An Empirical Dynamic Model of the Used Car Market

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1 Model

Let u_{ij} be the utility gross of cost that individual i gets from owning automobile j at time t where time is measured relative to the purchase time; $t = 0$ at the purchase time. Assume there is a finite number of brands J. Assume that

$$u_{ij} \sim iidF\left[\theta\left(b_j\right); \sigma_u^2\right]$$

with support $[0, U]$ Let a_{jt} be the age of j at t and b_j be the brand of j. Assume there are log operating costs of

$$\begin{aligned} \log c_{jt} &= \alpha_j + \lambda\left(b_j, a_{jt}\right), \\ \alpha_j &\sim iidN\left[\mu\left(b_j\right), \sigma_\mu^2\right], \end{aligned} \tag{1}$$

and $\lambda\left(b_j, a_{jt}\right)$ is a known function accounting for the effect of aging on operating costs with

$$\frac{\partial \lambda\left(b_j, a\right)}{\partial a} > \lambda^*$$

for some positive constant λ^*. Prior to the purchase of j, i does not know the value of α_j but learns it immediately after purchase. Then the flow i receives from owning j at t is

$$u_{ij} - c_{jt}.$$

Let

$$p_{jt} = p\left(b_j, a_{jt}\right)$$

be the price of automobile j at time t. Then, the value to i of owning j at t is

$$\begin{aligned} & V\left(u_{ij}, \alpha_j, b_j, a_{jt}\right) \\ = \; & u_{ij} - c_{jt} + \beta \max\left[W, p\left(b_j, a_{jt+1}\right) + W, V\left(u_{ij}, \alpha_j, b_j, a_{jt+1}\right)\right] \end{aligned}$$

where

$$W = \beta \max_k \left\{0, E_{\alpha_k} V\left(u_{ik}, \alpha_k, b_k, a_{kt+1}\right) - p\left(b_k, a_{kt+1}\right) \mid V\left(v_{ik}, \alpha_k, b_k, a_{kt+1}\right) \leq p\left(b_k, a_{kt+1}\right) + W\right\} \geq 0.$$

Note that i has the option to scrap j costlessly.

2 Equilibrium Properties

Theorem 1 *For all b_j and all finite α_j, there exists A large enough such that $V(u_{ij}, \alpha_j, b_j, A) = \beta W$.*

Proof. Conditional on b_j, α_j, and a_{jt},

$$V(U, \alpha_j, b_j, a_{jt}) = U - c_{jt} + \beta \max[W, V(U, \alpha_j, b_j, a_{jt+1})] \tag{2}$$

because there is no other individual who will value j as much. Equation (2) satisfies

$$V(U, \alpha_j, b_j, a_{jt}) \leq \max\left[W, U - c_{jt} + \frac{\beta}{1-\beta}(U - \exp\{\alpha_j + \lambda(b_j, a_{jt})\})\right].$$

Since $U - c_{jt}$ is growing (negatively) without bound, there exists t large enough that satisfies the condition. But if the condition is satisfied for U, then we should be able to show that it is satisfied for any $u_{ij} < U$. ■

Theorem 2 *For each pair (b_j, a_{jt}), $\frac{\partial V(u_{ij},\alpha_j,b_j,a_{jt})}{\partial \alpha_j} \leq 0$ and $\frac{\partial V(u_{ij},\alpha_j,b_j,a_{jt})}{\partial a_{jt}} \leq 0$, there exists an equilibrium price $p(b_j, a_{jt})$ with $\frac{\partial p(b_j,a_{jt})}{\partial a_{jt}} \leq 0$.*

Proof. Conditional on b_j and α_j, let A be the first time such that $V_i(u_{ij}, \alpha_j, b_j, a_{jt}) = W$. Then

$$\begin{aligned}
\frac{\partial V(u_{ij}, \alpha_j, b_j, A-1)}{\partial \alpha_j} &= -\frac{\partial c_{jt-1}}{\partial \alpha_j} < 0; \\
\frac{\partial V(u_{ij}, \alpha_j, b_j, A-1)}{\partial a_{jt-1}} &= -\frac{\partial c_{jt-1}}{\partial a_{jt-1}} < 0.
\end{aligned}$$

