HOUSING MARKET RISK

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1 Pricing housing market risk

This section develops a multi-factor asset pricing model for housing markets in which the national Case-Shiller housing price index measures national risk and Case-Shiller indexes for metropolitan areas are used to construct local risk factors orthogonal to the national risk factor. The model is formulated in continuous time as a system of stochastic differential equations (SDE's) driven by a multi-dimensional Wiener process, using as a framework the standard "multidimensional market model" in finance (see, for example, Björk (2004) or Shreve (2004)).

Our setting is a collection

$$\mathfrak{M} = \{m_1, m_2, \ldots, m_M\}$$

of metropolitan housing markets observed over a time interval $[0, T] \subset \mathbb{R}$, for example the 20-year period $[0, 20]$.[1] Assume each single-unit house can be classified into one of a finite number of housing types,

$$\mathcal{H} := \{h_1, h_2, \ldots, h_H\}$$

Housing types located in different metropolitan areas are treated as distinct. For example, if we define 10 categories of house for each metropolitan market and there are 50 metropolitan markets, then $H = 500$.

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[1] To simplify exposition, we assume for now that all metropolitan areas are observed over the same time period. In the empirical implementation, we allow the time periods to be of varying length.

Let

$$W = (W^{h_1}, \dots, W^{h_H}, W^{m_1}, \dots, W^{m_M}, W^*)$$

be an $H + M + 1$-dimensional Wiener process adapted to a filtered probability space $(\Omega, \mathfrak{F}, \mathbb{F}, \mathbb{P})$ where Ω is the sample space, $\mathfrak{F}$ a σ-algebra of measurable subsets of Ω, $\mathbb{P}$ a probability measure on $(\Omega, \mathfrak{F})$, and $\mathbb{F} := (\mathfrak{F}_t)_{t\in[0,T]}$ a filtration of sub-σ-algebras of $\mathfrak{F}$.[2] The element $\mathfrak{F}_t$ of the filtration $\mathbb{F}$ is interpreted as the "information set" known to market participants at time t. By definition, the components of W are independent 1-dimensional Wiener processes. The first H-components of W represent *idiosyncratic risk* (one component for each house type), the next M components represent *local systematic risk* (one component for each metropolitan area), and the final component represents *national systematic risk*.

Let $\widehat{V}_t^h$ denote the price of a house of type $h \in \mathcal{H}$ at time t, a random variable defined on the filtered probability space $(\Omega, \mathfrak{F}, \mathbb{F}, \mathbb{P})$. Then $\widehat{V}^h = (\widehat{V}_t^h)_{t\in[0,T]}$ represents the stochastic price process for houses of type h, a collection of random variables indexed by time t that is adapted to the flow of information represented by the filtration $\mathbb{F}$.

In finance it turns out to be useful to consider relative rather than absolute prices, expressing all asset prices relative to a numéraire asset price, usually the value of a bank account earning the risk-free rate. At time 0 the bank account has value $B_0 = 1$, and at time t

$$B_t = e^{rt} \qquad t \in [0, T] \tag{1}$$

where for now we assume that the risk-free rate $r_t = r$, a constant, for all $t \in [0, T]$. Using the bank account as numéraire, the discounted asset price at time t for a house of type h is

$$V_t^h = \frac{\widehat{V}_t^h}{B_t} \qquad t \in [0, T]$$

We use $V^h = (V_t^h)_{t\in[0,T]}$ to denote the discounted price process for houses of type h.

Discounted housing prices are characterized by stochastic differential equa-

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[2] The collection $\mathbb{F}$ of sub-sigma-algebras of $\mathfrak{F}$ is a filtration provided that $\mathfrak{F}_s \subset \mathfrak{F}_t$ for all s, t such that $0 \leq s \leq t \leq T$.

tions. For each housing type $h \in \mathcal{H}$ in metropolitan area $m \in \mathfrak{M}$,

$$\begin{aligned}
\frac{dV_t^h}{V_t^h} &= (\mu^h - \delta^h - r)dt + \sigma^{hm} dW_t^m + \sigma^{h*} dW_t^* + \sigma^{hh} dW_t^h \\
&= (\alpha^h - \delta^h)dt + \sigma^{hm} dW_t^m + \sigma^{h*} dW_t^* + \sigma^{hh} dW_t^h \qquad (2)
\end{aligned}$$

where μ^h is the expected total rate of return on housing of type h, δ^h is the rental yield, and $\alpha^h = \mu^h - r$ is the expected total rate of return discounted by the bank-account process. Shifting the rental yield to the left-hand side, we can write equation (2) in the form

$$\frac{dV_t^h}{V_t^h} + \delta^h dt = \alpha^h dt + \sigma^{hm} dW_t^m + \sigma^{h*} dW_t^* + \sigma^{hh} dW_t^h$$

or, equivalently,

$$\frac{dV_t^h}{V_t^h} + \frac{\delta^h V_t^h dt}{V_t^h} = \alpha^h dt + \sigma^{hm} dW_t^m + \sigma^{h*} dW_t^* + \sigma^{hh} dW_t^h$$

The left-hand sides of these equations represent the expected total return to housing asset h at time t, the sum of the rate of price appreciation dV_t^h / V_t^h and the rental yield δ^h (which is the ratio of the rental flow $\delta^h V_t^h dt$ to house value V_t^h). The right-hand side decomposes the expected total return into the *predictable component* (the expected discounted total return $\alpha^h dt$) and the *innovation* (the random component $\sigma^{hm} dW_t^m + \sigma^{h*} dW_t^* + \sigma^{hh} dW_t^h$). Equation (2) has solution

$$\log V_t^h = \log V_0^h + \left[\alpha^h - \delta^h - \frac{(\sigma^h)^2}{2}\right] t + \sigma^{hm} W_t^m + \sigma^{h*} W_t^* + \sigma^{hh} W_t^h \qquad (3)$$

where

$$(\sigma^h)^2 = (\sigma^{hm})^2 + (\sigma^{h*})^2 + (\sigma^{hh})^2 \qquad (4)$$

The stochastic differentials dW_t^*, dW^m and dW_t^h in equation (2) represent shocks to the national systematic-risk component W_t^*, the local systematic-risk component W_t^m and the idiosyncratic-risk component W_t^h respectively that cause the realized return dV_t^h / V_t^h for houses of type h to deviate from its expected value $\alpha^h - \delta^h$. The covariation parameters σ^{h*}, σ^{hm} and σ^{hh} measure the sensitivity of the realized return to changes in W^*, W^m and W^h respectively.

