Nonparametric and Semiparametric Analysis of a Dynamic Game Model

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1 Introduction

Game theory has had a profound effect on microeconomic theory and theoretical industrial organization in particular. Also, game theory has had an important impact on economic policy, especially in antitrust and regulation. It is therefore desirable to have empirical methods that are applicable when agents are behaving strategically as predicted by game theory. Following Bresnahan and Reiss (1991a,b,c) we study econometric models of gaming where players choose between a finite number of mutually exclusive actions. As in a standard discrete choice model, utility depend on exogenous covariates, preference parameters and random preference shocks. However, these models generalize standard discrete choice models by allowing an agent’s utility to depend on the actions of other agents.

The first goal of this paper is to study nonparametric identification of static and dynamic discrete games under the incomplete information assumption, where the error terms are private information to each agent. Identification arguments are valid regardless of whether the state variables are discrete or continuous. In a static game, individual exclusion restrictions are sufficient for identifying the latent mean utility functions. Identification of a dynamic discrete game follows from the identification of a single agent dynamic discrete choice model and of a static game.

The identification result for the dynamic discrete game naturally suggests a nonparametric estimator which is robust against misspecification of parametric assumptions but may suffer from the curse of dimensionality. The second goal of this paper is to develop a flexible semiparametric estimator when the static utility function is a linear index. This estimator is very easy to implement and does not require any numerical optimization, and allows for both discrete and continuous state variables. A unified large sample theory is developed that nests both continuous and discrete state variables as special cases.

Finally, we use our proposed method to analyze the empirical ......
an improvement over the existing literature by being more general and more flexible. Both identification results and estimation methods are applicable regardless of whether the state variables have a discrete or continuous distribution. Both nonparametric and semiparametric estimation methods are developed. The representation of the asymptotic distribution for the semiparametric estimator also unifies both discrete and continuous state variable distributions.

The usual approach in the nested fixed point algorithm is to discretize the state space, which is only required to be precise enough subject to the constraints imposed by the computing facility. However, increasing the number of grids in a nonparametric or two stage semiparametric method has two offsetting effects. It reduces the bias in the first stage estimation but also increases the variance. In fact, when the dimension of the continuous state variables is larger than four, it can be shown that it is not possible to obtain through discretization $\sqrt{T}$ consistent and asymptotically normal parameter estimates in the second stage, where $T$ is the sample size. Therefore, discretizing the state space does not provide a solution to continuous state variables, which requires a more refined econometric analysis.

Section 2 discusses identification in a static discrete game model. Section 3 extends the identification analysis to a dynamic game. Section 5 develops nonparametric and semiparametric estimation methods, ......

2 Nonparametric identification of static games

We begin by describing the model for the case of static games. This serves two purposes. First, this will allow us to discuss some key modeling assumptions in a simpler setting. Second, we sketch a proof of identification for the static model. This will highlight some key ideas in our identification of the full dynamic model and also will be used as a step in the identification of the more general dynamic model.

In the model, there are a finite number of players $i = 1, \ldots, n$. Each player simultaneously chooses an action $a_i \in \{0, 1, \ldots, K\}$ out of a finite set. We assume that the set of actions are identical across players. This is for notational convenience only and could easily be weakened at the cost of notational complexity. Let $A = \{0, 1, \ldots, K\}^n$ denote the set of possible actions for all players and $a = (a_1, \ldots, a_n)$ denote a generic element of $A$. We
shall let $a_{-i} = (a_1, \ldots a_{i-1}, a_{i+1}, \ldots, a_n)$ denote a vector of strategies for all players excluding player $i$.

Let $s_i \in S_i$ denote the state variable for player $i$. The set $S_i$ can be discrete, continuous or both. Also, define $S = \Pi_i S_i$ and let $s = (s_1, \ldots, s_n) \in S$ denote a vector of state variables for all $n$ players. We assume that $s$ is common knowledge to all players in the game and is observable to the econometrician.