Let $g_\alpha(\alpha \mid b, a,)$ be the density of α and $g_u(u \mid \alpha, b, a,)$ be the conditional density of u given α among owners of brand b cars of age a. Let

$$S(b, a) = \{(\alpha, u) : W \leq V(u, \alpha, b, a) \leq p + W\}$$

be the set of values of (α, u) where a seller wants to sell and

$$\begin{aligned}
B(b, a) = \; & \{(\alpha, u, v, u') : (\alpha, v) \in S(b, a), \\
& E_{\tilde{\alpha}}[V(u, \tilde{\alpha}, b, a) - p(b, a) - W \mid (\tilde{\alpha}, v) \in S(b, a)] \\
\geq \; & \max_{j, a'_j > 0} E_{\tilde{\alpha}'}\left[V\left(u'_j, \tilde{\alpha}'_j, b'_j, a'_j\right) - p\left(b'_j, a'_j\right) - W \mid W \leq V\left(v'_j, \tilde{\alpha}'_j, b'_j, a'_j\right) \leq p\left(b'_j, a'_j\right) + W\right], \\
& E_{\tilde{\alpha}}[V(u, \tilde{\alpha}, b, a) - p(b, a) - W] \geq \max_j E_{\tilde{\alpha}'}[V(u'_j, \tilde{\alpha}'_j, b'_j, 0) - p\left(b'_j, 0\right) - W], \\
& E_{\tilde{\alpha}}[V(u, \tilde{\alpha}, b, a) - p(b, a) - W \mid (\tilde{\alpha}, v) \in S(b, a)] \geq 0\}
\end{aligned}$$

be the set of values of (α, u, v, u') where a buyer wants to buy. Then, at age A, $p(b_j, A)$ satisfies

$$\gamma \iint\limits_{S(b,a)} g_u(u \mid \alpha, b, a,)\, g_\alpha(\alpha \mid b, a)\, du d\alpha \tag{3}$$

$$= \left[\iint\limits_{S(b,a)} g_u(u \mid \alpha, b, a,)\, g_\alpha(\alpha \mid b, a)\, du d\alpha \prod_j \iint\limits_{S\left(b_j', a_j'\right)} g_u\left(u_j' \mid \alpha, b, a\right) du_j' d\alpha_j' \right]^{-1} \cdot$$

$$\int \cdots \int\limits_{B(b,a)} g_u(v \mid \alpha, b, a)\, g_\alpha(\alpha_j \mid b, a)\, g_u(u \mid \alpha, b, a) \cdot$$

$$\prod_j \left[g_u\left(u_j' \mid \alpha, b, a\right) g_u\left(v_j' \mid \alpha, b, a\right) g_\alpha\left(\alpha_j' \mid b, a\right) \right] du' du d\alpha$$

where γ is the ratio of potential sellers to potential buyers. As $p \to \infty$, the left hand side (LHS) of equation (3) approaches γ and the right hand side (RHS) approaches 0, and as $p \to 0$, the LHS approaches $0 \leq \zeta \leq \gamma$ and the RHS approaches $\xi \geq 0$. Also, equation (3) is continuous in p. If $\zeta < \xi$, then there exists $p > 0$ that satisfies equation (3) with positive trade, and if $\zeta \geq \xi$, $p = 0$ maybe with some trade. If $\zeta < \xi$, then, since $\frac{\partial V(u_{ij}, \alpha_j, b_j, A-1)}{\partial a_{jt-1}} < 0$, $S(b, a)$ is increasing in a and $B(b, a)$ is decreasing in a. So p must be decreasing in a.

