} > 1$, then, since ε_t is positively serially correlated, equation (6) is positive. But, since $1 \geq \partial E\left[\varepsilon_t \mid \widehat{\theta}_{d_{t^*}}, \varepsilon_{t^*}\right] / \partial \varepsilon_{t^*}$,

$$\begin{aligned}
& \frac{\partial V_{t^*}[m_{t^*}, m_{t^*-1}, \widehat{\theta}_{d_{t^*}}, \varepsilon_{t^*}]}{\partial \varepsilon_{t^*}} \\
= \; & 1\left(m_{t^*} > 1\right) + 1\left(m_{t^*} > 1\right)\sum_{t=t^*+1}^{t^{**}} \beta^{t-t^*}\frac{\partial E\left[\varepsilon_t \mid \widehat{\theta}_{d_{t^*}}, \varepsilon_{t^*}\right]}{\partial \varepsilon_{t^*}} \\
< \; & 1\left(m_{t^*} > 1\right)\frac{1-\beta^{t^{**}+1-t^*}}{1-\beta} = 1\left(m_{t^*} > 1\right) B_{t^*}.
\end{aligned}$$

Also,

$$\frac{\partial V_{t^*}[m_{t^*}, m_{t^*-1}, \widehat{\theta}_{d_{t^*}}, \varepsilon_{t^*}]}{\partial \widehat{\theta}_{d_{t^*}}} = 1\left(m_{t^*} > 1\right)\frac{\partial E\left[\sum_{t=t^*+1}^{t^{**}} \beta^{t-t^*}\varepsilon_t \mid \widehat{\theta}_{d_{t^*}}, \varepsilon_{t^*}\right]}{\partial \widehat{\theta}_{d_{t^*}}}. \tag{7}$$

If $m_{t^*} = 1$, then equation (7) is identically zero. If $m_{t^*} > 1$, then, since the mean of ε_t (for $t > t^*$) is increasing in $\widehat{\theta}_{d_{t^*}}$, equation (7) is positive. But it is bounded by B_{t^*} because $1 \geq \partial E\left[\varepsilon_t \mid \widehat{\theta}_{d_{t^*}}, \varepsilon_{t^*}\right] / \partial\widehat{\theta}_{d_{t^*}}$. Finally,

$$\begin{aligned} V_{t^*}[m_{t^*}, m_{t^*-1}, 0, 0] &= \sum_{t=t^*}^{t^{**}} \beta^{t-t^*} f_t(m_t) + \\ &\quad 1(m_{t^*} > 1) \sum_{t=t^*+1}^{t^{**}} \beta^{t-t^*} E[\varepsilon_t \mid 0, 0] \end{aligned}$$

which is finite because $E[\varepsilon_t \mid 0, 0]$ is finite.

Now assume that there exists a t' such that

$$\begin{aligned} \partial V_t[m_t, m_{t-1}, \widehat{\theta}_{d_t}, \varepsilon_t]/\partial\varepsilon_t &= 0 \text{ and} \\ \partial V_t[m_t, m_{t-1}, \widehat{\theta}_{d_t}, \varepsilon_t]/\partial\widehat{\theta}_{d_t} &= 0 \quad \text{if } m_t = 1; \end{aligned}$$

$$\begin{aligned} B_t &\geq \partial V_t[m_t, m_{t-1}, \widehat{\theta}_{d_t}, \varepsilon_t]/\partial\varepsilon_t \geq 1 \text{ and} \\ B_t &\geq \partial V_t[m_t, m_{t-1}, \widehat{\theta}_{d_t}, \varepsilon_t]/\partial\widehat{\theta}_{d_t} \geq 0 \quad \text{if } m_t > 1 \end{aligned}$$

for all $t > t'$; $t^* - 1$ is such a age. Then

$$\begin{aligned} &\frac{\partial V_{t'}[m_{t'}, m_{t'-1}, \widehat{\theta}_{d_{t'}}, \varepsilon_{t'}]}{\partial\varepsilon_{t'}} \qquad (8) \\ &= 1(m_{t'} > 1) + \beta \frac{\partial E_{c_{t'}, \varepsilon_{t'+1}}\left\{\max_{m_{t'+1} \in F(m_{t'})} (V_{t'+1}[S_{t'+1}]) \mid \widehat{\theta}_{d_{t'}}, \varepsilon_{t'}, c_{t'-1}\right\}}{\partial\varepsilon_{t'}} \end{aligned}$$

from equation (3.6) in the paper. The first term on the right is positive if $m_{t'} > 1$, and it is zero if $m_{t'} = 1$. Consider the second term:

$$\partial V_{t'+1}[S_{t'+1}]/\partial\varepsilon_{t'+1} = 0 \quad \text{if } m_t = 1;$$

$$B_{t'+1} \geq \partial V_{t'+1}[S_{t'+1}]/\partial\varepsilon_{t'+1} \geq 1 \quad \text{if } m_t > 1$$

by assumption. Therefore

$$B_{t'+1} \geq \partial \max_{m_{t'+1} \in F(m_{t'})} (V_{t'+1}[S_{t'+1}, X_{t'+1}])/\partial\varepsilon_{t'+1} \geq 0$$

everywhere the derivative exists. Therefore, since $c_{t'}$ and $\varepsilon_{t'+1}$ conditional on $\varepsilon_{t'}$ are independent, ε_t is positively serially correlated,

$$B_{t'+1} \geq \partial E_{\varepsilon_{t'+1}}\left\{\max_{m_{t'+1} \in F(m_{t'})} (V_{t'+1}[S_{t'+1}]) \mid \widehat{\theta}_{d_{t'}}, \varepsilon_{t'}, c_{t'}\right\}/\partial\varepsilon_{t'} \geq 0$$

and

$$B_{t'+1} \geq \partial E_{\varepsilon_{t'+1}} \left\{ \max_{m_{t'+1}\in F(m_{t'})} \left(V_{t'+1}\left[S_{t'+1}\right]\right) \mid \widehat{\theta}_{d_{t'}}, \varepsilon_{t'}, c_{t'} \right\} / \partial \widehat{\theta}_{d_{t'}} \geq 0$$

for all $c_{t'+1}$ (because the mean of $\varepsilon_{t'+1}$ is increasing in $\widehat{\theta}_{d_{t'}}$) which implies that

$$\begin{aligned} B_{t'+1} &\geq \partial E_{c_{t'},\varepsilon_{t'+1}} \left\{ \max_{m_{t'+1}\in F(m_{t'})} \left(V_{t'+1}\left[S_{t'+1}\right]\right) \mid \widehat{\theta}_{d_{t'}}, \varepsilon_{t'}, c_{t'-1} \right\} / \partial \varepsilon_{t'} \geq 0; \\ B_{t'+1} &\geq \partial E_{c_{t'},\varepsilon_{t'+1}} \left\{ \max_{m_{t'+1}\in F(m_{t'})} \left(V_{t'+1}\left[S_{t'+1}\right]\right) \mid \widehat{\theta}_{d_{t'}}, \varepsilon_{t'}, c_{t'-1} \right\} / \partial \widehat{\theta}_{d_{t'}} \geq 0. \end{aligned}$$

Therefore, since the first term in equation (8) is always unity if $m_t > 1$ and $B_t = 1 + \beta B_{t+1}$,

$$\begin{aligned} \partial V_{t'}\left[S_{t'}\right] / \partial \varepsilon_{t'} &= 0 \text{ and} \\ \partial V_{t'}\left[S_{t'}\right] / \partial \widehat{\theta}_{d_{t'}} &= 0 \quad \text{if } m_t = 1; \end{aligned}$$

$$\begin{aligned} B_{t'} &\geq \partial V_{t'}\left[S_{t'}\right] / \partial \varepsilon_{t'} \geq 1 \text{ and} \\ B_{t'} &\geq \partial V_{t'}\left[S_{t'}\right] / \partial \widehat{\theta}_{d_{t'}} \geq 0 \quad \text{if } m_t > 1. \end{aligned}$$

Also,

$$\begin{aligned} & V_{t'}[m_{t'}, m_{t'-1}0, 0] \\ = \; & f_{t'}\left(m_{t'}\right) - D_{m_{t'-1}} 1\left(m_{t'} = 1\right) \\ & + \beta E_{c_{t'},\varepsilon_{t'+1}} \left\{ \max_{m_{t'+1}\in F(m_{t'})} \left(V_{t'+1}\left[S_{t'+1}\right]\right) \mid 0, 0, c_{t-1} \right\}. \end{aligned}$$

The first term is finite. Consider

$$\begin{aligned} & E_{\varepsilon_{t'+1}} \left\{ \max_{m_{t'+1}\in F(m_{t'})} \left(V_{t'+1}\left[S_{t'+1}\right]\right) \mid 0, 0, c_{t'} \right\} \qquad (9) \\ = \; & \int_{-\infty}^{\infty} \left[\max_{m_{t'+1}\in F(m_{t'})} V_{t'+1}\left[S_{t'+1}\right] \right] \phi\left(\varepsilon_{t'+1} \mid 0, 0\right) d\varepsilon_{t'+1} \end{aligned}$$

where $\phi\left(\varepsilon_{t'+1} \mid \widehat{\theta}_{d_{t'}}, \varepsilon_{t'}\right)$ is the (normal) density of $\varepsilon_{t'+1}$ conditional on $\widehat{\theta}_{d_{t'}}$ and $\varepsilon_{t'}$. Since $\max_{m_{t'+1}\in F(m_{t'})} V_{t'+1}\left[S_{t'+1}, X_{t'+1}\right]$ grows linearly and $\phi\left(\varepsilon_{t'+1} \mid \widehat{\theta}_{d_{t'}}, \varepsilon_{t'}\right)$ declines faster than exponentially, equation (9) is finite implying that

$$E_{c_{t'},\varepsilon_{t'+1}} \left\{ \max_{m_{t'+1}\in F(m_{t'})} \left(V_{t'+1}\left[S_{t'+1}\right]\right) \mid 0, 0, c_{t-1} \right\}$$

is also finite. Thus $V_{t'}[m_{t'}, m_{t'-1}, 0, 0]$ is finite. ■

C Reservation Value Proofs

For the proofs in this section, we suppress the following arguments of the value functions for brevity of notation: c_{t-1}, d_t, and X_t.