We can also express the SDE for the price process for houses of type h in terms of a 1-dimensional Wiener process. If we define the 1-dimensional Wiener process $\mathbb{W}^h$ by setting

$$\mathbb{W}_t^h := \frac{\sigma^{hm}}{\sigma^h} W_t^m + \frac{\sigma^{h*}}{\sigma^h} W_t^* + \frac{\sigma^{hh}}{\sigma^h} W_t^h \qquad t \in [0, T]$$

or, in differential form,

$$d\,\mathbb{W}_t^h := \frac{\sigma^{hm}}{\sigma^h} dW_t^m + \frac{\sigma^{h*}}{\sigma^h} dW_t^* + \frac{\sigma^{hh}}{\sigma^h} dW_t^h \qquad t \in [0, T]$$

then we can rewrite equation (2) in the form

$$\frac{dV_t^h}{V_t^h} = (\alpha^h - \delta^h)dt + \sigma^h d\,\mathbb{W}_t^h \tag{5}$$

Although 1-dimensional, the Wiener process $\mathbb{W}^h$ for housing type h is a compound process, a linear combination of the independent 1-dimensional Wiener processes W^m, W^* and W^h. If the covariation parameter σ^{h*} or σ^{hm} is non-zero, then the compound process $\mathbb{W}^h$ that drives prices for houses of type h will be correlated with the national systematic-risk component W^* or the local systematic-risk component W^m respectively.

We assume that the national Case-Shiller housing price index $V^* = (V_t^*)_{t \in [0,T]}$ is driven solely by the national risk component W^*. The price process V^* is characterized by the SDE

$$\begin{aligned} \frac{dV_t^*}{V_t^*} &= (\mu^* - \delta^* - r)dt + \sigma^* dW_t^* \\ &= (\alpha^* - \delta^*)dt + \sigma^* dW_t^* \end{aligned} \tag{6}$$

where $\alpha^* = \mu^* - r$. Equation (6) has solution

$$\log V_t^* = \log V_0^* + \left[\alpha^* - \delta^* - \frac{(\sigma^*)^2}{2}\right] t + \sigma^* W_t^* \tag{7}$$

We assume that the local Case-Shiller housing price index $V^m = (V_t^m)_{t \in [0,T]}$ for metropolitan housing market $m \in \mathfrak{M}$ is driven by both the national risk component W^* and the local risk component W^m specific to market m. The price process V^m is characterized by the SDE

$$\begin{aligned} \frac{dV_t^m}{V_t^m} &= (\mu^m - \delta^m - r)dt + \sigma^{mm} dW_t^m + \sigma^{m*} dW_t^* \\ &= (\alpha^m - \delta^m)dt + \sigma^{mm} dW_t^m + \sigma^{m*} dW_t^* \end{aligned} \tag{8}$$

where $\alpha^m = \mu^m - r$. Equation (8) has solution

$$\log V_t^m = \log V_0^m + \left[\alpha^m - \delta^m - \frac{(\sigma^m)^2}{2}\right] t + \sigma^{mm} W_t^m + \sigma^{m*} W_t^* \tag{9}$$

where

$$(\sigma^m)^2 = (\sigma^{mm})^2 + (\sigma^{m*})^2 \tag{10}$$

If we define the 1-dimensional Wiener process $\mathbb{W}^m$ by setting

$$\mathbb{W}_t^m := \frac{\sigma^{mm}}{\sigma^m} W_t^m + \frac{\sigma^{m*}}{\sigma^m} W_t^* \qquad t \in]0, T]$$

or, in differential form,

$$d\,\mathbb{W}_t^m := \frac{\sigma^{mm}}{\sigma^m} dW_t^m + \frac{\sigma^{m*}}{\sigma^m} dW_t^*$$

then we can rewrite equation (8) in the form

$$\frac{dV_t^m}{V_t^m} = (\alpha^m - \delta^m)dt + \sigma^m d\,\mathbb{W}_t^m \tag{11}$$

If the covariation parameter σ^{m*} is non-zero, then the compound process $\mathbb{W}^m$ that drives the local Case-Shiller index will be correlated with the national systematic-risk component W^* and hence the national Case-Shiller index.

By definition, for $t \geq s$ the conditional expectation of a Wiener process is zero: $\mathbb{E}[W_t^* \mid \mathfrak{F}_s] = 0$ and, for $h \in \mathfrak{H}$ and $m \in \mathfrak{M}$, $\mathbb{E}[W_t^h \mid \mathfrak{F}_s] = 0$ and $\mathbb{E}[W_t^m \mid \mathfrak{F}_s] = 0$. Consequently the unconditional expectations are also zero, and so from equation (3) we conclude that

$$\mathbb{E}[\log V_t^h] = \log V_0^h + \left[\alpha^h - \delta^h - \frac{(\sigma^h)^2}{2}\right] t \tag{12}$$

which means that the expected value of the discounted log price process for housing type h is an affine function of time, with constant term $\log V_0^h$ and slope $\alpha^h - \delta^h - \frac{(\sigma^h)^2}{2}$. From equation (2) we interpret $\alpha^h - \delta^h$ as the (conditional) expected rate of return net of rental yield for houses of type h, discounted by the bank account process. The trend in the log price is lower than the expected rate of return $\alpha^h - \delta^h$ because of the "Itô correction" $(\sigma^h)^2/2$, a correction implied by the stochastic calculus used to solve the SDE. The same argument applied to equations (7) and (9) respectively implies that the expectation of logarithm of the discounted national and local Case-Shiller indexes are also affine functions of time,

$$\mathbb{E}[\log V_t^*] = \log V_0^* + \left[\alpha^* - \delta^* - \frac{(\sigma^*)^2}{2}\right] t \tag{13}$$

and

$$\mathbb{E}[\log V_t^m] = \log V_0^m + \left[\alpha^m - \delta^m - \frac{(\sigma^m)^2}{2}\right] t \tag{14}$$

Recapitulating, we have described the metropolitan housing market in terms of a system of $H + M + 1$ SDE's, one SDE for each house type, one for each metropolitan Case-Shiller index and one for the national Case-Shiller index:

$$\begin{aligned}
\frac{dV_t^h}{V_t^h} &= (\alpha^h - \delta^h)dt + \sigma^{hm} dW_t^m + \sigma^{h*} dW_t^* + \sigma^{hh} dW_t^h \qquad (h \in \mathcal{H}) \\
\frac{dV_t^m}{V_t^m} &= (\alpha^m - \delta^m)dt + \sigma^{mm} dW_t^m + \sigma^{m*} dW_t^* \qquad (m \in \mathfrak{M}) \\
\frac{dV_t^*}{V_t^*} &= (\alpha^* - \delta^*)dt + \sigma^* dW_t^*
\end{aligned}$$

Up to this point, we have imposed no restrictions on the parameters that appear in these equations. We now impose the hypothesis that the housing market offers no opportunities for arbitrage: no self-financing[3] portfolio comprised of houses and the bank account process can make a positive profit with no risk of loss unless the initial investment is strictly positive; i.e,. there is no free lunch. Remarkably, this relatively weak hypothesis is strong enough to impose very strong restrictions on the parameters of our housing market model, a special case of the multi-dimensional market model of finance. When applied to the housing market model, the *Fundamental Theorem of Asset Pricing* asserts that the market satisfies the hypothesis of no arbitrage if and only if there exists a vector of *risk prices*

$$\lambda = (\lambda^{h_1}, \ldots, \lambda^{h_H}, \lambda^{m_1}, \ldots, \lambda^{m_H}, \lambda^*) \tag{15}$$

with norm

$$\|\lambda\| := \left[\sum_{h \in \mathcal{H}} (\lambda^h)^2 + \sum_{m \in \mathfrak{M}} (\lambda^m)^2 + (\lambda^*)^2\right]^{1/2} > 0$$