For each agent, there are $K + 1$ private shocks $\epsilon_i(a_i)$ indexed by the actions $a_i$. Let $\epsilon_i = (\epsilon_i(0), \ldots, \epsilon_i(K))$ have a density $f(\epsilon_i)$ and assume that the shocks $\epsilon_i$ are i.i.d across agents and actions $a_i$. In this paper, we shall assume that $\epsilon_i(a_i)$ is distributed extreme value, that is, it has a density $f(\epsilon_i) = \exp(-\exp(\epsilon_i))$. We could easily weaken this assumption. However, it is commonly used in the applied literature and will allow us to write a number of formulas in closed form which will be useful for coding our estimator.

A1 The error terms $\epsilon_i(a_i)$ are distributed i.i.d. across actions and agents. Furthermore, the error term has an extreme value distribution with density $f(\epsilon_i) = \exp(-\exp(\epsilon_i))$

The vNM utility function for player $i$ is:

$$u_i(a, s, \epsilon_i) = \Pi_i(a_i, a_{-i}, s) + \epsilon_i(a_i).$$

In the above, $\Pi_i(a_i, a_{-i}, s)$ is a scalar which depends on $i$’s own actions, the actions of all other agents $a_{-i}$ and the entire vector of state variables $s$. Utility is a function of this known function and player $i$’s private information $\epsilon_i(a_i)$. Note that the assumption that the error term $\epsilon_i(a_i)$ is private information is not universal in the literature. For example, Bresnahan and Reiss (XXXX), Berry (XXXX), Tamer (XXXX) and Bajari, Hong and Ryan (XXXX) assume that the error terms are common knowledge. Such a model requires quite different econometric methods which explicitly account for the presence of multiple equilibrium. The models with private information are typically simpler analytically as we shall demonstrate below.

A strategy for agent $i$ can be written as $a_i = \delta_i(s, \epsilon_i)$. That is, a strategy is a map which is a function of the state $s$ and agent $i$’s private information $\epsilon_i$. Note that agent $i$’s strategy does not depend on $\epsilon_{-i}$ since this is assumed to be private information to the other
agents in the game. Define

\[ \sigma_i(a_i = k | s) = \int 1 \{ \delta_i(s, \epsilon_i) = k \} f(\epsilon_i) d\epsilon_i. \]

This is the probability that agent \( i \) will play strategy \( k \) after we marginalize out the error term \( \epsilon_i \). In equilibrium, player \( j \neq i \) does not know \( \epsilon_i \) since it is private information to agent \( i \). Therefore, \( j \)'s belief is that \( i \) will play strategy \( k \) with probability \( \sigma_i(a_i = k | s) \).

The utility maximizing strategy will be a best response to this set of beliefs.

The expected utility to player \( i \) from choosing strategy \( a_i \) can be written as

\[
\sum_{a_{-i}} \Pi_i(a_i, a_{-i}, s) \sigma_{-i}(a_{-i} | s) + \epsilon_i(a_i)
\]  

(1)

Players \(-i\) use the strategies \( a_{-i} \) with probability \( \sigma_{-i}(a_{-i} | s) \). The term

\[
\sum_{a_{-i}} \Pi_i(a_i, a_{-i}, s) \sigma_{-i}(a_{-i} | s)
\]

is the expected value of \( \Pi_i(a_i, a_{-i}, s) \) averaging over the actions of the other players. Expected utility is therefore the sum of the expected value of \( \Pi_i(a_i, a_{-i}, s) \) and player \( i \)'s private information \( \epsilon_i(a_i) \). Moving forward, it will be useful to define the choice specific value function as

\[
\Pi_i(a_i, s) = \sum_{a_{-i}} \Pi_i(a_i, a_{-i}, s) \sigma_{-i}(a_{-i} | s).
\]  

(2)