Proof. (Theorem 4.2)

$V_t[1,1,\widehat{\theta}_{d_t}\left(\varepsilon_t\right),\varepsilon_t]$ is finite. From Theorem 4.1, $\partial V_{t'}\left[S_{t'}\right]/\partial\varepsilon_{t'}=0$ if $m_t=1$. Then, also by Theorem 4.1, since $\partial V_{t'}\left[S_{t'}\right]/\partial\varepsilon_{t'}\geq 1$ if $m_t>1$, the result follows. ■

Proof. (Theorem 4.3)

$V_t[1,m_{t-1},\widehat{\theta}_{d_t},\varepsilon_t]$ is finite for all $\widehat{\theta}_{d_t}$ and ε_t, and $\partial V_t[m_{t-1},m_{t-1},\widehat{\theta}_{d_t},\varepsilon_t]/\partial\varepsilon_t\geq 1$. The result follows. ■

Proof. (Theorems 4.4 and 4.5)

One can show that

$$V_{t^*}\left[3,m_{t^*-1},\widehat{\theta}_{d_{t^*}},\varepsilon_{t^*}\right]-V_{t^*}\left[2,m_{t^*-1},\widehat{\theta}_{d_{t^*}},\varepsilon_{t^*}\right]=0$$

because $f_t\left(3\right)=f_t\left(2\right)$ and no choices can be made beyond t^*. Assume that there exists a t such that $V_{t'}\left[2,m_{t'-1},\widehat{\theta}_{d_{t'}},\varepsilon_{t'}\right]\geq V_{t'}\left[3,m_{t'-1},\widehat{\theta}_{d_{t'}},\varepsilon_{t'}\right]$ for all $t^{'}>t$ (except when $m_{t'-1}=3$). Then

$$\begin{aligned}&V_t\left[3,m_{t-1},\widehat{\theta}_{d_t},\varepsilon_t\right]-V_t\left[2,m_{t-1},\widehat{\theta}_{d_t}\varepsilon_t\right] \qquad (10)\\ =\;&\beta E_{c_t,\varepsilon_{t+1}}\left\{\max_{m_{t+1}\in F(3)}\left(V_{t+1}\left[S_{t+1}\right]\right)\mid\widehat{\theta}_{d_t}\varepsilon_t,c_{t-1}\right\}\\ &-\beta E_{c_t,\varepsilon_{t+1}}\left\{\max_{m_{t+1}\in F(2)}\left(V_{t+1}\left[S_{t+1}\right]\right)\mid\widehat{\theta}_{d_t},\varepsilon_t,c_{t-1}\right\}.\end{aligned}$$

Now one can consider all possible values of ε_{t+1} subject to the condition that $V_t\left[2,m_t,\widehat{\theta}_{d_{t+1}},\varepsilon_{t+1}\right]\geq V_{t+1}\left[3,m_t,\widehat{\theta}_{d_{t+1}},\varepsilon_{t+1}\right]$ and show that, for each value, the relevant part of equation (10) is nonnegative and, for some parts, it is negative. Thus equation (10) is negative. The proof for Theorem 4.5 has the same structure. ■

Proof. (Theorem 4.6)

At t^*,

$$\begin{aligned}V_{t^*}\left[3,3,\widehat{\theta}_{d_{t^*}},\varepsilon_{t^*}\right]-V_{t^*}\left[2,2,\widehat{\theta}_{d_{t^*}},\varepsilon_{t^*}\right]\quad&=\\ \sum_{t=t^*}^{t^{**}}\beta^{(t-t^*)}\left[f_t\left(3\right)-f_t\left(2\right)\right]\quad&>\quad 0.\end{aligned}$$

At $\varepsilon^*_{t^*}\left(1,2\right)$,

$$\begin{aligned}&\left\{V_{t^*}\left[3,3,\widehat{\theta}_{d_{t^*}},\varepsilon_{t^*}\right]-V_{t^*}\left[1,3,\widehat{\theta}_{d_{t^*}},\varepsilon_{t^*}\right]\right\}\\ &-\left\{V_{t^*}\left[2,2,\widehat{\theta}_{d_{t^*}},\varepsilon_{t^*}\right]-V_{t^*}\left[1,2,\widehat{\theta}_{d_{t^*}},\varepsilon_{t^*}\right]\right\}\\ =\;&V_{t^*}\left[3,3,\widehat{\theta}_{d_{t^*}},\varepsilon_{t^*}\right]-V_{t^*}\left[2,2,\widehat{\theta}_{d_{t^*}},\varepsilon_{t^*}\right]+D_3-D_2>0\end{aligned}$$

which implies that, at $\varepsilon^*_{t^*}(1,2)$, $V_{t^*}\left[3,3,\widehat{\theta}_{d_{t^*}},\varepsilon_{t^*}\right] - V_{t^*}\left[1,3,\widehat{\theta}_{d_{t^*}},\varepsilon_{t^*}\right] > 0 \Rightarrow$ $\varepsilon^*_{t^*}(1,3) < \varepsilon^*_{t^*}(1,2)$ (because $\partial V_{t^*}\left[3,3,\widehat{\theta}_{d_{t^*}},\varepsilon_{t^*}\right]/\partial\varepsilon_{t^*} > 0$). At t^*-1,

$$\begin{aligned}
& V_{t^*-1}\left[3,3,\widehat{\theta}_{d_{t^*-1}},\varepsilon_{t^*-1}\right] - V_{t^*-1}\left[2,2,\widehat{\theta}_{d_{t^*-1}},\varepsilon_{t^*-1}\right] \\
= \;& f_{t^*-1}(3) - f_{t^*-1}(2) + \\
& \beta E \max_{m\neq 2} V_{t^*}\left[m,3,\widehat{\theta}_{d_{t^*-1}},\varepsilon_{t^*}\right] - \beta E \max_{m} V_{t^*}\left[m,2,\widehat{\theta}_{d_{t^*-1}},\varepsilon_{t^*}\right] \\
> \;& f_{t^*-1}(3) - f_{t^*-1}(2) + \beta\left(D_2 - D_3\right)\int_{-\infty}^{\varepsilon^*_{t^*}(1,2)} d\Phi\left(\varepsilon_{t^*}\right) \\
& +\beta\int_{\varepsilon^*_{t^*}(1,2)}^{\infty}\left\{V_{t^*}\left[3,3,\widehat{\theta}_{d_{t^*-1}},\varepsilon_{t^*}\right] - V_{t^*}\left[2,2,\widehat{\theta}_{d_{t^*-1}},\varepsilon_{t^*}\right]\right\} d\Phi\left(\varepsilon_{t^*}\right) \\
> \;& \beta\left(D_2 - D_3\right),
\end{aligned}$$

and

$$\begin{aligned}
& \left\{V_{t^*-1}\left[3,3,\widehat{\theta}_{d_{t^*-1}},\varepsilon_{t^*-1}\right] - V_{t^*-1}\left[1,3,\widehat{\theta}_{d_{t^*-1}},\varepsilon_{t^*-1}\right]\right\} \\
& -\left\{V_{t^*-1}\left[2,2,\widehat{\theta}_{d_{t^*-1}},\varepsilon_{t^*-1}\right] - V_{t^*-1}\left[1,2,\widehat{\theta}_{d_{t^*-1}},\varepsilon_{t^*-1}\right]\right\} \\
= \;& V_{t^*-1}\left[3,3,\widehat{\theta}_{d_{t^*-1}},\varepsilon_{t^*-1}\right] - V_{t^*-1}\left[2,2,\widehat{\theta}_{d_{t^*-1}},\varepsilon_{t^*-1}\right] + D_3 - D_2 \\
> \;& \beta\left(D_2 - D_3\right) + D_3 - D_2 \\
= \;& \left(D_3 - D_2\right)\left(1-\beta\right) > 0
\end{aligned}$$

which implies that, at $\varepsilon^*_{t^*-1}(1,2)$, $V_{t^*-1}\left[3,3,\widehat{\theta}_{d_{t^*-1}},\varepsilon_{t^*-1}\right] - V_{t^*-1}\left[1,3,\widehat{\theta}_{d_{t^*-1}},\varepsilon_{t^*-1}\right] >$ $0 \Rightarrow \varepsilon^*_{t^*-1}(1,3) < \varepsilon^*_{t^*-1}(1,2)$ (because $\partial V_{t^*-1}\left[3,3,\widehat{\theta}_{d_{t^*-1}},\varepsilon_{t^*-1}\right]/\partial\varepsilon_{t^*-1} > 0$).

Now assume that $\exists\, t : V_{t'}\left[3,3,\widehat{\theta}_{d_{t'}},\varepsilon_{t'}\right] - V_{t'}\left[2,2,\widehat{\theta}_{d_{t'}},\varepsilon_{t'}\right] > \beta\left(D_2 - D_3\right)$ and $\varepsilon^*_{t'}(1,3) < \varepsilon^*_{t'}(1,2) \quad \forall t' > t$. Then

$$\begin{aligned}
& V_t\left[3,3,\widehat{\theta}_{d_t},\varepsilon_t\right] - V_t\left[2,2,\widehat{\theta}_{d_t},\varepsilon_t\right] \\
= \;& f_t(3) - f_t(2) \\
& +\beta E \max_{m\neq 2} V_{t+1}\left[m,3,\widehat{\theta}_{d_{t+1}},\varepsilon_{t+1}\right] - \beta E \max_{m} V_{t+1}\left[m,2,\widehat{\theta}_{d_{t+1}},\varepsilon_{t+1}\right] \\
> \;& f_t(3) - f_t(2) + \beta\left[\left(D_2 - D_3\right)\int_{-\infty}^{\varepsilon^*_{t+1}(1,2)} d\Phi\left(\varepsilon_t\right) + \right. \\
& \left.\int_{\varepsilon^*_{t+1}(1,2)}^{\infty}\left\{V_{t+1}\left[3,3,\widehat{\theta}_{d_{t+1}},\varepsilon_{t+1}\right] - V_{t+1}\left[2,2,\widehat{\theta}_{d_{t+1}},\varepsilon_{t+1}\right]\right\} d\Phi\left(\varepsilon_{t+1}\right)\right] \\
> \;& \beta\left[\left(D_2 - D_3\right)\int_{-\infty}^{\varepsilon^*_{t+1}(1,2)} d\Phi\left(\varepsilon_t\right) + \beta\left(D_2 - D_3\right)\int_{\varepsilon^*_{t+1}(1,2)}^{\infty} d\Phi\left(\varepsilon_{t+1}\right)\right] \\
> \;& \beta\left(D_2 - D_3\right),
\end{aligned}$$

and

$$\begin{aligned}
& \left\{V_t\left[3,3,\widehat{\theta}_{d_t},\varepsilon_t\right]-V_t\left[1,3,\widehat{\theta}_{d_t},\varepsilon_t\right]\right\}- \\
& \left\{V_t\left[2,2,\widehat{\theta}_{d_t},\varepsilon_t\right]-V_t\left[1,2,\widehat{\theta}_{d_t},\varepsilon_t\right]\right\} \\
= \; & V_t\left[3,3,\widehat{\theta}_{d_t},\varepsilon_t\right]-V_t\left[2,2,\widehat{\theta}_{d_t},\varepsilon_t\right]+D_3-D_2 \\
> \; & \beta\left(D_2-D_3\right)+D_3-D_2 \\
= \; & \left(D_3-D_2\right)(1-\beta)>0
\end{aligned}$$

which implies that, at $\varepsilon_t^*(1,2)$, $V_t\left[3,3,\widehat{\theta}_{d_t},\varepsilon_t\right]-V_t\left[1,3,\widehat{\theta}_{d_t},\varepsilon_t\right]>0 \Rightarrow \varepsilon_t^*(1,3)<\varepsilon_t^*(1,2)$ (because $\partial V_t\left[3,3,\widehat{\theta}_{d_t},\varepsilon_t\right]/\partial\varepsilon_t>0$). ■