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[3] A portfolio of assets is self financing if, except for the initial outlay, no outside funds are added to the portfolio and no funds are withdrawn until the date at which the risk-less profit is claimed.

such that

$$\alpha^h = \lambda^h \sigma^{hh} + \lambda^m \sigma^{hm} + \lambda^* \sigma^{h*} \qquad (m \in \mathfrak{M}) \tag{16}$$

$$\alpha^m = \lambda^m \sigma^{mm} + \lambda^* \sigma^{m*} \qquad (h \in \mathfrak{H}) \tag{17}$$

and

$$\alpha^* = \lambda^* \sigma^* \tag{18}$$

where in equation (16) it is understood that the index m refers to the metropolitan area in which house type h is located. Equations (16)–(18), called the *market-price-of-risk equations*, have a natural economic interpretation. The left-hand sides are expected discounted returns: α^h the expected discounted return for a house of type h, α^m the expected discounted return for the Case-Shiller index for metropolitan area m, and α^* the expected discounted rate of return for the national Case-Shiller index (with rental yield included in each case). The right-hand sides compute the value of risk exposure by summing over types of risk the product of the quantity of risk (measured by the covariance parameters) times the price of risk. Thus, the market-price-of-risk equations make the very reasonable claim that markets provide no opportunity for arbitrage if and only if for every asset the *compensation for risk* (as measured by its expected return relative to the bank-account process) matches the *value of risk exposure* (as measured by price of risk times quantity of risk).

Equations (16)–(18) constitute a system of $H + M + 1$ linear equations in $H + M + 1$ unknowns, the market prices of risk. If the matrix of covariation parameters

$$\Sigma := \begin{pmatrix} \sigma^{h_1 h_1} & \dots & 0 & \sigma^{h_1 m_1} & \dots & 0 & \sigma^{h_1 *} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & \sigma^{h_H h_H} & 0 & \dots & \sigma^{h_H m_M} & \sigma^{h_H *} \\ 0 & \dots & 0 & \sigma^{m_1 m_1} & \dots & 0 & \sigma^{m_1 *} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & \dots & \sigma^{m_H m_M} & \sigma^{m_H *} \\ 0 & \dots & 0 & 0 & \dots & 0 & \sigma^* \end{pmatrix}$$

is invertible, then the market prices of risk are uniquely determined. The asset market is then said to be *complete*.

If $\sigma^{hh} \neq 0$ for all $h \in \mathfrak{H}$, $\sigma^{mm} \neq 0$ for all $m \in \mathfrak{M}$ and $\sigma^* \neq 0$, then Σ is invertible. Under this assumption, it is easy to solve for the unique market prices of risk. Equation (18) implies that

$$\lambda^* = \frac{\alpha^*}{\sigma^*} \tag{19}$$

An expression like the right-hand side of equation (19), the ratio of the equity premium of an asset to its volatility, is called a *Sharpe ratio*. Equation (19) says that the market price of the systematic risk component W^* is the Sharpe ratio of the national Case-Shiller index.

Equation (17) implies that the price of local systematic risk is

$$\lambda^m = \frac{\alpha^m - \lambda^* \sigma^{m*}}{\sigma^{mm}} \tag{20}$$

In contrast to the price of the national component of systematic risk, the price of the local component of systematic risk is *not* the Sharpe ratio of the local Case-Shiller index, the ratio α^m/σ^{mm}. By subtracting off from the expected discounted return α^m the portion $\lambda^* \sigma^{m*}$ attributional to national risk exposure, we obtain the correct measure λ^m for the price of exposure to the local risk component W^m, which by construction is orthogonal to the national risk component W^*.

By definition, if risk is *idiosyncratic* rather than systematic, then exposure to that risk contributes nothing to the expected discounted return of the asset. From equation (16), this implies that the risk component W^h is idiosyncratic if and only if its price $\lambda^h = 0$. Imposing this restriction, equation (16) reduces to

$$\alpha^h = \lambda^m \sigma^{hm} + \lambda^* \sigma^{h*} \qquad (m \in \mathfrak{M}) \tag{21}$$

which expresses the equity premium for houses of type h as the value of exposure to local risk plus the value of exposure to national risk.

For a deeper interpretation of this model, we need to introduce a few more concepts from modern asset-pricing theory. Two probability measures $\mathbb{P}$ and $\widetilde{\mathbb{P}}$ defined on the same measurable space $(\Omega, \mathfrak{F})$ are *equivalent* if $\mathbb{P}$ and $\widetilde{\mathbb{P}}$ have the same sets of measure zero. The *Fundamental Theorem of Asset Pricing* asserts that a financial market leaves no arbitrage opportunity unexploited if and only if there exists a probability measure $\widetilde{\mathbb{P}}$ equivalent to the true probability measure $\mathbb{P}$ generating the asset price processes such that the value of *every* self-financing portfolio is a martingale under the measure $\widetilde{\mathbb{P}}$. Applied to our model of a housing market, this means that for all housing types $h \in \mathcal{H}$

$$\widetilde{\mathbb{E}}[V_t^h D_t^h \mid \mathfrak{F}_s] = V_s^h D_s^h \qquad \text{for all } s, t \in [0, T],\, s \le t \tag{22}$$

where D_t^h is the cumulative rental return earned up to time t. The process $D^h = (D_t^h)_{t\in[0,T]}$ is defined by

$$D_t^h = e^{\delta^h t} \qquad t \in [0,T]$$

The tilde over the expectation sign indicates that the conditional expectation is taken with respect to the *equivalent martingale measure* (EMM) $\widetilde{\mathbb{P}}$.

If $\widetilde{\mathbb{P}}$ is an EMM for $\mathbb{P}$, then there exists a stochastic process $Z = (Z_t)_{t\in[0,T]}$ such that for every $h \in \mathcal{H}$ the process $Z(V^h D^h) = (Z_t(V_t^h D_t^h))_{t\in[0,T]}$ is a martingale under the true probability measure:

$$\mathbb{E}[Z_t(V_t^h D_t^h) \mid \mathcal{F}_s] = Z_s(V_s^h D_s^h) \qquad \text{for all } s,t \in [0,T],\, s \le t \tag{23}$$

where the conditional expectation is taken with respect to the true probability measure $\mathbb{P}$. Furthermore, the Z process takes a very simple form, a geometric Brownian motion without drift with covariation parameters equal to the market prices of risk, generated by the SDE:

$$\begin{aligned} \frac{dZ_t}{Z_t} &= -\left[\sum_{h\in\mathcal{H}} \lambda^h dW_t^h + \sum_{m\in\mathfrak{M}} \lambda^m dW_t^h + \lambda^* dW_t^*\right] \\ &= -[\sum_{m\in\mathfrak{M}} \lambda_t^m dW_t^m + \lambda^* dW_t^*] \end{aligned} \tag{24}$$

where in the second line we use the assumption that all specific risk is idiosyncratic (i.e., that $\lambda^h = 0$ for all $h \in \mathcal{H}$) to eliminate all terms on the right-hand side except the systematic risk. In finance, the Z process is called the *pricing kernel*. When Z is discounted by the bank account process, we obtain another process, called the *stochastic discount factor* (SDF): $\Lambda = (\Lambda_t)_{t\in[0,T}$ defined by

$$\Lambda_t = \frac{Z_t}{B_t} \qquad t \in [0,T]$$

From equation (24), we conclude that the SDF is generated by the SDE

$$\frac{d\Lambda_t}{\Lambda_t} = -rdt - [\sum_{m\in\mathfrak{M}} \lambda^m dW_t^m + \lambda^* dW_t^*] \tag{25}$$