This term is the expected utility from choosing \( a_i \) when the preference shock is excluded. To summarize, we can write the expected utility from choosing \( a_i \) as

\[
\Pi_i(a_i, s) + \epsilon_i(a_i)
\]  

(3)

Recall that the error terms are distributed extreme value. Standard results about the logit model imply that

\[
\sigma_i(a_i = k | s) = \frac{\exp(\Pi_i(a_i, s))}{\sum_{a'_i \in A_i} \exp(\Pi_i(a'_i, s))}
\]  

(4)

An attractive feature of this model is that the probability with which agent \( i \) chooses action \( k \) can be expressed as a simple function of \( \Pi_i(a_i, s) \) reminiscent of the logit model.
Definition Fix the state $s$. A Bayes-Nash equilibrium is a collection of probabilities, $\sigma^*_i(a_i = k|s)$ for $i = 1, ..., n$ and $k = 0, ..., K$ such that for all $i$ and all $k$

$$\sigma^*_i(a_i = k|s) = \frac{\exp(\Pi^*_i(a_i, s))}{\sum_{a'_i \in A_i} \exp(\Pi^*_i(a'_i, s))}$$

$$\Pi^*_i(a_i, s) = \sum_{a_{-i}} \Pi_i(a_i, a_{-i}, s)\sigma^*_{-i}(a_{-i}|s).$$

The definition of equilibrium is standard. An equilibrium requires the actions of all players to be a best response to the actions of all other players. However, we find it convenient to define an equilibrium in terms of $\sigma_i(a_i = k|s)$ instead of using the more primitive $\delta_i(s, \epsilon_i)$. Obviously, given $\sigma_i(a_i = k|s)$ we could derive $\delta_i(s, \epsilon_i)$ almost trivially using the analysis above.

2.1 Simple Example

In this subsection, we display the simplest possible example of our discrete games in order to illustrate the key ideas. Following Bresnahan and Reiss (XXXX) and Tamer (XXXX), we consider a two player entry game. Suppose that two firm (e.g. Target and WalMart) simultaneously decide whether not to enter a particular market. In applied, such a market usually corresponds to the geographic boundaries of a city. Let there be $t = 1, ..., T$ distinct markets. We let $s_t$ denote the state variables for market $t$ and $a_{it}$ denote the entry decision for player $i$ in market $t$. We shall follow the convention that $a_{it} = 1$ means that firm $i$ chooses to enter market $t$ and $a_{it} = 0$ means no entry. For the purposes of our example, we shall assume that $i$'s payoffs take the form:

$$u(a_{it}, s_t, \epsilon_{it}) = \begin{cases} 
\beta_1 \text{Pop}_t + \beta_2 \text{DIST}_{it} + \beta_3 a_{-i} + \epsilon_i(a_i) & \text{if } a_{it} = 1 \\
0 & \text{if } a_{it} = 0 
\end{cases}$$

(5)

In empirical work on entry, $u(a_{it}, s_t, \epsilon_{it})$ is meant to be a reduced form approximation to firm $i$'s profits from entering market $t$. Theoretical models of oligopoly suggest that the profits from entry should depend on cost shifters, demand shifters and market power. In applied work, the number of people living in a market, which we denote as $\text{Pop}_t$, is often used as a demand shifter. This seems natural since the number of people potentially entering a big box store such as Walmart or Target is certainly a first order determinant
of their final demand. The term $\beta_1$ is a free parameter which captures the importance of this variable. The distance of firm $i$ from headquarters, $DIST_{it}$, is a cost shifter used in empirical work. Previous empirical work on this industry demonstrates that Walmart first entered in markets near its corporate headquarters in Bentonville, Arkansas. Target is much more likely to be present in markets near its corporate headquarters of Minneapolis. Researchers normally attribute this to the importance of proximity to distribution centers in transportation costs. Also, we would expect managerial efforts to be more effect in stores that are close to headquarters. The term $a_{-i}$ is meant to capture competitive effects. If $-i$ is in market $t$, we would expect $i$’s market power to be lower. The term $\epsilon_i(a_i)$ is meant to capture idiosyncratic shocks to $i$’s profits or entry decisions. These could be omitted supply and demand variables. Our modeling assumptions imply that these omitted factors are private information to firm $i$. Note that we normalize the profits from not entering to 0 as is standard in the literature.