Proof. (Theorem 4.7)

Let $m_t>1$. Then, by equation (3.6) in the paper,

$$\begin{aligned}
& V_t[m_t,m_{t-1},\widehat{\theta}_{d_t},\varepsilon_t] \\
= \; & f_t\left(m_t\right)+1\left(m_t>1\right)\varepsilon_t+ \\
& \beta E_{c_t}[\int_{\varepsilon_{t+1}^*(1,m_t)}^{\infty}\left\{\max_{\substack{m_{t+1}\in F(m_t)\\ m_{t+1}\neq 1}} V_{t+1}\left[S_{t+1}\right]\right\}\phi\left(\varepsilon_{t+1}\mid\bar{\theta}_t,\varepsilon_t\right)d\varepsilon_{t+1}+ \\
& \Phi\left(\varepsilon_{t+1}^*\left(1,m_t\right)\mid\bar{\theta}_t,\varepsilon_t\right)V_t[1,m_t,0,0]\mid c_{t-1}],
\end{aligned}$$

and its partial derivative is

$$\begin{aligned}
& \partial V_t[m_t,m_{t-1},\widehat{\theta}_{d_t},\varepsilon_t]/\partial\varepsilon_t=1+ \\
& \beta E_{c_t}[\int_{\varepsilon_{t+1}^*(1,m_t)}^{\infty}\left\{\max_{\substack{m_{t+1}\in F(m_t)\\ m_{t+1}\neq 1}} V_{t+1}\left[S_{t+1}\right]\right\}\frac{\partial\phi\left(\varepsilon_{t+1}\mid\bar{\theta}_t,\varepsilon_t\right)}{\partial\varepsilon_t}d\varepsilon_{t+1}+ \qquad (11) \\
& \frac{\partial\Phi\left(\varepsilon_{t+1}^*\left(1,m_t\right)\mid\widehat{\theta}_{d_t},\varepsilon_t\right)}{\partial\varepsilon_t}V_{t+1}[1,m_t,0,0]\mid c_{t-1}].
\end{aligned}$$

Note:

$$\frac{\partial\phi\left(\varepsilon_{t+1}\mid\widehat{\theta}_{d_t},\varepsilon_t\right)}{\partial\varepsilon_t}=-\rho\frac{\partial\phi\left(\varepsilon_{t+1}\mid\widehat{\theta}_{d_t},\varepsilon_t\right)}{\partial\varepsilon_{t+1}}.$$

Thus, the integral in the second line of equation (11) can be written as

$$-\rho\int_{\varepsilon_{t+1}^*(1,m_t)}^{\infty}\left\{\max_{\substack{m_{t+1}\in F(m_t)\\ m_{t+1}\neq 1}} V_{t+1}\left[S_{t+1}\right]\right\}\frac{\partial\phi\left(\varepsilon_{t+1}\mid\widehat{\theta}_{d_t},\varepsilon_t\right)}{\partial\varepsilon_{t+1}}d\varepsilon_{t+1}$$

which, through integration by parts, can be written as

$$\rho \int_{\varepsilon^*_{t+1}(1,m_t)}^{\infty} \frac{\partial}{\partial \varepsilon_{t+1}} \left\{ \max_{\substack{m_{t+1}\in F(m_t) \\ m_{t+1}\neq 1}} V_{t+1}\left[m_{t+1}, m_t, \widehat{\theta}_{d_{t+1}}, \varepsilon_{t+1}\right] \right\} \bullet \tag{12}$$

$$\phi\left(\varepsilon_{t+1} \mid \widehat{\theta}_{d_t}, \varepsilon_t\right) d\varepsilon_{t+1}$$

$$+\rho \left\{ \max_{\substack{m_{t+1}\in F(m_t) \\ m_{t+1}\neq 1}} V_{t+1}\left[m_{t+1}, m_t, \widehat{\theta}_{d_{t+1}}, \varepsilon^*_{t+1}(1, m_t)\right] \right\} \bullet$$

$$\phi\left(\varepsilon^*_{t+1}(1, m_t) \mid \widehat{\theta}_{d_t}, \varepsilon_t\right)$$

The last term in equation (11) can be written as

$$-\rho\phi\left(\varepsilon^*_{t+1}(1, m_t) \mid \widehat{\theta}_{d_t}, \varepsilon_t\right) V_{t+1}[1, m_t, 0, 0]. \tag{13}$$

Combining equations (12) and (13) leads to

$$\rho \int_{\varepsilon^*_{t+1}(1,m_t)}^{\infty} \frac{\partial}{\partial \varepsilon_{t+1}} \left\{ \max_{\substack{m_{t+1}\in F(m_t) \\ m_{t+1}\neq 1}} V_{t+1}\left[m_{t+1}, m_t, \widehat{\theta}_{d_{t+1}}, \varepsilon_{t+1}\right] \right\} \phi\left(\varepsilon_{t+1} \mid \widehat{\theta}_{d_t}, \varepsilon_t\right) d\varepsilon_{t+1} \tag{14}$$

(the second term in equation (12) cancels with equation (13) by the definition of $\varepsilon^*_{t+1}(1, m_t)$). The integrand in equation (14) is the same for $m_t = 2$ or $m_t = 3$, but, since $\varepsilon^*_{t+1}(1, 3) < \varepsilon^*_{t+1}(1, 2)$,

$$\frac{\partial V_t[3, m_{t-1}, \widehat{\theta}_{d_t}, \varepsilon_t]}{\partial \varepsilon_t} - \frac{\partial V_t[2, m_{t-1}, \widehat{\theta}_{d_t}, \varepsilon_t]}{\partial \varepsilon_t} =$$

$$\rho \int_{\varepsilon^*_{t+1}(1,3)}^{\varepsilon^*_{t+1}(1,2)} \frac{\partial}{\partial \varepsilon_{t+1}} \left\{ \max_{\substack{m_{t+1}\in F(m_t) \\ m_{t+1}\neq 1}} V_{t+1}\left[m_{t+1}, m_t, \widehat{\theta}_{d_{t+1}}, \varepsilon_{t+1}\right] \right\} \phi\left(\varepsilon_{t+1} \mid \widehat{\theta}_{d_t}, \varepsilon_t\right) d\varepsilon_{t+1} > 0.$$

By a similar argument, we can show that

$$\frac{\partial V_t[3, m_{t-1}, \widehat{\theta}_{d_t}, \varepsilon_t]}{\partial \widehat{\theta}_{d_t}} - \frac{\partial V_t[2, m_{t-1}, \widehat{\theta}_{d_t}, \varepsilon_t]}{\partial \widehat{\theta}_{d_t}} > 0.$$

Therefore, since $\partial\widehat{\theta}_{d_t}/\partial\varepsilon_t > 0$ when $d_t = 1$, the result follows. ■

Proof. (Theorem 4.8)

a) This follows from the fact that $\partial V_t[2, m_{t-1}, \widehat{\theta}_{d_t}, \varepsilon_t]/\partial\varepsilon_t$ is bounded from below and above and that $\partial V_t[1, m_{t-1}, \widehat{\theta}_{d_t}, \varepsilon_t]/\partial\varepsilon_t = 0$.

b) This follows from the same argument.

c) Consider

$$
\begin{aligned}
& V_{t^*}\left[3, m_{t^*-1}, \widehat{\theta}_{d_{t^*}}, \varepsilon_{t^*}\right] - V_{t^*}\left[2, m_{t^*-1}, \widehat{\theta}_{d_{t^*}}, \varepsilon_{t^*}\right] \\
= & \sum_{t=t^*}^{t^{**}} \beta^{t-t^*}\left(f_t'(3) - f_t'(2)\right) > 0
\end{aligned}
$$

by assumption. There exists a $\varepsilon_{t^*}^{**}(3, m_{t^*-1})$ such that $V_{t^*}\left[3, m_{t^*-1}, \widehat{\theta}_{d_{t^*}}, \varepsilon_{t^*}\right] >$ $V_{t^*}\left[1, m_{t^*-1}, \widehat{\theta}_{d_{t^*}}, \varepsilon_{t^*}\right]$ $\forall \varepsilon_{t^*} > \varepsilon_{t^*}^{**}(3, m_{t^*-1})$ and $V_{t^*}\left[3, m_{t^*-1}, \widehat{\theta}_{d_{t^*}}, \varepsilon_{t^*}\right] < V_{t^*}\left[1, m_{t^*-1}, \widehat{\theta}_{d_{t^*}}, \varepsilon_{t^*}\right]$ $\forall \varepsilon_{t^*} < \varepsilon_{t^*}^{**}(3, m_{t^*-1})$ by the previous arguments about bounded slopes.