If we use the SDF Λ in place of the pricing kernel Z, then for every $h \in \mathcal{H}$ the un-discounted value process $\Lambda \widehat{V}^h D^h$ is a $\mathbb{P}$ martingale:

$$\mathbb{E}[\Lambda_t \widehat{V}_t^h D_t^h \mid \mathcal{F}_s] = \Lambda_s \widehat{V}_s^h D_s^h \qquad \text{for all } s,t \in [0,T],\, s \le t \tag{26}$$

In particular, if we define $M_{t+\Delta} = \Lambda_{t+\Delta}/\Lambda_t$ and apply equation (26) over the time period $[t, t+\Delta]$, we conclude that

$$V_t^h = \mathbb{E}[M_{t+\Delta}(V_{t+\Delta}^h e^{\delta^h \Delta}) \mid \mathfrak{F}_t]$$

In words, the value of a house of type h at time t equals the conditional expectation of the payoff $V_{t+\Delta}^h e^{\delta^h \Delta}$ to owning the house over the period $[t, t+\Delta]$, discounted by the relative stochastic discount factor $M_{t+\Delta}$. This fundamental asset-pricing relationship forms the centerpiece of Cochrane (2005), a book on asset pricing in which $M_{t+\Delta}$ is interpreted as the marginal rate of substitution for a representative consumer in a discrete-time macro-economic model of the economy.

To illustrate our asset-pricing model for housing markets, consider the special case of two metropolitan areas ($M = 2$) with two housing types in each metropolitan area ($H = 4$). Assume the matrix of covariation parameters is given by

$$\Sigma = \begin{pmatrix} \sigma^{h_1h_1} & 0 & 0 & 0 & \sigma^{h_1m_1} & 0 & \sigma^{h_1*} \\ 0 & \sigma^{h_2h_2} & 0 & 0 & \sigma^{h_2m_1} & 0 & \sigma^{h_2*} \\ 0 & 0 & \sigma^{h_3h_3} & 0 & 0 & \sigma^{h_3m_2} & \sigma^{h_3*} \\ 0 & 0 & 0 & \sigma^{h_4h_4} & 0 & \sigma^{h_4m_2} & \sigma^{h_4*} \\ 0 & 0 & 0 & 0 & \sigma^{m_1m_1} & 0 & \sigma^{m_1*} \\ 0 & 0 & 0 & 0 & 0 & \sigma^{m_2m_2} & \sigma^{m_2*} \\ 0 & 0 & 0 & 0 & 0 & 0 & \sigma^{*} \end{pmatrix}$$

$$= \begin{pmatrix} 0.04 & 0 & 0 & 0 & 0.04 & 0 & 0.02 \\ 0 & 0.04 & 0 & 0 & 0.02 & 0 & 0.04 \\ 0 & 0 & 0.02 & 0 & 0 & 0.04 & 0.04 \\ 0 & 0 & 0 & 0.06 & 0 & 0.03 & 0.02 \\ 0 & 0 & 0 & 0 & 0.04 & 0 & 0.03 \\ 0 & 0 & 0 & 0 & 0 & 0.03 & 0.04 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.04 \end{pmatrix}$$

Suppose the market prices of risk take the following values:

$$\lambda = (\lambda^{h_1}, \lambda^{h_2}, \lambda^{h_3}, \lambda^{h_4}, \lambda^{m_1}, \lambda^{m_2}, \lambda^*) = (0, 0, 0, 0, 0.08, 0.08, 0.04)$$

We now work backwards to determine the parameter values consistent with this arbitrarily chosen set of equilibrium prices. From equation (16), the expected discounted returns for the Case-Shiller indexes must take the following values:

$$\alpha^{m_1} = 0.0044 \qquad \alpha^{m_2} = 0.0040 \qquad \alpha^* = 0.0032$$

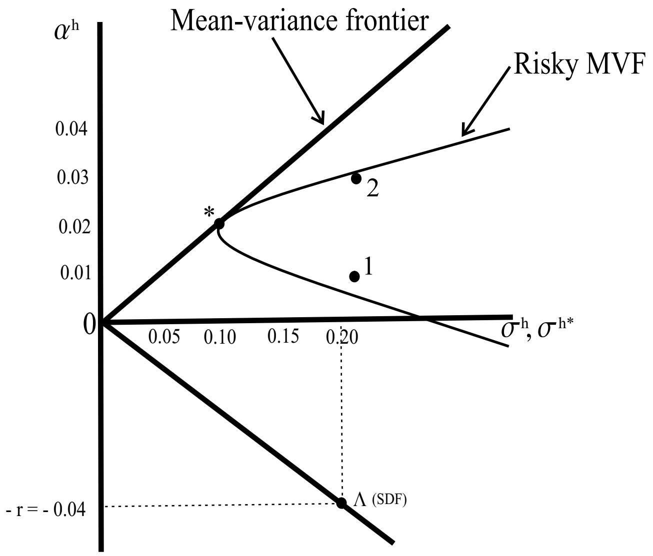

Figure 1: MVF and risky MVF

From equation (17), the expected discounted returns for the 4 housing types must be

$$\alpha^{h_1} = 0.0040 \qquad \alpha^{h_2} = 0.0032 \qquad \alpha^{h_3} = 0.0048 \qquad \alpha^{h_4} = 0.0032$$

Equation (10) gives the volatility parameters for the metropolitan Case-Shiller indexes:

$$\sigma^{m_1} = 0.05 \qquad \sigma^{m_2} = 0.05$$

The covariance parameters for idiosyncratic risk can be chosen arbitrarily, say

$$\sigma^{h_1 h_1} = 0.04 \qquad \sigma^{h_2 h_2} = 0.04 \qquad \sigma^{h_3 h_3} = 0.02 \qquad \sigma^{h_4 h_4} = 0.06$$

Equation (4) then yields the volatility parameters for the 4 housing types:

$$\sigma^{h_1} = 0.06 \qquad \sigma^{h_2} = 0.06 \qquad \sigma^{h_3} = 0.06 \qquad \sigma^{h_4} = 0.07$$

Figure 1 displays the (volatility, expected discounted return) pairs for housing types (σ^h, α^h), for the metropolitan Case-Shiller indexes (σ^m, α^m), for the national Case-Shiller index (σ^*, α^*), and for the stochastic discount factor $(\lambda^*, -r)$.