Let $\sigma_i(a_i = 1|s_t)$ denote the probability that $i$ enters market $t$. The choice specific value function for $i$ can be written as

$$\Pi_i(a_i = 1, s) = \beta_1 Pop_t + \beta_2 DIST_{it} + \beta_3 \sigma_{-i}(a_{-i} = 1|s_t)$$  \hspace{1cm} (6)

The difference between (5) and (6) is that we replace $a_{-i}$ with $\sigma_{-i}(a_{-i} = 1|s_t)$ and we omit $\epsilon_i(a_i)$. The choice specific value function is the expected value of $u(a_{it}, s_t, \epsilon_{it})$ when we average over $i$’s beliefs about $-i$’s actions. The choice specific value function omits $i$’s private information.

We can now express the equilibrium to the model as the following system of nonlinear equations:

$$\sigma_1(a_1 = 0|s_t) = \frac{1}{1 + \exp(\beta_1 Pop_t + \beta_2 DIST_{1t} + \beta_3 \sigma_2(a_2 = 1|s_t))}$$

$$\sigma_1(a_1 = 1|s_t) = \frac{\exp(\beta_1 Pop_t + \beta_2 DIST_{1t} + \beta_3 \sigma_2(a_2 = 1|s_t))}{1 + \exp(\beta_1 Pop_t + \beta_2 DIST_{1t} + \beta_3 \sigma_2(a_2 = 1|s_t))}$$

$$\sigma_2(a_2 = 0|s_t) = \frac{1}{1 + \exp(\beta_1 Pop_t + \beta_2 DIST_{2t} + \beta_3 \sigma_1(a_1 = 1|s_t))}$$

$$\sigma_2(a_2 = 1|s_t) = \frac{\exp(\beta_1 Pop_t + \beta_2 DIST_{2t} + \beta_3 \sigma_1(a_1 = 1|s_t))}{1 + \exp(\beta_1 Pop_t + \beta_2 DIST_{2t} + \beta_3 \sigma_1(a_1 = 1|s_t))}$$

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The equilibrium probabilities, $\sigma_1(a_1 = 0|s_t), \sigma_1(a_1 = 1|s_t), \sigma_2(a_2 = 0|s_t), \sigma_2(a_2 = 1|s_t)$ are the solution to for nonlinear equations in these four unknowns. The choice probabilities are quite similar to the logit model. A main difference is that the right hand side of $i$’s choice probabilities involve the choice probabilities for $-i$. This framework has a number of attractive features. First, becaus because we build our model by generalizing the logit, agents are indifferent between actions with probability zero. Therefore, we do not need to worry about the technical difficulties implied by mixed strategies. Second, the equilibrium strategies can be expressed as the solution to a closed form system of equations. Finally, our model is a strict generalization of the commonly used logit model which is commonly used in discrete choice. The main difference is that we include agent $i$’s beliefs about $-i$’s entry decision as a right hand side variable.

### 2.2 Identification of the static model

Obviously, an important question is whether it is possible for us to identify the parameters of our model. If we are willing to use the parameteric specification in (5), the model is likely to be identified simply because the parametric restrictions will imply a unique maximum to an appropriately defined maximum likelihood estimator. However, one might argue that a strategy is not desirable because it relies heavily on correct specification of the functional form of the payoffs. We choose a more general approach and attempt to identify $\Pi_i(a_i, a_{-i}, s_i)$ by making only minimal restrictions on this function.