Now assume that there is some t' and $\varepsilon_{t'+1}^{**}(3, m_{t'})$ for $m_{t'} = 1, 2$ such that $V_{t'+1}\left[3, 2, \widehat{\theta}_{d_{t'+1}}, \varepsilon_{t'+1}\right] > \max_{m=1}^{2} V_{t'+1}\left[m, 2, \widehat{\theta}_{d_{t'+1}}, \varepsilon_{t'+1}\right]$ $\forall \varepsilon_{t'+1} > \varepsilon_{t'+1}^{**}(3, 2)$ and $V_{t'+1}\left[3, 2, \widehat{\theta}_{d_{t'+1}}, \varepsilon_{t'+1}\right] < \max_{m=1}^{2} V_{t'+1}\left[m, 2, \widehat{\theta}_{d_{t'+1}}, \varepsilon_{t'+1}\right]$ $\quad \forall \varepsilon_{t'+1} < \varepsilon_{t'+1}^{**}(3, 2)$. Then

$$
V_{t'}\left[3, m_{t'-1}, \widehat{\theta}_{d_{t'}}, \varepsilon_{t'}\right] - V_{t'}\left[2, m_{t'-1}, \widehat{\theta}_{d_{t'}}, \varepsilon_{t'}\right] =
$$

$$
f_{t'}(3) - f_{t'}(2) + \beta \int_{-\infty}^{\varepsilon_{t'+1}^{**}(3,2)} [\max_{m_{t'+1} \in F(3)} \left(V_{t'+1}\left[m_{t'+1}, 3, \widehat{\theta}_{d_{t'+1}}, \varepsilon_{t'+1}\right]\right) -
$$

$$
\max_{m_{t'+1} \in F(2)} \left(V_{t'+1}\left[m_{t'+1}, 2, \widehat{\theta}_{d_{t'+1}}, \varepsilon_{t'+1}\right]\right) \mid \widehat{\theta}_{d_{t'}}, \varepsilon_{t'}, c_{t'-1}] \phi\left(\varepsilon_{t'+1} \mid \widehat{\theta}_{d_{t'}}, \varepsilon_{t'}\right) d\varepsilon_{t'+1} + \tag{15}
$$

$$
\beta \int_{\varepsilon_{t'+1}^{**}(3,2)}^{\infty} [\max_{m_{t'+1} \in F(3)} \left(V_{t'+1}\left[m_{t'+1}, 3, \widehat{\theta}_{d_{t'+1}}, \varepsilon_{t'+1}\right]\right) -
$$

$$
\max_{m_{t+1} \in F(2)} \left(V_{t'+1}\left[m_{t'+1}, 2, \widehat{\theta}_{d_{t'+1}}, \varepsilon_{t'+1}\right]\right) \mid \widehat{\theta}_{d_{t'}}, \varepsilon_{t'}, c_{t'-1}] \phi\left(\varepsilon_{t'+1} \mid \widehat{\theta}_{d_{t'}}, \varepsilon_{t'}\right) d\varepsilon_{t'+1}.
$$

Note that $\varepsilon_{t'+1}^{**}(3, 2) \geq \varepsilon_{t'+1}^{*}(1, 2)$ because $\partial V_{t'+1}\left[m_{t'+1}, 2, \widehat{\theta}_{d_{t'+1}}, \varepsilon_{t'+1}\right] / \partial \varepsilon_{t'+1}$ ≥ 1. Therefore, for any $\varepsilon_{t'+1} > \varepsilon_{t'+1}^{**}(3, 2)$, the agent will choose marriage whether $m_{t'-1}$ was 2 or 3. Therefore, the last integral in equation (15) is equal to zero. The integrand of the first integral is bounded from above because $V_{t'+1}[m_{t'+1}, 3, 0, 0]$ is finite, the partial derivatives of $V_{t'+1}\left[m_{t'+1}, 3, \widehat{\theta}_{d_{t'+1}}, \varepsilon_{t'+1}\right]$ with respect to $\bar{\theta}_{t'+1}$ and $\varepsilon_{t'+1}$ are bounded from below and above and $\phi\left(\varepsilon_{t'+1} \mid \widehat{\theta}_{d_{t'}}, \varepsilon_{t'}\right)$

is declining faster than exponentially as $\varepsilon_{t'+1} \to -\infty$. Let the bound be Γ. Then the first integral can be written as

$$\beta\Gamma\Phi\left(\varepsilon^{**}_{t'+1}(3,2) \mid \widehat{\theta}_{d_{t'}}, \varepsilon_{t'}\right) \to 0 \text{ as } \varepsilon_{t'} \to \infty.$$

Thus, since the first term in equation (15) is positive, there exists a $\varepsilon_{t'}$ where $V_{t'}\left[3, m_{t'-1}, \widehat{\theta}_{d_{t'}}, \varepsilon\right] - V_{t'}\left[2, m_{t'-1}, \widehat{\theta}_{d_{t'}}, \varepsilon\right] > 0$ for all $\varepsilon > \varepsilon_{t'}$. But there is an $\varepsilon_{t'}$ small enough such that $V_{t'}\left[3, m_{t'-1}, \widehat{\theta}_{d_{t'}}, \varepsilon\right] - V_{t'}\left[2, m_{t'-1}, \widehat{\theta}_{d_{t'}}, \varepsilon\right] < 0$ for all $\varepsilon < \varepsilon_{t'}$ because of the bounded partial derivative of $V_{t'}\left[3, m_{t'-1}, \widehat{\theta}_{d_{t'}}, \varepsilon\right]$. Thus, because $\partial V_{t'}\left[3, m_{t'-1}, \widehat{\theta}_{d_{t'}}, \varepsilon\right]/\partial\varepsilon > \partial V_{t'}\left[2, m_{t'-1}, \widehat{\theta}_{d_{t'}}, \varepsilon\right]/\partial\varepsilon$, there exists a $\varepsilon^{**}_{t'}(3, m_{t'-1})$ for $m_{t'-1} = 1, 2$. ■

Proof. (Theorem 4.9)

>From Theorem 4.4, we know that if $f_t(3) = f_t(2)$, then $V_t\left[2, m_{t-1}, \widehat{\theta}_{d_t}, \varepsilon_t\right] > V_t\left[3, m_{t-1}, \widehat{\theta}_{d_t}, \varepsilon_t\right]$ and that there exists $\underline{\varepsilon}^{**}_t(2, m_{t-1})$ such that $V_t\left[2, m_{t-1}, \widehat{\theta}_{d_t}, \varepsilon_t\right] > V_t\left[1, m_{t-1}, \widehat{\theta}_{d_t}, \varepsilon_t\right]$ for all $\varepsilon_t > \underline{\varepsilon}^{**}_t(2, m_{t-1})$. We also know that if $f_t(3) > f_t(2)$, there is some $\overline{\varepsilon}^{**}_t(2, m_{t-1})$ such that $V_t\left[2, m_{t-1}, \widehat{\theta}_{d_t}, \varepsilon_t\right] < V_t\left[3, m_{t-1}, \widehat{\theta}_{d_t}, \varepsilon_t\right]$ for all $\varepsilon_t > \overline{\varepsilon}^{**}_t(2, m_{t-1})$.

The arguments in part (b) of the proof of Theorem 4.8 imply that $\varepsilon^{**}_t(3, m_{t-1})$ is a continuous function of $\Delta f_t = f_t(3) - f_t(2)$, and we already showed that $\varepsilon^{**}_t(3, m_{t-1}) = \infty$ when $\Delta f_t = 0$. Thus, for any $D_3 - D_2 > 0$ and any $\varepsilon^{**}_t(3, m_{t-1})$, there will be a positive Δf_t small enough. Set $\overline{\varepsilon}^{**}_t(2, m_{t-1}) = \varepsilon^{**}_t(3, m_{t-1})$ and $\underline{\varepsilon}^{**}_t(2, m_{t-1})$ equal to the value of ε_t where $V_t\left[2, m_{t-1}, \widehat{\theta}_{d_t}, \varepsilon_t\right] = V_t\left[1, m_{t-1}, \widehat{\theta}_{d_t}, \varepsilon_t\right]$. ■

D Marital Instability Proofs

For the proofs in this section, we suppress the following arguments of the value functions for brevity of notation: c_{t-1}, d_t, and X_t. The following lemma is used in the proof.

Lemma 1 *Let* $(\varsigma_1, \varsigma_2, \varsigma_3)' \sim N[0, \Omega]$ *where* $\Omega_{jk} > 0 \;\forall\; j, k$.[1] *Then*

$$\Pr[\varsigma_3 < \alpha_{31} \mid \alpha_{11} < \varsigma_1 < \alpha_{12}, \alpha_{22} < \varsigma_2] > \Pr[\varsigma_3 < \alpha_{31} \mid \alpha_{12} < \varsigma_1, \alpha_{22} < \varsigma_2]$$

and

$$\Pr[\varsigma_3 < \alpha_{31} \mid \alpha_{11} < \varsigma_1 < \alpha_{12}] > \Pr[\varsigma_3 < \alpha_{31} \mid \alpha_{12} < \varsigma_1]$$

---

[1] Note that $(\varsigma_1, \varsigma_2, \varsigma_3)'$ is totally unrelated to the ς variables in Appendix A.

Proof.

$$\begin{aligned}
& \Pr\left[\varsigma_3 < \alpha_{31} \mid \varsigma_1, \alpha_{22} < \varsigma_2\right] \\
= \; & \frac{\int_{\alpha_{22}}^{\infty} \int_{-\infty}^{\alpha_{31}} \phi_{3|12}\left(\varsigma_3 \mid \varsigma_1, \varsigma_2\right) \phi_{2|1}\left(\varsigma_2 \mid \varsigma_1\right) \phi_1\left(\varsigma_1\right) d\varsigma_3 d\varsigma_2}{\int_{\alpha_{22}}^{\infty} \int_{-\infty}^{\infty} \phi_{3|12}\left(\varsigma_3 \mid \varsigma_1, \varsigma_2\right) \phi_{2|1}\left(\varsigma_2 \mid \varsigma_1\right) \phi_1\left(\varsigma_1\right) d\varsigma_3 d\varsigma_2} \\
= \; & \frac{\int_{\alpha_{22}}^{\infty} \Phi_{3|12}\left(\alpha_{31} \mid \varsigma_1, \varsigma_2\right) \phi_{2|1}\left(\varsigma_2 \mid \varsigma_1\right) d\varsigma_2}{\int_{\alpha_{22}}^{\infty} \phi_{2|1}\left(\varsigma_2 \mid \varsigma_1\right) d\varsigma_2}
\end{aligned}$$

and

$$\begin{aligned}
& \frac{\partial \Pr\left[\varsigma_3 < \alpha_{31} \mid \varsigma_1, \alpha_{22} < \varsigma_2\right]}{\partial \varsigma_1} \\
= \; & \int_{\alpha_{22}}^{\infty} \left[\frac{\phi_{2|1}\left(\varsigma_2 \mid \varsigma_1\right)}{\int_{\alpha_{22}}^{\infty} \phi_{2|1}\left(\varsigma_2 \mid \varsigma_1\right) d\varsigma_2} \frac{\partial}{\partial \varsigma_1} \Phi_{3|12}\left(\alpha_{31} \mid \varsigma_1, \varsigma_2\right) + \right. \\
& \left. \Phi_{3|12}\left(\alpha_{31} \mid \varsigma_1, \varsigma_2\right) \frac{\partial}{\partial \varsigma_1} \frac{\phi_{2|1}\left(\varsigma_2 \mid \varsigma_1\right)}{\int_{\alpha_{22}}^{\infty} \phi_{2|1}\left(\varsigma_2 \mid \varsigma_1\right) d\varsigma_2} \right] d\varsigma_2 .
\end{aligned}$$