By construction, for these values the market price of risk equations $\Sigma\lambda = \alpha$ take the form

$$\begin{pmatrix} 0.04 & 0 & 0 & 0 & 0.04 & 0 & 0.02 \\ 0 & 0.04 & 0 & 0 & 0.02 & 0 & 0.04 \\ 0 & 0 & 0.02 & 0 & 0 & 0.04 & 0.04 \\ 0 & 0 & 0 & 0.06 & 0 & 0.03 & 0.02 \\ 0 & 0 & 0 & 0 & 0.04 & 0 & 0.03 \\ 0 & 0 & 0 & 0 & 0 & 0.03 & 0.04 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.04 \end{pmatrix} \begin{pmatrix} \lambda^{h_1} \\ \lambda^{h_2} \\ \lambda^{h_3} \\ \lambda^{h_4} \\ \lambda^{m_1} \\ \lambda^{m_2} \\ \lambda^{*} \end{pmatrix} = \begin{pmatrix} 0.0040 \\ 0.0032 \\ 0.0048 \\ 0.0032 \\ 0.0044 \\ 0.0040 \\ 0.0032 \end{pmatrix}$$

These equations have the unique solution

$$\lambda = (\lambda^{h_1}, \lambda^{h_2}, \lambda^{h_3}, \lambda^{h_4}, \lambda^{m_1}, \lambda^{m_2}, \lambda^{*}) = (0, 0, 0, 0, 0.08, 0.08, 0.04)$$

Because $\lambda^h = 0$ for all $h \in \mathcal{H}$, the idiosyncratic risk has no impact on the equity premium α^h of the asset.

The slope of the *mean-variance frontier* displayed in Figure 1 is given by

$$\|\lambda\| = \sqrt{(0.08)^2 + (0.04)^2 + (0.04)^2} = 0.12$$

for the portion of the MVF above the horizontal axis and by -0.12 for the portion below the horizontal axis. The MVF represents the locus of volatility, expected discounted return pairs for all portfolios of assets that minimize portfolio volatility for a given level of portfolio expected discounted return. Just as with the discrete-time theory described in Cochrane (2004), the SDF for this continuous-time housing market model is necessarily on the mean-variance frontier. From equation (25), we know that the SDF is itself a geometric Brownian motion generated by the SDE

$$\frac{d\Lambda_t}{\Lambda_t} = -r dt - \lambda^{m_1} dW_t^{m_1} - \lambda^{m_2} dW_t^{m_2} - \lambda^{*} dW_t^{*}$$

Because this market is complete, the SDF can be replicated by a portfolio of the traded housing assets and the Case-Shiller indexes. The market-price-of-risk equation for this replicating portfolio asserts that

$$-r = \lambda^{m_1}(-\lambda^{m_1}) + \lambda^{m_2}(-\lambda^{m_2}) + (\lambda^{*})(-\lambda^{*}) = -\|\lambda\|^2$$

The left-hand-side is the expected discounted return of this asset. The middle expression sums over the components of risk the product of the price of risk for each risk component times the covariance of Λ^* with the risk component. The final equality follows from the definition of the norm $\|\lambda\|$. Consequently, the risk-free rate equals the squared norm of the SDF,

$$r = \|\lambda\|^2 = 0.0144 \tag{27}$$

Therefore, the Sharpe ratio of the SDF is

$$\frac{-r}{\|\lambda\|} = -\frac{(\|\lambda\|)^2}{\|\lambda\|} = -\|\lambda\|$$

which places the SDF on the bottom branch of the MVF, as shown in Figure 1.

Notice that neither the metropolitan Case-Shiller indexes nor the national Case-Shiller index are on the MVF for this housing market. Nevertheless, as we have seen, in this model these factors suffice to price all housing assets, portfolios of housing assets, and derivatives based on housing assets, assuming of course that the market has eliminated all arbitrage opportunities.

The curved line labeled *risky MVF* in Figure 1 is the Markowitz mean-variance frontier, which differs from the true MVF because (by definition) it excludes the risk-free asset. Markowitz derived this frontier for a static model by minimizing the variance of portfolios of all risky assets subject to the constraint that the expected return of the portfolio equals a given constant k. (Here we modernize Markowitz's approach by using discounted returns rather than returns.) The same calculation used by Markowitz can be applied to the continuous-time multidimensional market model we are employing in this paper: the instantaneous variance-covariance matrix

$$V = \Sigma \Sigma^\top$$

takes the place of the static variance-covariance matrix, and expected discounted returns α and α^* replace expected returns. The equation for the risky MVF is given by

$$\sigma^2 = \frac{C}{D}\left(e - \frac{A}{C}\right)^2 + \frac{1}{C} \tag{28}$$

where σ is the volatility of the minimum-volatility portfolio with expected discounted return e. A, B, C and D are constants given by[4]

$$A = \mathbf{e}^\top V^{-1}\mathbf{1} \qquad B = \mathbf{e}^\top V^{-1}\mathbf{e} \qquad C = \mathbf{1}^\top V^{-1}\mathbf{1} \qquad D = BC - A^2$$

---

[4] The formula (23) and the expressions for the constants A, B, C and D are identical in form to the corresponding results for Markowitz's model as presented in Huang and Litzenberger (1986).

where **e** and **1** are the $H + M + 1$ column vectors of expected discounted returns and ones respectively:

$$\begin{aligned}\mathbf{e}^\top &= (\alpha^{h_1}, \dots, \alpha^{h_H}, \alpha^{m_1}, \dots, \alpha^{m_M}, \alpha^*)\\ \mathbf{1}^\top &= (1, \dots\dots, 1)\end{aligned}$$

Figure 1 plots the risky MVF implied by the parameters we have chosen.

Up to this point, we have assumed that the drift, volatility and covariation parameters of the housing market model are time invariant. However, the model generalizes immediately to allow these parameters to be arbitrary stochastic processes adapted to the same filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ on which the $H + M + 1$-dimensional Wiener process is defined. Specifically, we could assume that the price processes for the H house types and for the $M+1$ Case-Shiller indexes are generated by the system of SDE's[5]

$$\begin{aligned}\frac{dV_t^h}{V_t^h} &= (\alpha_t^h - \delta_t^h)dt + \sigma_t^{hm} dW_t^m + \sigma_t^{h*} dW_t^* + \sigma_t^{hh} dW_t^h \qquad (h \in \mathcal{H})\\ \frac{dV_t^m}{V_t^m} &= (\alpha_t^m - \delta_t^m)dt + \sigma_t^{mm} dW_t^m + \sigma_t^{m*} dW_t^* \qquad (m \in \mathfrak{M})\\ \frac{dV_t^*}{V_t^*} &= (\alpha_t^* - \delta_t^*)dt + \sigma_t^* dW_t^*\end{aligned}$$

where

- the risk-free rate $r = (r_t)_{t \in [0,T]}$,
- the drift $\mu^h = (\mu_t^h)_{t \in [0,T]}$ and covariation parameters $\sigma^{h*} = (\sigma_t^{h*})_{t \in [0,T]}$, $\sigma^{hm} = (\sigma_t^{hm})_{t \in [0,T]}$, and $\sigma^{hh} = (\sigma_t^{hh})_{t \in [0,T]}$ for every house type $h \in \mathcal{H}$,
- the drift $\mu^m = (\mu_t^m)_{t \in [0,T]}$ and volatility $\sigma^m = (\sigma_t^m)_{t \in [0,T]}$ for the metropolitan Case-Shiller indexes
- the drift $\mu^* = (\mu_t^*)_{t \in [0,T]}$ and volatility $\sigma^* = (\sigma_t^*)_{t \in [0,T]}$ for the Case-Shiller index
- the rental yields $\delta^h = (\delta_t^h)_{t \in [0,T]}$, $\delta^m = (\delta_t^*)_{t \in [0,T]}$ and $\delta^* = (\delta_t^*)_{t \in [0,T]}$

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[5] Properly speaking, these equations are stochastic differential equations only in special circumstances. Nevertheless, even when they are not SDE's, they still define stochastic differentials that can be solved "in principle" by stochastic integration because they are semimartingales. See Protter (2008) for a presentation of stochastic integration for semimartingales.

are stochastic processes adapted to the filtration $\mathbb{F}$. All of the results we have presented generalize immediately, simply by adding subscripts.