**Definition** Let $\Pi_i(a_i, a_{-i}, s_i)$ and $\bar{\Pi}_i(a_i, a_{-i}, s_i)$ be two different specifications of the payoffs that are not identical, i.e. $\Pi_i(a_i, a_{-i}, s_i) \neq \bar{\Pi}_i(a_i, a_{-i}, s_i)$. Also let $\sigma_i(a_i = k|s)$ and $\bar{\sigma}_i(a_i = k|s)$ be the corresponding equilibrium choice probabilities for $i = 1, \ldots, n$. We say that our model is identified if $\Pi_i(a_i, a_{-i}, s_i) \neq \bar{\Pi}_i(a_i, a_{-i}, s_i)$ implies that $\sigma_i(a_i = k|s) \neq \bar{\sigma}_i(a_i = k|s)$.

Our proof of identification is constructive. The strategy is to start by assuming that the econometric knows the population probabilities $\sigma_i(a_i = k|s)$ for all $k, i$ and $s$. We then “reverse engineer” the $\Pi_i(a_i, a_{-i}, s_i)$ that rationalize the data. Starting from (4), simple
algebra implies that

\[
\sigma_i(a_i = k|s) = \frac{\exp(\Pi_i(a_i, s))}{\sum_{a_i' \in A_i} \exp(\Pi_i(a_i', s))}
\]

(7)

\[
\log(\sigma_i(a_i = k|s)) - \log(\sigma_i(a_i = 0|s)) = \Pi_i(a_i = k, s) - \Pi_i(a_i = 0, s)
\]

(8)

Equation (8) is the well known Hotz-Miller inversion. This equation implies that it is possible to learn the choice specific value functions, \( \Pi_i(a_i = k, s) \) up to a first difference from knowledge of the choice probabilities \( \sigma_i(a_i = k|s) \). Since these choice-specific value functions can only be learned up to a first difference, we need to impose the normalization that an “outside good” action always yields zero utility:

\[
\Pi_i(a_i = 0, a_{-i}, s) = 0.
\]

(9)

In our simple example of the previous subsection, the outside good assumption is imposed by assuming that the mean utility from not entering the market is zero. Given that we have imposed (9), we can now learn \( \Pi_i(a, s) \) from (8) since the choice specific value from choosing action \( a_i = 0 \) must be equal to zero.

\( A2 \) For all \( i \), all \( a_{-i} \) and all \( s \), \( \Pi_i(a_i = 0, a_{-i}, s) = 0. \)

Having identified the choice specific value functions \( \Pi_i(a, s) \), we next turn to the problem of identification of primitive mean utilities \( \Pi_i(a_i, a_{-i}, s) \). The definition of the choice specific value function implies that these two objects are related by the following equation:

\[
\Pi_i(a_i, s) = \sum_{a_{-i}} \sigma_{-i}(a_{-i}|s) \Pi_i(a_i, a_{-i}, s), \quad \forall i = 1, \ldots, n, a_i = 1, \ldots, K.
\]

(10)

Given \( s \), this is a system of \( n \times K \) equations. To see why, recall that there are \( n \) agents and for each agent, there are \( K + 1 \) choices. However, since choice probabilities must sum to one, we omit the equation for \( k = 0 \) since it is redundant. There are \( n \times K \times (K + 1)^{n-1} \) free parameters \( \Pi_i(a_i, a_{-i}, s) \) in equation (10). To see why, recall that for each agent, we have normalized the utility for the action \( a_i = 0 \) to zero regardless of the actions of the other players. Therefore, we are left with as free parameters \( i \)'s mean utility for \( K \) actions of agent \( i \) and the \( K + 1 \) actions of the other agents.
Clearly, $n \times K \times (K+1)^{n-1} > n \times K$, which implies that the model is underidentified in general. In order to identify the model, we will impose exclusion restrictions on $i$’s payoffs. Partition $s = (s_i, s_{-i})$, and assume that