We know that

$$\frac{\partial}{\partial \varsigma_1} \Phi_{3|12}\left(\alpha_{31} \mid \varsigma_1, \varsigma_2\right) < 0,$$

so

$$\int_{\alpha_{22}}^{\infty} \frac{\phi_{2|1}\left(\varsigma_2 \mid \varsigma_1\right)}{\int_{\alpha_{22}}^{\infty} \phi_{2|1}\left(\varsigma_2 \mid \varsigma_1\right) d\varsigma_2} \frac{\partial}{\partial \varsigma_1} \Phi_{3|12}\left(\alpha_{31} \mid \varsigma_1, \varsigma_2\right) d\varsigma_2 < 0.$$

We can write the second term as[2]

$$\begin{aligned}
& \int_{\alpha_{22}}^{\infty} \Phi_{3|12}\left(\alpha_{31} \mid \varsigma_1, \varsigma_2\right) \frac{\partial}{\partial \varsigma_1} \frac{\phi_{2|1}\left(\varsigma_2 \mid \varsigma_1\right)}{\int_{\alpha_{22}}^{\infty} \phi_{2|1}\left(\varsigma_2 \mid \varsigma_1\right) d\varsigma_2} d\varsigma_2 \\
= \; & -\Phi_{3|12}\left(\alpha_{31} \mid \varsigma_1, \varsigma_2\right) \frac{\partial}{\partial \varsigma_1} \frac{\left[1 - \Phi_{2|1}\left(\varsigma_2 \mid \varsigma_1\right)\right]}{\left[1 - \Phi_{2|1}\left(\alpha_{22} \mid \varsigma_1\right)\right]} \Big|_{\alpha_{22}}^{\infty} \\
& + \int_{\alpha_{22}}^{\infty} \frac{\partial}{\partial \varsigma_1} \frac{\left[1 - \Phi_{2|1}\left(\varsigma_2 \mid \varsigma_1\right)\right]}{\left[1 - \Phi_{2|1}\left(\alpha_{22} \mid \varsigma_1\right)\right]} \frac{\partial}{\partial \varsigma_2} \Phi_{3|12}\left(\alpha_{31} \mid \varsigma_1, \varsigma_2\right) d\varsigma_2 \\
= \; & \int_{\alpha_{22}}^{\infty} \frac{\partial}{\partial \varsigma_1} \frac{\left[1 - \Phi_{2|1}\left(\varsigma_2 \mid \varsigma_1\right)\right]}{\left[1 - \Phi_{2|1}\left(\alpha_{22} \mid \varsigma_1\right)\right]} \frac{\partial}{\partial \varsigma_2} \Phi_{3|12}\left(\alpha_{31} \mid \varsigma_1, \varsigma_2\right) d\varsigma_2 .
\end{aligned}$$

---

[2]

$$\begin{aligned}
u &= \Phi_{3|12}\left(\alpha_{31} \mid \varepsilon_1, \varepsilon_2\right) \Rightarrow du = \frac{\partial}{\partial \varepsilon_2} \Phi_{3|12}\left(\alpha_{31} \mid \varepsilon_1, \varepsilon_2\right) d\varepsilon_2 \\
dv &= \frac{\partial}{\partial \varepsilon_1} \frac{\phi_{2|1}\left(\varepsilon_2 \mid \varepsilon_1\right)}{\int_{\alpha_{22}}^{\infty} \phi_{2|1}\left(\varepsilon_2 \mid \varepsilon_1\right) d\varepsilon_2} d\varepsilon_2 \Rightarrow v = \frac{\partial}{\partial \varepsilon_1} \frac{-\left[1 - \Phi_{2|1}\left(\varepsilon_2 \mid \varepsilon_1\right)\right]}{\left[1 - \Phi_{2|1}\left(\alpha_{22} \mid \varepsilon_1\right)\right]}
\end{aligned}$$

We know that

$$\frac{\partial}{\partial \varsigma_2} \Phi_{3|12} \left(\alpha_{31} \mid \varsigma_1, \varsigma_2 \right) < 0$$

and

$$\frac{\partial}{\partial \varsigma_1} \frac{\left[1 - \Phi_{2|1} \left(\varsigma_2 \mid \varsigma_1 \right) \right]}{\left[1 - \Phi_{2|1} \left(\alpha_{22} \mid \varsigma_1 \right) \right]} > 0.$$

So

$$\frac{\partial \Pr \left[\varsigma_3 < \alpha_{31} \mid \varsigma_1, \alpha_{22} < \varsigma_2 \right]}{\partial \varsigma_1} < 0.$$

Now define

$$H \left(\varsigma_1 \right) = \Pr \left[\varsigma_3 < \alpha_{31} \mid \varsigma_1, \alpha_{22} < \varsigma_2 \right]$$

as a function of ς_1, and write $\Pr \left[\varsigma_3 < \alpha_{31} \mid \alpha_{11} < \varsigma_1 < \alpha_{12}, \alpha_{22} < \varsigma_2 \right]$ as

$$\int H \left(\varsigma_1 \right) g^* \left(\varsigma_1 \right) d\varsigma_1$$

where

$$g^* \left(\varsigma_1 \right) = \frac{\phi \left(\varsigma_1 \right)}{\Phi \left(\alpha_{12} \right) - \Phi \left(\alpha_{11} \right)} 1 \left[\alpha_{11} < \varsigma_1 < \alpha_{12} \right],$$

$G^* \left(\varsigma_1 \right)$ is its distribution function, and $\Pr \left[\varsigma_3 < \alpha_{31} \mid \alpha_{12} < \varsigma_1, \alpha_{22} < \varsigma_2 \right]$ as

$$\int H \left(\varsigma_1 \right) g^{**} \left(\varsigma_1 \right) d\varsigma_1$$

where

$$g^{**} \left(\varsigma_1 \right) = \frac{\phi \left(\varsigma_1 \right)}{1 - \Phi \left(\alpha_{12} \right)} 1 \left[\alpha_{12} < \varsigma_1 \right]$$

and $G^{**} \left(\varsigma_1 \right)$ is its distribution. It is clear that $G^* \left(\varsigma_1 \right) > G^{**} \left(\varsigma_1 \right)$, and we have already shown that $H' \left(\varsigma_1 \right) < 0$. Thus, the result follows by properties of stochastic dominance. The second condition follows by setting $\alpha_{22} = -\infty$. ■

Proof. (Theorem 4.11)

Step1: Show that $\tilde{P}_t \left(k, \tau \right) > \tilde{P}_t \left(k - 1, \tau \right) \ \forall \ k > 1$.

Using the notation from Lemma 1, let $\varsigma_{3-s} = \varepsilon_{t-s}, s = k-1, k, k+1$. Let $\alpha_{31} = \varepsilon_t^* \left(1, 3 \right)$, $\alpha_{11} = \varepsilon_{t-2}^* \left(2, 1 \right)$, $\alpha_{12} = \overline{\varepsilon}_{t-2}^{**} \left(2, 1 \right)$, $\alpha_{22} = \overline{\varepsilon}_{t-1}^{**} \left(2, 1 \right)$. The condition that all the covariances are positive is satisfied because the two ways in which early errors affect later errors is through positive serial correlation and through updates of θ. Thus the result follows.

Step2: The theorem follows from induction on the previous step. ■

Proof. (Theorem 4.12)

There are two differences between cohabitation and marriage: a) $\varepsilon_t^*(2,1) < \varepsilon_t^*(3,1)$ (by Theorem 4.7) and b) $\varepsilon_t^*(1,2) > \varepsilon_t^*(1,3)$ (by Theorem 4.6). Condition (a) and Lemma 1 alone would imply the result. Condition (b) would also imply the result. The two together strengthen each other. ■

Proof. (Theorem 4.13)

For this proof, we suppress the following arguments of the value functions for brevity of notation: c_{t-1} and X_t.

a) Let $P_{31}^*(t+s,t)$ be the probability that a first divorce occurs at $t+s$ conditional on being married at t:

$$P_{31}^*(t+s,t) = P_{t+s}[1 \mid 3, d_{t+s}]\,\Pi_{r=0}^{s-1} P_{t+r}[3 \mid 3, d_{t+s}],$$

and let $P_k^{**}(t+s,t)$ be the probability that the kth remarriage occurs at $t+s$ conditional on being single at t. Then

$$\begin{aligned}
&\frac{\partial V_t\left[1,3,1,\widehat{\theta}_{d_t},\varepsilon_t\right]}{\partial D_3} - \frac{\partial V_t\left[3,3,d_t,\widehat{\theta}_{d_t},\varepsilon_t\right]}{\partial D_3} \qquad (16)\\
= \;& -1 + \left\{\sum_{s=0}^{\infty}\beta^{s+1}P_{31}^*(t+s,t)\right\}\\
& - \sum_{k=1}^{\infty}\{\sum_{j=2+k}^{\infty}\beta^j\sum_{r=1}^{j-1}P_k^{**}(t+r,t)\,P_{31}^*(j+t,t+r) - \\
& \sum_{s=0}^{\infty}\beta^{s+j+1}P_{31}^*(t+s,t)\sum_{r=1}^{j-1}P_k^{**}(t+s+r,t+s)\,P_{31}^*(j+t+s,t+s+r)\}.
\end{aligned}$$

So we need to be able to say something about

$$\begin{aligned}
&\sum_{r=1}^{j-1}P_k^{**}(t+r,t)\,P_{31}^*(j+t,t+r) - \qquad (17)\\
&\sum_{s=0}^{\infty}\beta^{s+1}P_{31}^*(t+s,t)\sum_{r=1}^{j-1}P_k^{**}(t+s+r,t+s)\,P_1^*(j+t+s,t+s+r)
\end{aligned}$$

for all s and j. But given our assumption about no age and children effects, $P_k^{**}(t+r,t) = P_k^{**}(t+s+r,t+s)$ and $P_{31}^*(j+t,t+r) = P_{31}^*(j+t+s,t+s+r)$ for all s. Thus equation (17) becomes

$$\begin{aligned}
&\sum_{r=1}^{j-1}P_k^{**}(t+r,t)\,P_{31}^*(j+t,t+r)\left[1-\sum_{s=0}^{\infty}\beta^{s+1}P_{31}^*(t+s,t)\right]\\
< \;& 1-\sum_{s=0}^{\infty}\beta^{s+1}P_{31}^*(t+s,t).
\end{aligned}$$