In our empirical implementation of the model we will take advantage of this flexibility by allowing the risk-free rate r_t to vary over time. We assume that the parameters μ_t^h, μ_t^m and μ_t^* vary over time as well, but in such a way that the discounted parameters $\alpha^h := \mu_t^h - r_t$ (for $h \in \mathcal{H}$), $\alpha^m := \mu_t^m - r_t$ (for $m \in \mathfrak{M}$) and $\alpha^* := \mu_t^* - r_t$ are time invariant. In this way we can allow for fluctuations in the interest rate while retaining the practical advantages of a model with constant parameters.

2 Moment estimation

To implement the model of Section 1 empirically, we require estimates of the parameters appearing in the matrix Σ and the vectors α and λ. These parameters are either first or second moments of the instantaneous returns dV_t^*/V_t^* or dV^h/V_h or, in the case of the risk prices λ, a continuous function of first and second moments.[6] Because the key implications of asset-pricing involve moment conditions, estimation using GMM is particularly appropriate. However, this requires us to translate instantaneous returns over infinitesimal intervals dt into returns over intervals $[t, t + \Delta]$ of positive duration Δ. We assume that the period of observation $[0, T]$ is divided into N intervals, each of length $\Delta = T/N$.

Estimation of the first moments is straightforward. We start with the national Case-Shiiler index. Evaluating equation (7) at the endpoints of the time interval $[t, t + \Delta]$, we obtain

$$\log V_t^* = \log V_0^* + \left[\alpha^* - \delta^* - \frac{(\sigma^*)^2}{2}\right] t + \sigma^* W_t^*$$

and

$$\log V_{t+\Delta}^* = \log V_0^* + \left[\alpha^* - \delta^* - \frac{(\sigma^*)^2}{2}\right] (t + \Delta) + \sigma^* W_{t+\Delta}^*$$

Subtracting the first equation from the second and simplifying yields

$$\log\left(\frac{V_{t+\Delta}^*}{V_t^*}\right) = \left[\alpha^* - \delta^* - \frac{(\sigma^*)^2}{2}\right] \Delta + \sigma^* (W_{t+\Delta}^* - W_t^*) \tag{29}$$

---

[6] Lurking in the background is the risk-free rate r and the parameters δ^m, δ^* and δ^h, the rental yields for the Case-Shiller indexes and for individual housing types respectively. We ignore these for now, but we will return to them later.

where $\log(V^*_{t+\Delta}/V^*_t)$ is the *log return* for the Case-Shiller index over the period $[t, t + \Delta]$. Letting $r^*_n \Delta$ denote the log return for the n^{th} interval, $\mathfrak{F}_{n-1}$ the information set at the beginning of the interval, and $\varepsilon^*_n \sqrt{\Delta}$ the increment $W^*_{t+\Delta} - W^*_t$ to the Wiener process W^* over the interval, we can rewrite equation (28) in the form

$$r^*_n \Delta = \left[\alpha^* - \delta^* - \frac{(\sigma^*)^2}{2} \right] \Delta + \sigma^* \sqrt{\Delta}\, \varepsilon^*_n \tag{30}$$

Because W^* is a Wiener process, $\varepsilon = (\varepsilon_n)_{n=1}^N$ is an indepndent sequence of standard normal random variables. Consequently,

$$\mathbb{E}\left[r^*_n - \left(\alpha^* - \delta^* - \frac{(\sigma^*)^2}{2} \right) \mid \mathfrak{F}_{n-1} \right] = 0$$

gives the first-moment condition for the Case-Shiller index, where we have used the fact that

$$\mathbb{E}[\sigma^* \sqrt{\Delta}\, \varepsilon^*_n \mid \mathfrak{F}_{n-1}] = \sigma^* \sqrt{\Delta}\, \mathbb{E}[\varepsilon^*_n \mid \mathfrak{F}_{n-1}] = 0$$

Because ε is an i.i.d. sequence of standard normal variables, the corresponding sample moment converges $\mathbb{P}$-a.s. and hence in probability to the population moment: letting

$$\hat{\alpha}^*_N - \hat{\delta}^*_N - \frac{(\hat{\sigma}^*_N)^2}{2} := \frac{1}{T} \sum_{n=1}^{N} r^*_n \tag{31}$$

denote the estimator, we have

$$\hat{\alpha}^*_N - \hat{\delta}^*_N - \frac{(\hat{\sigma}^*_N)^2}{2} \xrightarrow{p} \alpha^* - \delta^* - \frac{(\sigma^*)^2}{2}$$

as $T \to \infty$ and hence $N = T/\Delta \to \infty$.

Estimators for the other first moments are constructed in the same way. For the metropolitan Case-Shiller indexes, evaluating equation (9) at the endpoints of the interval $[t, t + \Delta]$ and differencing yields

$$\log\left(\frac{V^m_{t+\Delta}}{V^m_t} \right) = \left[\alpha^m - \delta^m - \frac{(\sigma^m)^2}{2} \right] \Delta + \sigma^{mm}(W^m_{t+\Delta} - W^m_t) + \sigma^{m*}(W^*_{t+\Delta} - W^*_t) \tag{32}$$

where $\log(V^m_{t+\Delta}/V^m_t)$ is the log return for the Case-Shiller index over the period $[t, t + \Delta]$. Let $r^m_n \Delta$ denote the log return for the n^{th} interval, $\mathfrak{F}_{n-1}$ the information set at the beginning of the interval, and $\varepsilon_n^m \sqrt{\Delta}$ the increment $W_{t+\Delta}^m - W_t^m$ to the Wiener process W^m over the interval. We then can rewrite equation (32) in the form

$$r_n^m \Delta = \left[\alpha^m - \delta^m - \frac{(\sigma^m)^2}{2}\right] \Delta + \sigma^{mm} \sqrt{\Delta}\, \varepsilon_n^m + \sigma^{m*} \sqrt{\Delta}\, \varepsilon_n^* \tag{33}$$

where $\varepsilon^m = (\varepsilon_n^m)_{n=1}^N$ and ε^* are independent sequences of standard normal random variables. Consequently,

$$\mathbb{E}\left[r_n^m - \left(\alpha^m - \delta^m - \frac{(\sigma^m)^2}{2}\right) \mid \mathfrak{F}_{n-1}\right] = 0$$

gives the first-moment condition for the Case-Shiller index for metropolitan area m. Letting

$$\hat{\alpha}_N^m - \hat{\delta}_N^m - \frac{(\hat{\sigma}_N^m)^2}{2} := \frac{1}{T} \sum_{n=1}^N r_n^m \tag{34}$$

denote the estimator, we have

$$\hat{\alpha}_N^m - \hat{\delta}_N^m - \frac{(\hat{\sigma}_N^m)^2}{2} \xrightarrow{p} \alpha^m - \delta^m - \frac{(\sigma^m)^2}{2}$$

as $T \to \infty$ and hence $N = T/\Delta \to \infty$.