$$
\Pi_i(a_i, a_{-i}, s) = \Pi_i(a_i, a_{-i}, s_i)
$$

depends only on the subvector $s_i$. In other words, $s_i$ are the variables that only enter the profit function of firm $i$ but no other firms. In our entry game example, $s_i$ corresponds to $DIST_{it}$. That is, we exclude firm $-i$’s distance from $i$’s profits. This seems reasonable since $DIST_{it}$ was included in profits because it is a cost shifter. Our identifying assumption is that competitor’s costs can be excluded from $i$’s profits. This seems reasonable since it is unlikely that the distance from Bentonville Arkansas will lower or raise Target’s costs. One might argue that there could be an indirect effect on Target’s profits because Walmart may be a tougher competitor close to its headquarters. However, note that the specification of profits allows us to control for the strategies of competing firms. Most oligopoly models predict that after we control for $-i$’s strategies, $i$’s profits are not influenced by $-i$’s costs.

If we impose these exclusion restrictions, we can rewrite (10) as

$$
\Pi_i(a_i, s_{-i}, s_i) = \sum_{a_{-i}} \sigma_{-i}(a_{-i}|s_{-i}, s_i) \Pi_i(a_i, a_{-i}, s_i),
$$

Our strategy for identifying the model is similar to the identification of a system of simultaneous equations. Holding $s_i$ fixed, we shift $s_{-i}$. This generates new values of $\sigma_{-i}(a_{-i}|s_{-i}, s_i)$ and $\Pi_i(a_i, a_{-i}, s_i)$ by changing the equilibrium to the model. We assume that the population probabilities $\sigma_{-i}(a_{-i}|s_{-i}, s_i)$ are known to the econometrician so that we can use the Hotz-Miller inversion to learn $\Pi_i(a_i, s_{-i}, s_i)$. However, the unknown term $\Pi_i(a_i, a_{-i}, s_i)$ in equation (12) does not change as we vary $s_{-i}$. If there are $(K + 1)^{n-1}$ points in the support of the conditional distribution of $s_{-i}$ given $s_i$, we will have more equations than unknowns.

**Theorem** Suppose that A1 and A2 hold. Also suppose that for each $s_i$, there exists $(K+1)^{n-1}$ points in the support of the conditional distribution of $s_{-i}$ given $s_i$. Assume that the $(K + 1)^{n-1}$ equations defined by (12) are linearly independent. Then the latent utilities $\Pi_i(a_i, s_{-i}, s_i)$ are identified.
We can alternatively state a rank condition, similar to the linear least squares regression model, that is sufficient for identification. This rank condition requires that given each $s_i$, the second moment matrix of the “regressors” $\sigma_{-i}(a_{-i}|s_{-i}, s_i)$,

$$E\sigma_{-i}(a_{-i}|s_{-i}, s_i)\sigma_{-i}(a_{-i}|s_{-i}, s_i)'$$

is nonsingular. Intuitively, we interpret $\Pi_i(a_i, s_{-i}, s_i)$ as the dependent variable in an ols regression and $\sigma_{-i}(a_{-i}|s_{-i}, s_i)$ as a regressor. Then (13) is the familiar rank condition from the theory of ordinary least squares.

3 Nonparametric identification of dynamic games

3.1 Dynamic game of incomplete information

In this section, we extend our model to allow for non-trivial dynamics. Our model is similar to the framework proposed by Aguirregabiria and Mira (XXX) and Pesendorfer and Schmidt-Dengler. Period returns are defined using a static logit model. However, the current decision and state influence the future evolution of the state variable. As a result, agents must solve a dynamic programming problem. We shall restrict attention to Markov perfect equilibrium as in Fudenberg and Matzkin (XXXX). In principal, the methods that we propose here could be applied to other dynamic games, such as a finite horizon where payoffs and the low of motion are time dependent. However, considering these extentions at the same time would be at the cost of considerable notational complexity.