Now assume that there exists a a such that, for all $s > a$,

$$\begin{aligned} \frac{\partial V(u_{ij}, \alpha_j, b_j, s)}{\partial \alpha_j} &\leq 0; \\ \frac{\partial V(u_{ij}, \alpha_j, b_j, s)}{\partial s} &\leq 0; \\ \frac{\partial p(b_j, s)}{\partial s} &\leq 0. \end{aligned}$$

Then

$$\begin{aligned} \frac{\partial V(u_{ij}, \alpha_j, b_j, a_{jt})}{\partial \alpha_j} &= -\frac{\partial c_{jt}}{\partial \alpha_j} + \frac{\partial}{\partial \alpha_j} \beta \max\left[W, p(b_j, a_{jt}+1) + W, V(u_{ij}, \alpha_j, b_j, a_{jt}+1)\right] \leq 0; \\ \frac{\partial V(u_{ij}, \alpha_j, b_j, a_{jt})}{\partial a_{jt}} &= -\frac{\partial c_{jt}}{\partial a_{jt}} + \frac{\partial}{\partial a_{jt}} \beta \max\left[W, p(b_j, a_{jt}+1) + W, V(u_{ij}, \alpha_j, b_j, a_{jt}+1)\right] \leq 0; \end{aligned}$$

and $\frac{\partial p(b_j, a_{jt})}{\partial a_{jt}} \leq 0$ because $\frac{\partial V(u_{ij}, \alpha_j, b_j, a_{jt})}{\partial a_{jt}} \leq 0$. By induction, the result is proven for all $a \geq 0$. ■

3 Construction of Equilibrium Prices

Step 1: Pick an age A^* such that $p(b, A^*) = 0$ for all b_j.

Step 2: Look for a fixed point in R^{J+A^*}. Standard iteration methods can be used.

4 Empirical Work

Assume we can get data on average price of a brand b car at age a, $p(b,a)$ (from the "Blue Book"). Assume we can get average cost of maintainence of a brand b car at age a,

$$\exp\left\{\mu(b) + \frac{1}{2}\sigma_{\mu}^{2} + \lambda(b,a)\right\} \tag{4}$$

(from equation (1)). Assume we can get data on the proportion of brand b cars from vintage t sold in year s,

$$\widehat{\Psi}(s-t,b) = n_{stb}/n_{ttb}$$

where n_{stb} is the number of brand b automobiles of vintage t sold in year s. The $plim\widehat{\Psi}(s-t,b)$ is

$$\begin{aligned}\Psi(s-t,b) &= \Pr[(u,\alpha) \in S(b,a)] \\ &= \iint\limits_{S(b,a)} g_u(u \mid \alpha, b, a,)\, g_{\alpha}(\alpha \mid b, a)\, du d\alpha\end{aligned}$$

(maybe from DMV data). Equation (4) identifies $\mu(b)$ and $\lambda(b,a)$, and $p(b,a)$ is identified from "Blue Book" data. The remaining parameters to estimate are $\pi_2 = \left[\sigma_u^2, \beta, U, \{\theta(b)\}_{b=1}^{J}\right]$. It is pretty likely that the DMV data overidentifies π.

We can estimate all of the parameters by first estimating $\pi_1 = \left\{\{\mu(b), \lambda(b,a), p(b,a)\}_{a=1}^{A^*}\right\}_{b=1}^{J}$ using cost and blue book data and then estimating π_2 in a MLE routine using survivor data. The log likelihood function is

$$L = \sum_{b=1}^{J} L_b$$

with

$$L_b = \sum_{a=1}^{A^*} n_{ba} \log\left[\iint\limits_{S(b,a)} g_u(u \mid \alpha, b, a,)\, g_{\alpha}(\alpha \mid b, a)\, du d\alpha\right].$$

5 Policy Issues

1) How much do equilibrium prices vary from full information equilibrium prices?

2) How much does the equilibrium survivor function vary from the full information survivor function?