Thus, equation (16) is less than

$$\begin{aligned} & -1+\left\{\sum_{s=0}^{\infty}\beta^{s+1}P_{31}^{*}(t+s,t)\right\}-\sum_{k=1}^{\infty}\{\sum_{j=2+k}^{\infty}\beta^{j}\sum_{s=1}^{j-1}\left\{1-\sum_{s=0}^{\infty}\beta^{s+1}P_{31}^{*}(t+s,t)\right\} \\ = & -\left\{1-\sum_{s=0}^{\infty}\beta^{s+1}P_{31}^{*}(t+s,t)\right\}\left\{1+\sum_{k=1}^{\infty}\sum_{j=2+k}^{\infty}\beta^{j}\right\}<0. \end{aligned}$$

b) If $m \notin F(m_{t-1})$, then $P_t[m \mid m_{t-1}, d_t] = 0$ independent of f_m, so $\partial P_t[m \mid m_{t-1}, d_t]/\partial f_m = 0$. Consider the case where $m \in F(m_{t-1})$, and let $m' \in F(m_{t-1})$. Let $P_m^{***}(t+s,t)$ be the probability that one is in state m at age $t+s$ conditional on having been in state m at age t (possibly with transitions in between), and $P_{m'm}^{*}(t+s,t)$ be the probability that the first transition into state m occurs at age $t+s$ conditional on being in state m' at age t. Then

$$\begin{aligned} & \frac{\partial V_t\left[m,m_{t-1},1,\widehat{\theta}_{d_t},\varepsilon_t\right]}{\partial f_m}-\frac{\partial V_t\left[m',m_{t-1},d_t,\widehat{\theta}_{d_t},\varepsilon_t\right]}{\partial f_m} \qquad (18) \\ = & \left\{1+\sum_{s=1}^{t^*-1-t}\beta^{s}P_m^{***}(t+s,t)+P_m^{***}(t^*,t)\sum_{s=t^*}^{t^{**}}\beta^{s-t}\right\}- \\ & \sum_{r=1}^{t^*-1-t}P_{m'm}^{*}(t+r,t)\beta^{r}\left\{1+\sum_{s=1}^{t^*-1-(t+r)}\beta^{s}P_m^{***}(t+r+s,t+r)\right. \\ & \left.+P_m^{***}(t^*,t+r)\sum_{s=t^*}^{t^{**}}\beta^{s-t-r}\right\}. \end{aligned}$$

Since $P_m^{***}(t+s,t)=P_m^{***}(t+r+s,t+r)$ for all r, equation (18) becomes

$$\begin{aligned} & \left\{1+\sum_{s=1}^{t^*-1-t}\beta^{s}P_m^{***}(t+s,t)+P_m^{***}(t^*,t)\sum_{s=t^*}^{t^{**}}\beta^{s-t}\right\}- \\ & \sum_{r=1}^{t^*-1-t}P_{m'm}^{*}(t+r,t)\beta^{r}\bullet \\ & \left\{1+\sum_{s=1}^{t^*-1-(t+r)}\beta^{s}P_m^{***}(t+s,t)+P_m^{***}(t^*,t+r)\sum_{s=t^*}^{t^{**}}\beta^{s-t-r}\right\} \end{aligned}$$

which can be written as

$$\left\{1+\sum_{s=1}^{t^*-1-t}\beta^s P_m^{***}\left(t+s,t\right)\right\}\left\{1-\sum_{r=1}^{t^*-1-t}P_{m'm}^{*}\left(t+r,t\right)\beta^r\right\}+$$
$$\left\{P_m^{***}\left(t^*,t\right)-\sum_{r=1}^{t^*-1-t}P_{m'm}^{*}\left(t+r,t\right)P_m^{***}\left(t^*,t+r\right)\right\}\sum_{s=t^*}^{t^{**}}\beta^{s-t}+$$
$$\sum_{r=1}^{t^*-1-t}P_{m'm}^{*}\left(t+r,t\right)\beta^r\left\{\sum_{s=t^*-(t+r)+1}^{t^*-1-t}\beta^s P_m^{***}\left(t+s,t+r\right)\right\}.$$

The first term is always positive because $1-\sum_{r=1}^{t^*-1-t}P_{m'm}^{*}\left(t+r,t\right)\beta^r > 1-\sum_{r=1}^{t^*-1-t}P_{m'm}^{*}\left(t+r,t\right) > 0$. The last term is positive. The middle term is proportional to the difference between $P_m^{***}\left(t^*,t\right)$ and the probability that the agent will be in state m at t^* conditional on being in state m' at t which can not be signed. But the proportionality constant, $\sum_{s=t^*}^{t^{**}}\beta^{s-t}=\beta^{t^*-t}\sum_{s=0}^{t^{**}-t^*}\beta^s$ $\to 0$ uniformly as $t^*-t\to\infty$. Note: this qualification occurs only because $t^{**}-t^*>0$ which is the particular way we end lives in this model. ■

Proof. (Theorem 4.14)

Step1: Derive the functional describing how the distribution of ε_t is changing over age.

Consider the case where $d_t \geq \tau_d$; at such a point, there is no more learning about θ. Therefore, $\varepsilon_t^*\left(1,3\right)$ is not changing with d_t or t (call this level $\varepsilon_\infty^*\left(1,3\right)$). Consider the distribution of ε_t conditional on other (fixed) state variables (and explicitly on $\hat{\theta}$):

$$\begin{aligned}\Psi_t\left[x\mid\hat{\theta}\right] &= \Pr\left[\varepsilon_t<x\mid\hat{\theta}\right] \qquad (19)\\ &= \Pr\left[\hat{\theta}+\rho\left(\varepsilon_{t-1}-\hat{\theta}\right)+\eta_t<x\right]\\ &= \frac{\int_{\varepsilon_\infty^*(1,3)}^{\infty}\Pr\left[\hat{\theta}+\rho\left(\varepsilon_{t-1}-\hat{\theta}\right)+\eta_t<x\mid\varepsilon_{t-1}\right]d\Psi_{t-1}\left(\varepsilon_{t-1}\mid\hat{\theta}\right)}{1-\Psi_{t-1}\left[\varepsilon_\infty^{**}\left(1,3\right)\mid\hat{\theta}\right]}\\ &= \frac{\int_{\varepsilon_\infty^*(1,3)}^{\infty}\Phi\left[\frac{x-\bar{\theta}-\rho\left(\varepsilon_{t-1}-\hat{\theta}\right)}{\sigma_\eta}\right]d\Psi_{t-1}\left(\varepsilon_{t-1}\mid\hat{\theta}\right)}{1-\Psi_{t-1}\left[\varepsilon_\infty^{*}\left(1,3\right)\mid\hat{\theta}\right]}\end{aligned}$$

(the distribution is truncated at $\varepsilon_\infty^*\left(1,3\right)$ because anyone with $\varepsilon_{t-1}<\varepsilon_\infty^*\left(1,3\right)$ divorces). Using integration by parts (and defining $\Psi_t^*\left[x\mid\hat{\theta}\right]=1-\Psi_t\left[x\mid\hat{\theta}\right]$), equation (19) can be written as

$$\Psi_t^*\left[x \mid \hat{\theta}\right] = \Phi\left[\frac{\hat{\theta}-x+\rho\left(\varepsilon_\infty^*(1,3)-\hat{\theta}\right)}{\sigma_\eta}\right] -\rho\int_{\varepsilon_\infty^*(1,3)}^{\infty}\frac{\Psi_{t-1}^*\left[\varepsilon_{t-1} \mid \hat{\theta}\right]}{\Psi_{t-1}^*\left[\varepsilon_\infty^*(1,3) \mid \hat{\theta}\right]}\frac{1}{\sigma_\eta}\phi\left[\frac{x-\hat{\theta}-\rho\left(\varepsilon_{t-1}-\hat{\theta}\right)}{\sigma_\eta}\right]d\varepsilon_{t-1}. \tag{20}$$

Step 2: Derive the asymptotic (steady state) distribution of ε_t.

Let $\Upsilon\left\{\Psi_{t-1}^*\right\}$ be defined by equation (20); i.e., $\Psi_t^* = \Upsilon\left\{\Psi_{t-1}^*\right\}$. Consider two potential distribution functions, Ψ^* and Ψ^{**}. Then it is straightforward to show that

$$\sup_x\left|\Upsilon\left\{\Psi^*\left[x \mid \hat{\theta}\right]\right\}-\Upsilon\left\{\Psi^{**}\left[x \mid \hat{\theta}\right]\right\}\right| = \rho\sup_x\left|\Psi^*\left[x \mid \hat{\theta}\right]-\Psi^{**}\left[x \mid \hat{\theta}\right]\right|$$

which is the (contraction mapping) condition necessary for the existence of a unique asymptotic conditional distribution, $\Psi^*\left[x \mid \hat{\theta}\right]$ and for the asymptotic convergence of $\Psi_t^*\left[x \mid \hat{\theta}\right]$ to $\Psi^*\left[x \mid \hat{\theta}\right]$.

Step 3: Show that $d\Psi^*\left[x \mid \hat{\theta}\right]/d\hat{\theta} > 0$.

Note that

$$\frac{d\Psi^*\left[x \mid \hat{\theta}\right]}{d\hat{\theta}} = \frac{\partial\Psi^*\left[x \mid \hat{\theta}\right]}{\partial\hat{\theta}} + \frac{\partial\Psi^*\left[x \mid \hat{\theta}\right]}{\partial\varepsilon_\infty^*(1,3)}\frac{\partial\varepsilon_\infty^*(1,3)}{\partial\hat{\theta}}.$$