For houses of type h, evaluating equation (3) at the endpoints of the interval $[t, t+\Delta]$ and differencing yields

$$\begin{aligned} \log\left(\frac{V_{t+\Delta}^h}{V_t^h}\right) &= \left[\alpha^h - \delta^h - \frac{(\sigma^h)^2}{2}\right] \Delta + \sigma^{hm} (W_{t+\Delta}^m - W_t^m) \\ &\quad + \sigma^{h*} (W_{t+\Delta}^* - W_t^*) + \sigma^{hh} (W_{t+\Delta}^h - W_t^h) \end{aligned} \tag{35}$$

where $\log(V_{t+\Delta}^h / V_t^h)$ is the log return for housing of type h over the interval $[t, t+\Delta]$. Let $r_n^h \Delta$ denote the log return for the n^{th} interval, $\mathfrak{F}_{n-1}$ the information set at the beginning of the interval, and $\varepsilon_n^h \sqrt{\Delta}$ the increment $W_{t+\Delta}^h - W_t^h$ to the Wiener process W^h over the interval. We then can rewrite equation (35) in the form

$$r_n^h \Delta = \left[\alpha^h - \delta^h - \frac{(\sigma^h)^2}{2}\right] \Delta + \sigma^{hm} \sqrt{\Delta}\, \varepsilon_n^m + \sigma^{h*} \sqrt{\Delta}\, \varepsilon_n^* + \sigma^{hh} \sqrt{\Delta}\, \varepsilon_n^h \tag{36}$$

where $\varepsilon^h = (\varepsilon_n^h)_{n=1}^N$, ε^m and ε^* are independent sequences of standard normal random variables. Consequently,

$$\mathbb{E}\left[r_n^h - \left(\alpha^h - \delta^h - \frac{(\sigma^h)^2}{2}\right) \mid \mathfrak{F}_{n-1}\right] = 0$$

gives the first-moment condition for housing of type h. Letting

$$\hat{\alpha}_N^h - \hat{\delta}_N^h - \frac{(\hat{\sigma}_N^h)^2}{2} := \frac{1}{T}\sum_{n=1}^{N} r_n^h \tag{37}$$

denote the estimator, we have

$$\hat{\alpha}_N^h - \hat{\delta}_N^h - \frac{(\hat{\sigma}_N^h)^2}{2} \xrightarrow{p} \alpha^h - \delta^h - \frac{(\sigma^h)^2}{2}$$

as $T \to \infty$ and hence $N = T/\Delta \to \infty$.

A remarkable fact about finance models driven by Brownian motion is that, with high-frequency data, second moments of returns data can be estimated much more precisely than their first moments. The key to the estimation strategy involves auxiliary stochastic processes called quadratic variation or quadratic covariation. We begin with quadratic variation. Applying Itô's Lemma to equation (7) implies that the log price process for the Case-Shiller index is the solution to the SDE

$$d \log V_t^* = \left[\alpha^* - \delta^* - \frac{(\sigma^*)^2}{2}\right] dt + \sigma^* d W_t^* \tag{38}$$

Consequently, the quadratic variation of the process $\log V^*$ is the stochastic process $[\log V^*, \log V^*] = ([\log V^*, \log V^*]_t)_{t\in[0,T]}$ defined by

$$[\log V^*, \log V^*]_t = (\sigma^*)^2 t \qquad t \in [0, T] \tag{39}$$

A heuristic proof goes as follows.[7] The differential $d[\log V^*, \log V^*]_t$ can be interpreted as the square of the differentials $d \log V_t^*$. Therefore,

$$\begin{aligned} d[\log V^*, \log V^*]_t &= d \log V_t^* d \log V_t^* \\ &= \left\{\left[\alpha^* - \delta^* - \frac{(\sigma^*)^2}{2}\right] dt + \sigma^* d W_t^*\right\}^2 \\ &= (\sigma^*)^2 d W_t^* d W_t^* \\ &= (\sigma^*)^2 dt \end{aligned}$$

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[7] For a formal proof, see Protter (2004) or Shreve (2004).

where we use the "Itô rules" of stochastic calculus,

$$dtdt = 0 \qquad dtdW_t^* = 0 \qquad dW_t^* dW_t^* = dt$$

Integrating the SDE $d[\log V^*, \log V^*]_t = (\sigma^*)^2 dt$ gives equation (39). Thus, if the Case-Shiller index is driven by Brownian motion, then its quadratic-variation process is a *deterministic* function of time, a linear function with slope equal to $(\sigma^*)^2$, the square of the volatility parameter. What makes this result so useful is that, with high-frequency data, this function can be approximated very accurately: as $\Delta \longrightarrow 0$, the empirical process defined by the cumulative square returns converges uniformly on compacts in probability to the stochastic volatility.[8] Suppose $[0, T]$ is divided into N intervals of length Δ. Letting $\log R_n^* = r_n^* \Delta$ denote the log return of the Case-Shiller index over the n^{th} interval,

$$\sum_{n=1}^{N} (\log R_n^*)^2 \xrightarrow{p} [\log V^*, \log V^*]_T = (\sigma^*)^2 T$$

as $N \to \infty$, or equivalently as $\Delta \to 0$, and so

$$\hat{\sigma}_N^* := \sqrt{\frac{1}{T} \sum_{n=1}^{N} (\log R_n^*)^2} \xrightarrow{p} \sigma^*$$

In other words, the sample volatility on the left is a consistent estimator of the volatility parameter σ^*.

The same argument applies to the volatility parameter for the Case-Shiller price index for metropolitan area m. Its volatility process $[\log V^m, \log V^m]$ is defined by

$$[\log V^m, \log V^m]_t = (\sigma^m)^2 t \qquad t \in [0, T] \tag{40}$$

which is a straight line with slope $(\sigma^m)^2$. The empiric process of cumulative squared log returns converges uniformly on compacts in probability to this deterministic function,

$$\sum_{n=1}^{N} (\log R_n^m)^2 \xrightarrow{p} [\log V^m, \log V^m]_T = (\sigma^h)^2 T$$

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[8] See Protter (2004), Theorem II.22

and hence

$$\hat{\sigma}_N^m := \sqrt{\frac{1}{T}\sum_{n=1}^{N}(\log R_n^m)^2} \xrightarrow{p} \sigma^m$$

as $N \to \infty$ or, equivalently, $\Delta \to 0$. The sample volatility on the left is a consistent estimator of the volatility parameter on the right.