Consider first

$$\begin{aligned}
&\frac{\partial \Psi^{*}\left[x \mid \hat{\theta}\right]}{\partial \hat{\theta}} \\
= \quad & \frac{\partial}{\partial \hat{\theta}} \Phi\left[\frac{\hat{\theta}-x+\rho\left(\varepsilon_{\infty}^{*}(1,3)-\hat{\theta}\right)}{\sigma_{\eta}}\right] \\
& +\frac{\partial}{\partial \hat{\theta}} \rho \int_{\varepsilon_{\infty}^{*}(1,3)}^{\infty} \frac{\Psi^{*}\left[\varepsilon_{t-1} \mid \hat{\theta}\right]}{\Psi^{*}\left[\varepsilon_{\infty}^{*}(1,3) \mid \hat{\theta}\right]} \frac{1}{\sigma_{\eta}} \phi\left[\frac{x-\hat{\theta}-\rho\left(\varepsilon_{t-1}-\hat{\theta}\right)}{\sigma_{\eta}}\right] d \varepsilon_{t-1} \\
= \quad & \frac{\partial}{\partial \hat{\theta}} \Phi\left[\frac{\hat{\theta}-x+\rho\left(\varepsilon_{\infty}^{*}(1,3)-\hat{\theta}\right)}{\sigma_{\eta}}\right] \\
& +\rho \int_{\varepsilon_{\infty}^{*}(1,3)}^{\infty} \frac{\partial}{\partial \hat{\theta}}\left\{\frac{\Psi^{*}\left[\varepsilon_{t-1} \mid \hat{\theta}\right]}{\Psi^{*}\left[\varepsilon_{\infty}^{*}(1,3) \mid \hat{\theta}\right]}\right\} \frac{1}{\sigma_{\eta}} \phi\left[\frac{x-\hat{\theta}-\rho\left(\varepsilon_{t-1}-\hat{\theta}\right)}{\sigma_{\eta}}\right] d \varepsilon_{t-1} \\
& +\rho \int_{\varepsilon_{\infty}^{*}(1,3)}^{\infty} \frac{\Psi^{*}\left[\varepsilon_{t-1} \mid \hat{\theta}\right]}{\Psi^{*}\left[\varepsilon_{\infty}^{*}(1,3) \mid \hat{\theta}\right]} \frac{1}{\sigma_{\eta}} \frac{\partial}{\partial \hat{\theta}} \phi\left[\frac{x-\hat{\theta}-\rho\left(\varepsilon_{t-1}-\hat{\theta}\right)}{\sigma_{\eta}}\right] d \varepsilon_{t-1} \\
= \quad & \frac{\partial}{\partial \hat{\theta}} \Phi\left[\frac{\hat{\theta}-x+\rho\left(\varepsilon_{\infty}^{*}(1,3)-\hat{\theta}\right)}{\sigma_{\eta}}\right] \\
& +\rho \int_{\varepsilon_{\infty}^{*}(1,3)}^{\infty} \frac{\partial}{\partial \hat{\theta}}\left\{\frac{\Psi^{*}\left[\varepsilon_{t-1} \mid \hat{\theta}\right]}{\Psi^{*}\left[\varepsilon_{\infty}^{*}(1,3) \mid \hat{\theta}\right]}\right\} \frac{1}{\sigma_{\eta}} \phi\left[\frac{x-\hat{\theta}-\rho\left(\varepsilon_{t-1}-\hat{\theta}\right)}{\sigma_{\eta}}\right] d \varepsilon_{t-1} \\
& -(1-\rho) \int_{\varepsilon_{\infty}^{*}(1,3)}^{\infty} \frac{\Psi^{*}\left[\varepsilon_{t-1} \mid \hat{\theta}\right]}{\Psi^{*}\left[\varepsilon_{\infty}^{*}(1,3) \mid \hat{\theta}\right]} \frac{1}{\sigma_{\eta}} \frac{\partial}{\partial \varepsilon_{t-1}} \phi\left[\frac{x-\hat{\theta}-\rho\left(\varepsilon_{t-1}-\hat{\theta}\right)}{\sigma_{\eta}}\right] d \varepsilon_{t-1}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{(1-\rho)}{\sigma_\eta}\phi\left[\frac{\hat{\theta}-x+\rho\left(\varepsilon_\infty^*(1,3)-\hat{\theta}\right)}{\sigma_\eta}\right] \\
&\quad +\rho\int_{\varepsilon_\infty^*(1,3)}^{\infty}\frac{\partial}{\partial\hat{\theta}}\left\{\frac{\Psi^*\left[\varepsilon_{t-1}\mid\hat{\theta}\right]}{\Psi^*\left[\varepsilon_\infty^*(1,3)\mid\hat{\theta}\right]}\right\}\frac{1}{\sigma_\eta}\phi\left[\frac{x-\hat{\theta}-\rho\left(\varepsilon_{t-1}-\hat{\theta}\right)}{\sigma_\eta}\right]d\varepsilon_{t-1} \\
&\quad -(1-\rho)\frac{1}{\sigma_\eta}\phi\left[\frac{x-\hat{\theta}-\rho\left(\varepsilon_\infty^*(1,3)-\hat{\theta}\right)}{\sigma_\eta}\right] \\
&\quad -(1-\rho)\int_{\varepsilon_\infty^*(1,3)}^{\infty}\frac{\partial}{\partial\varepsilon_{t-1}}\frac{\Psi^*\left[\varepsilon_{t-1}\mid\hat{\theta}\right]}{\Psi^*\left[\varepsilon_\infty^*(1,3)\mid\hat{\theta}\right]}\frac{1}{\sigma_\eta}\phi\left[\frac{x-\hat{\theta}-\rho\left(\varepsilon_{t-1}-\hat{\theta}\right)}{\sigma_\eta}\right]d\varepsilon_{t-1} \\
&= +\rho\int_{\varepsilon_\infty^*(1,3)}^{\infty}\frac{\partial}{\partial\hat{\theta}}\left\{\frac{\Psi^*\left[\varepsilon_{t-1}\mid\hat{\theta}\right]}{\Psi^*\left[\varepsilon_\infty^*(1,3)\mid\hat{\theta}\right]}\right\}\frac{1}{\sigma_\eta}\phi\left[\frac{x-\hat{\theta}-\rho\left(\varepsilon_{t-1}-\hat{\theta}\right)}{\sigma_\eta}\right]d\varepsilon_{t-1} \\
&\quad -(1-\rho)\int_{\varepsilon_\infty^*(1,3)}^{\infty}\frac{\partial}{\partial\varepsilon_{t-1}}\frac{\Psi^*\left[\varepsilon_{t-1}\mid\hat{\theta}\right]}{\Psi^*\left[\varepsilon_\infty^*(1,3)\mid\hat{\theta}\right]}\frac{1}{\sigma_\eta}\phi\left[\frac{x-\hat{\theta}-\rho\left(\varepsilon_{t-1}-\hat{\theta}\right)}{\sigma_\eta}\right]d\varepsilon_{t-1}
\end{aligned}
$$

The second term is a weighted average of $\partial\Psi^*\left[\varepsilon_{t-1}\mid\hat{\theta}\right]/\partial\varepsilon_{t-1}$ over ε_{t-1} which is negative, so it is negative (times a negative). If $\inf\frac{\partial\Psi^*\left[\varepsilon_{t-1}|\bar{\theta}\right]}{\partial\hat{\theta}}\geq 0$, then $\inf\frac{\partial\Psi^*\left[x|\bar{\theta}\right]}{\partial\hat{\theta}}>0$.[3] Since there is a unique solution to $\Psi^*\left[x\mid\hat{\theta}\right]$, there is a

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[3] The argument that is being made here conjectures a property of equilibrium, $\inf\frac{\partial\Psi^*\left[\varepsilon_{t-1}|\bar{\theta}\right]}{\partial\hat{\theta}}\geq 0$, and shows that it implies $\inf\frac{\partial\Psi^*\left[x|\bar{\theta}\right]}{\partial\hat{\theta}}>0$. Since the contraction mapping argument implies a unique equilibrium, the result follows.

unique solution to $\frac{\partial\Psi^*\left[x|\bar{\theta}\right]}{\partial\hat{\theta}}$. Thus $\partial\Psi^*\left[x \mid \hat{\theta}\right]/\partial\hat{\theta} > 0$. Also,

$$\begin{aligned}
&\frac{\partial\Psi_t^*\left[x \mid \hat{\theta}\right]}{\partial\varepsilon_\infty^*(1,3)} \\
= \quad & \frac{\partial}{\partial\varepsilon_\infty^*(1,3)}\Phi\left[\frac{\hat{\theta}-x+\rho\left(\varepsilon_\infty^*(1,3)-\hat{\theta}\right)}{\sigma_\eta}\right] \\
& -\rho\frac{\partial}{\partial\varepsilon_\infty^*(1,3)}\int_{\varepsilon_\infty^*(1,3)}^{\infty}\frac{\Psi_{t-1}^*\left[\varepsilon_{t-1} \mid \hat{\theta}\right]}{\Psi_{t-1}^*\left[\varepsilon_\infty^*(1,3) \mid \hat{\theta}\right]}\frac{1}{\sigma_\eta}\phi\left[\frac{x-\hat{\theta}-\rho\left(\varepsilon_{t-1}-\hat{\theta}\right)}{\sigma_\eta}\right]d\varepsilon_{t-1} \\
= \quad & \frac{\rho}{\sigma_\eta}\phi\left[\frac{\hat{\theta}-x+\rho\left(\varepsilon_\infty^*(1,3)-\hat{\theta}\right)}{\sigma_\eta}\right] \\
& +\frac{\rho}{\sigma_\eta}\phi\left[\frac{x-\hat{\theta}-\rho\left(\varepsilon_\infty^*(1,3)-\hat{\theta}\right)}{\sigma_\eta}\right] \\
& -\rho\int_{\varepsilon_\infty^*(1,3)}^{\infty}\frac{\partial}{\partial\varepsilon_\infty^*(1,3)}\left\{\frac{\Psi_{t-1}^*\left[\varepsilon_{t-1} \mid \hat{\theta}\right]}{\Psi_{t-1}^*\left[\varepsilon_\infty^*(1,3) \mid \hat{\theta}\right]}\right\}\frac{1}{\sigma_\eta}\phi\left[\frac{x-\hat{\theta}-\rho\left(\varepsilon_{t-1}-\hat{\theta}\right)}{\sigma_\eta}\right]d\varepsilon_{t-1} \\
= \quad & -\rho\int_{\varepsilon_\infty^*(1,3)}^{\infty}\frac{\partial}{\partial\varepsilon_\infty^*(1,3)}\left\{\frac{\Psi_{t-1}^*\left[\varepsilon_{t-1} \mid \hat{\theta}\right]}{\Psi_{t-1}^*\left[\varepsilon_\infty^*(1,3) \mid \hat{\theta}\right]}\right\}\frac{1}{\sigma_\eta}\phi\left[\frac{x-\hat{\theta}-\rho\left(\varepsilon_{t-1}-\hat{\theta}\right)}{\sigma_\eta}\right]d\varepsilon_{t-1} \\
= \quad & \rho\int_{\varepsilon_\infty^*(1,3)}^{\infty}\frac{\Psi_{t-1}^*\left[\varepsilon_{t-1} \mid \hat{\theta}\right]}{\left\{\Psi_{t-1}^*\left[\varepsilon_\infty^*(1,3) \mid \hat{\theta}\right]\right\}^2}\frac{\partial\Psi_{t-1}^*\left[\varepsilon_\infty^*(1,3) \mid \hat{\theta}\right]}{\partial\varepsilon_\infty^*(1,3)}\bullet \\
& \frac{1}{\sigma_\eta}\phi\left[\frac{x-\hat{\theta}-\rho\left(\varepsilon_{t-1}-\hat{\theta}\right)}{\sigma_\eta}\right]d\varepsilon_{t-1} \\
< \quad & 0
\end{aligned}$$

and $\partial\varepsilon_\infty^*(1,3)/\partial\hat{\theta} < 0$ (because $\partial V_t[m_t, 1, c_{t-1}, 1, \hat{\theta}(\varepsilon_t), \varepsilon_t, X_t]/\partial\hat{\theta} > 0$, an agent is willing to tolerate worse ε's as $\hat{\theta}$ increases). Thus, $d\Psi^*\left[x \mid \hat{\theta}\right]/d\hat{\theta} > 0$ which implies that $d\Psi^*\left[\varepsilon_\infty^*(1,3) \mid \hat{\theta}\right]/d\hat{\theta} > 0$.