The volatility of house prices of type h is estimated in the same way. The volatility process $[\log V^h, \log V^h]$ is defined by

$$[\log V^h, \log V^h]_t = (\sigma^h)^2 t \qquad t \in [0, T] \tag{41}$$

which is a straight line with slope $(\sigma^h)^2$. The empiric process of cumulative squared log returns converges uniformly on compacts in probability to this deterministic function,

$$\sum_{n=1}^{N}(\log R_n^h)^2 \xrightarrow{p} [\log V^h, \log V^h]_T = (\sigma^h)^2 T$$

and hence

$$\hat{\sigma}_N^h := \sqrt{\frac{1}{T}\sum_{n=1}^{N}(\log R_n^h)^2} \xrightarrow{p} \sigma^h$$

as $N \to \infty$ or, equivalently, $\Delta \to 0$. The sample volatility on the left is a consistent estimator of the volatility parameter on the right.

Covariation processes play the same role in estimating covariance parameters that the quadratic variation process plays in estimating volatility parameters. To estimate σ^{m*}, we introduce the covariation process $[\log V^m, \log V^*] = ([\log V^m, \log V^*]_t)_{t\in[0,T]}$, which is defined by

$$[\log V^m, \log V^*]_t = \sigma^{m*}\sigma^* t \qquad t \in [0, T] \tag{42}$$

The heuristic proof is the following:

$$d[\log V^m, \log V^*]_t = d\log V_t^m d\log V_t^*$$

Substituting

$$d\log V_t^m = \left[\alpha^m - \delta^m - \frac{(\sigma^m)^2}{2}\right]dt + \sigma^{mm}dW_t^m + \sigma^{m*}dW_t^*$$

and

$$d \log V_t^* = \left[\alpha^* - \delta^* - \frac{(\sigma^*)^2}{2}\right] dt + \sigma^* dW_t^*$$

and using the Itô rules

$$dtdt = 0 \quad dtdW_t^m = 0 \quad dtdW_t^* = 0 \quad dW_t^m dW_t^* = 0 \quad dW_t^* dW_t^* = dt$$

we obtain the SDE

$$d[\log V^m, \log V^*]_t = \sigma^{m*}\sigma^* dt$$

which has as its solution equation (42). Thus, the covariation process is also a deterministic function of time, a linear equation with slope $\sigma^{m*}\sigma^*$. The cumulative sum of the products $\log R_n^h \log R_n^*$ converges uniformly in probability on compacts to this deterministic function.[9] Consequently,

$$\sum_{n=1}^{N} \log R_n^m \log R_n^* \xrightarrow{p} [\log V^m, \log V^*]_T = \sigma^{m*}\sigma^* T$$

and so

$$\hat{\sigma}_N^{h*}\hat{\sigma}_N^* := \frac{1}{T}\sum_{n=1}^{N} \log R_n^h \log R_n^* \xrightarrow{p} \sigma^{h*}\sigma^* T$$

as $N \to \infty$ (equivalently $\Delta \to 0$), providing a consistent estimator for $\sigma^{m*}\sigma^*$.

The instantaneous *correlation* between the Wiener processes $\mathbb{W}^m$ and W^* is

$$\rho^{m*} = \frac{\sigma^{m*}\sigma^*}{\sigma^m\sigma^*} = \frac{\sigma^{m*}}{\sigma^h} \tag{43}$$

Thus, the slope of the quadratic covariance process can be interpreted as the instantaneous covariance between the return on the Case-Shiller index for metropolitan area m and the national Case-Shiller index. Rewriting equation (9) in the form

$$\begin{aligned} \log V_t^m &= \log V_0^m + \left[\alpha^m - \delta^m - \frac{(\sigma^m)^2}{2}\right] t + \sigma^m \left[\rho^{m*} W_t^* + \sqrt{1 - (\rho^{m*})^2}\, W_t^m\right] \\ &= \log V_0^m + \left[\alpha^m - \delta^m - \frac{(\sigma^m)^2}{2}\right] t + \sigma^m \mathbb{W}_t^m \end{aligned} \tag{44}$$

brings out more clearly the interpretation of ρ^{m*} as the correlation between the instantaneous return on the Case-Shiller index for metropolitan area m and the instantaneous return to the national Case-Shiller index.

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[9] See Protter (2004), Theorem II.24.

Estimators for the covariation parameters σ^{hm} and σ^{h*} for houses of type h are obtained in the same way. It is straightforward to show that

$$d[\log V^h, \log V^m]_t = (\sigma^{hm}\sigma^{mm} + \sigma^{h*}\sigma^{m*})dt$$

and so

$$[\log V^h, \log V^m]_t = (\sigma^{hm}\sigma^{mm} + \sigma^{h*}\sigma^{m*})t \qquad t \in [0, T] \tag{45}$$

Consequently,

$$\sum_{n=1}^{N} \log R_n^h \log R_n^m \xrightarrow{p} [\log V^m, \log V^*]_T = (\sigma^{hm}\sigma^{mm} + \sigma^{h*}\sigma^{m*})T$$

and so

$$\hat{\sigma}_N^{hm}\hat{\sigma}_N^{mm} + \hat{\sigma}_N^{h*}\hat{\sigma}_N^{m*} := \frac{1}{T}\sum_{n=1}^{N} \log R_n^h \log R_n^m \xrightarrow{p} \sigma^{hm}\sigma^{mm} + \sigma^{h*}\sigma^{m*}$$

as $N \to \infty$ (equivalently $\Delta \to 0$), providing a consistent estimator for $\sigma^{hm}\sigma^{mm} + \sigma^{h*}\sigma^{m*}$.

Similarly,

$$d[\log V^h, \log V^*]_t = \sigma^{h*}\sigma^* dt$$

and so

$$[\log V^h, \log V^*]_t = \sigma^{h*}\sigma^* t \qquad t \in [0, T] \tag{46}$$

Consequently,

$$\sum_{n=1}^{N} \log R_n^h \log R_n^* \xrightarrow{p} [\log V^h, \log V^*]_T = \sigma^{h*}\sigma^* T$$

and so

$$\hat{\sigma}^{h*}\hat{\sigma}^* := \frac{1}{T}\sum_{n=1}^{N} \log R_n^h \log R_n^* \xrightarrow{p} \sigma^{h*}\sigma^*$$

as $N \to \infty$ (equivalently $\Delta \to 0$), providing a consistent estimator for $\sigma^{h*}\sigma^*$.

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HERE IS WHERE I AM STOPPING FOR NOW. A FEW REMAKRS:

- We again face an identification problem because of the rental yield. If one of the Case-Shiller yields is identified (for example, the dividend yield for San Francisco), then ALL of the parameters are identified. However, it does not help to have an estimate for one of the δ^h's.
- It should be easy to extend the estimates Paul and I did for San Francisco to this model. We can estimate all of the parameters by just adding the Case-Shiller national index to what we already did and then computing all the parameters (several of the computations will be the same).