Step 4: Use the effect of unobserved heterogeneity on duration dependence to prove the result.

The unconditional distribution of ε_t is

$$\Psi_t^*[x] = \int \Psi_t^*\left[x \mid \hat{\theta}\right]d\Im_t\left(\hat{\theta}\right)$$

where $\Im_t\left(\hat{\theta}\right)$ is the distribution of $\hat{\theta}$ at t,

$$\Im_t\left(\hat{\theta}\right) = \frac{\int_{-\infty}^{\hat{\theta}} \Psi_t^*\left[\varepsilon_\infty^*\left(1,3\right) \mid x\right] d\Im_{t-1}\left(x\right)}{\int_{-\infty}^{\infty} \Psi_t^*\left[\varepsilon_\infty^*\left(1,3\right) \mid x\right] d\Im_{t-1}\left(x\right)}.$$

The reciprocal of $\Im_t\left(\hat{\theta}\right)$ is

$$\begin{aligned}
& \frac{\int_{-\infty}^{\infty} \Psi_t^*\left[\varepsilon_\infty^*\left(1,3\right) \mid x\right] d\Im_{t-1}\left(x\right)}{\int_{-\infty}^{\hat{\theta}} \Psi_t^*\left[\varepsilon_\infty^*\left(1,3\right) \mid x\right] d\Im_{t-1}\left(x\right)} \\
= \quad & 1 + \frac{\int_{\hat{\theta}}^{\infty} \Psi_t^*\left[\varepsilon_\infty^*\left(1,3\right) \mid x\right] d\Im_{t-1}\left(x\right)}{\int_{-\infty}^{\hat{\theta}} \Psi_t^*\left[\varepsilon_\infty^*\left(1,3\right) \mid x\right] d\Im_{t-1}\left(x\right)} \\
< \quad & 1 + \frac{\int_{\hat{\theta}}^{\infty} \Psi_t^*\left[\varepsilon_\infty^*\left(1,3\right) \mid \hat{\theta}\right] d\Im_{t-1}\left(x\right)}{\int_{-\infty}^{\hat{\theta}} \Psi_t^*\left[\varepsilon_\infty^*\left(1,3\right) \mid \hat{\theta}\right] d\Im_{t-1}\left(x\right)} \\
= \quad & 1 + \frac{\Psi_t^*\left[\varepsilon_\infty^*\left(1,3\right) \mid \hat{\theta}\right] \left\{1 - \Im_{t-1}\left(\hat{\theta}\right)\right\}}{\Psi_t^*\left[\varepsilon_\infty^*\left(1,3\right) \mid \hat{\theta}\right] \Im_{t-1}\left(\hat{\theta}\right)} \\
= \quad & 1 + \frac{\left\{1 - \Im_{t-1}\left(\hat{\theta}\right)\right\}}{\Im_{t-1}\left(\hat{\theta}\right)} = \frac{1}{\Im_{t-1}\left(\hat{\theta}\right)}.
\end{aligned}$$

Therefore $\Im_{t-1}\left(\hat{\theta}\right) > \Im_t\left(\hat{\theta}\right)$. Since $\Psi_t^*\left[x \mid \hat{\theta}\right]$ is converging to $\Psi^*\left[x \mid \hat{\theta}\right]$ with $d\Psi^*\left[x \mid \hat{\theta}\right]/d\hat{\theta} > 0$ and $\Im_t\left(\hat{\theta}\right)$ is decreasing in t (shifting toward larger values of $\hat{\theta}$), $\Psi_t^*\left[x\right]$ is increasing in t. The probability of divorce is then $\Psi_t\left[\varepsilon_\infty^*\left(1,3\right)\right]$ which is declining in t. ■

E Numerical Methods for Evaluating the Value Functions

$\widehat{\theta}_{d_t}$ and ε_t are continuous state variables. Consider the case when $m_t > 1$ and $d_t < \bar{t}_d$ (all other cases are simpler versions of the same method). We suppress X_{t+1} from utility flow and value functions for notation brevity. When evaluating $V_t\left[S_t\right]$, $E\left\{\max \ \left(V_{t+1}\left[S_{t+1}\right]\right) \mid \widehat{\theta}_{d_t}, \varepsilon_t, c_{t-1}\right\}$ is

$$\begin{aligned}
& \frac{p_t}{\sigma_\eta} \int_{-\infty}^{\infty} \phi\left(\frac{\varepsilon_{t+1} - \left[\left(1-\rho\right)\widehat{\theta}_{d_t} + \rho\varepsilon_t\right]}{\sigma_\eta}\right) \left\{\max_{m_{t+1} \in F\left(m_t\right)} V_{t+1}\left[S_{t+1}^1\right]\right\} d\varepsilon_{t+1} \\
& \frac{1-p_t}{\sigma_\eta} \int_{-\infty}^{\infty} \phi\left(\frac{\varepsilon_{t+1} - \left[\left(1-\rho\right)\widehat{\theta}_{d_t} + \rho\varepsilon_t\right]}{\sigma_\eta}\right) \left\{\max_{m_{t+1} \in F\left(m_t\right)} V_{t+1}\left[S_{t+1}^0\right]\right\} d\varepsilon_{t+1}
\end{aligned} \tag{21}$$

if $m_t > 1$ where $S^1_{t+1} = \left(m_{t+1}, m_t, c^*_{1t}, 1, d_{t+1}, \widehat{\theta}_{d_{t+1}}, \varepsilon_{t+1}\right)$ is the vector of state variables conditional on a birth at t, $S^0_{t+1} = \left(m_{t+1}, m_t, c_{t-1}, c^*_{2t}, d_{t+1}, \widehat{\theta}_{d_{t+1}}, \varepsilon_{t+1}\right)$ is the vector of state variables conditional on no birth, c^*_{1t}, c^*_{2t}, and d_{t+1} move according to equations (3.5) and (3.4) in the paper respectively, and $\widehat{\theta}_{d_{t+1}}$ is given by equation (1). Note that, because $\widehat{\theta}_{d_{t+1}}$ is determined once ε_{t+1} is given, there is essentially only one continuous state variable to integrate over. The integral in equation (21) can be written as

$$\int_{-\infty}^{\infty} \frac{1}{\sigma_\eta \sqrt{2\pi}} \exp\left\{ \frac{-1}{2} \left[\frac{\varepsilon_{t+1} - \left[(1-\rho)\,\widehat{\theta}_{d_t} + \rho\varepsilon_t\right]}{\sigma_\eta} \right]^2 \right\} H\left[\varepsilon_{t+1}\right] d\varepsilon_{t+1} \qquad (22)$$

where

$$H\left[\varepsilon_{t+1}\right] = \max_{m_{t+1} \in F(m_t)} V_{t+1}\left[m_{t+1}, \bullet, \bullet, \bullet, \bullet, \widehat{\theta}_{d_{t+1}}\left(\varepsilon_{t+1}\right), \varepsilon_{t+1}\right]. \qquad (23)$$

Equation (22) can be written as

$$\frac{1}{\sigma_\eta \sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left\{ \frac{-1}{2} \left[\frac{\eta_{t+1}}{\sigma_\eta} \right]^2 \right\} H\left[(1-\rho)\,\widehat{\theta}_{d_t} + \rho\varepsilon_t + \eta_{t+1}\right] d\eta_{t+1} \qquad (24)$$

which, with a change of variables, can be written as

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp\left\{-u^2\right\} H\left[(1-\rho)\,\widehat{\theta}_{d_t} + \rho\varepsilon_t + \sqrt{2}\sigma_\eta u\right] du. \qquad (25)$$

This can be approximated using Gaussian quadrature as

$$\frac{1}{\sqrt{\pi}} \sum_{j=1}^{K} a_j H\left[(1-\rho)\,\widehat{\theta}_{d_t} + \rho\varepsilon_t + \sqrt{2}\sigma_\eta \vartheta_j\right] \qquad (26)$$

where $\left(a_j, \vartheta_j\right)_{j=1}^{K}$ are K-point Gaussian quadrature weights and points. But, since $H\left[\bullet\right]$ was not evaluated at $(1-\rho)\,\widehat{\theta}_{d_t} + \rho\varepsilon_t + \sqrt{2}\sigma_\eta\vartheta_j$, it must be interpolated. Let $\tilde{\theta} = \left(\tilde{\theta}_1, \tilde{\theta}_2, .., \tilde{\theta}_K\right)$ and $\tilde{\varepsilon} = \left(\tilde{\varepsilon}_1, \tilde{\varepsilon}_2, .., \tilde{\varepsilon}_K\right)$ be a grid of K^2 (Gaussian quadrature) points at which to evaluate $V_t\left[S_{t+1}\right]$. Let $y_j = (1-\rho)\,\widehat{\theta}_{d_t} + \rho\varepsilon_t + \sqrt{2}\sigma_\eta\vartheta_j$ or $y_j = E\theta + \sqrt{2\left(\sigma^2_\eta + \sigma^2_\theta\right)}\vartheta_j$ (depending on the case) and $z_j = \widehat{\theta}_{d_{t+1}}\left(y_j\right)$. Then each $H\left[\bullet\right]$ term in equation (26) can be interpolated as

$$h\left[y_j\right] = \frac{\sum_{k=[j]}^{[j]+1} \sum_{l=[j]}^{[j]+1} \max_{m_{t+1} \in F(m_t)} V_{t+1}\left[m_{t+1}, \bullet, \bullet, \bullet, \bullet, \tilde{\theta}_k, \tilde{\varepsilon}_l\right] R\left[\tilde{\theta}_k - z_j, \tilde{\varepsilon}_l - y_j\right]}{\sum_{k=1}^{K} \sum_{l=1}^{K} R\left[\tilde{\theta}_k - z_j, \tilde{\varepsilon}_l - y_j\right]} \qquad (27)$$

where

$$R\left[\tilde{\theta}_k - z_j, \tilde{\varepsilon}_l - y_j\right] = \left|\tilde{\theta}_{k'} - z_j\right|^{\xi} \left|\tilde{\varepsilon}_{l'} - y_j\right|^{\xi}, \tag{28}$$

where $k' = [j]$ if $k = [j] + 1$ and $k' = [j] + 1$ if $k = [j]$ (and l' has a similar definition), and $2 \geq \xi > 1$. Note that $R[\bullet, \bullet]$ is continuous and differentiable at seams (where $\tilde{\theta}_k = z_j$ or $\tilde{\varepsilon}_l = y_j$). A problem is that the derivative of $h[y_j]$ at seams is zero. A way to avoid this is to set $\xi = 1$. There is no way to have nonzero derivatives that are continuous at seams when only 4 points are used to evaluate $h[y_j]$.