3.2 The Environment

3.2.1 Payoffs

We begin by defining the agents period utility function. In the model, there are $t = 1, \ldots, \infty$ time periods. At each time period, we let $a_{it} \in \{0, 1, \ldots, K\}$ denote the set of choices for agent $i$. We shall assume that the choice set is identical for all agents and does not depend on the state variable. Both assumptions could easily be dropped at the cost of notational complexity.
Let \( s_{i,t} \in S_i \) denote the state variable for agent \( i \) at time \( t \). As in the previous section, \( S_i \) is a collection of real valued vectors and the state can either be continuous or discrete. In most of the paper, we shall assume that \( S_i \) is a compact subset of finite, real valued vectors.

Analogous to the previous section, we let \( \epsilon_{it} = (\epsilon_{it}(0), \ldots, \epsilon_{it}(K)) \) denote a vector of shocks to agent \( i \)'s payoffs. We shall assume that these shocks are iid across agents, actions and time periods. As in the previous section, we shall assume that the error terms are distributed extreme value. This has the advantage of simplifying the expressions for a number of dynamic programming formulas in the estimator. Of course, as before, we could generalize the model to allow for different distributions, such as normal error terms. However, the extreme value is by far the most common distribution of error terms in applied work and we believe focusing on this case will be of the greatest value for applied researchers.

Player \( i \)'s period utility function is

\[
u_i(a_{it}, a_{-it}, s_t, \epsilon_{it}) = \Pi_i(a_{it}, a_{-it}, s_t) + \epsilon_{it}(a_{it}).
\]

As in the previous section, the period return is a sum of the mean utilities \( \Pi_i(a_{it}, a_{-it}, s_t) \) and the idiosynratic error terms \( \epsilon_{it}(a_{it}) \). As in the previous section, we shall develop the model assuming that \( \Pi_i(a_{it}, a_{-it}, s_t) \) is a general function of the state variables rather than a member of a particular parametric family.

We will let \( \sigma_i(a_i | s) \) denote the probability that \( i \) plays \( s \) given that the state is \( s \). Since we will be stricting attention to Markov perfect equilibrium, we shall drop time subscripts from \( \sigma_i \). We shall establish that equilibrium in fact exists as one of our key theorems.

As in the previous section, we define \( \Pi_i(a_{it}, s) \) as

\[
\Pi_i(a_i, s) = \sum_{a_{-i}} \Pi_i(a_i, a_{-i}, s) \sigma_{-i}(a_{-i} | s).
\]

### 3.3 Value Functions

In the model, the evolution of the state variable depends on the current state and the actions of all players. We assume that the state variable evolves according to a first order Markov
process $\mathbf{g}(s'|s, a_i, a_{-i})$. Note that all of the state variables in our model are assumed to be observed. That is, $s$ is perfectly observed by the agent and the econometrician. We do not allow the unobserved state variable $\epsilon_i$ to influence the evolution of $s$, except indirectly through $a$. This assumption is quite standard in the literature. A more restrictive assumption is that $s$ does not contain a component that is not observed by the econometrician. After our discussion of identification and estimation, we shall discuss strategies that allow for unobserved heterogeneity. We will consider unobserved heterogeneity similar to fixed effect models and a serially correlated, unobserved state variable.

Each player $i$ is assumed to maximized expected discounted utility using a discount factor $\beta$. Let $W_i(s, \epsilon_i; \sigma)$ be player $i$’s value function given that the state is $s$ and the private information is $\epsilon_i$. The value function holds fixed as an argument the strategies of all agents $\sigma$. The value function then satisfies the following recursive relationship:

$$W_i(s, \epsilon_i; \sigma) = \max_{a_i \in A_i} \left\{ \Pi_i(a_i, s) + \epsilon_i(a_i) \right\} + \beta \int \sum_{a_{-i}} W_i(s', \epsilon_i'; \sigma) \mathbf{g}(s'|s, a_i, a_{-i}) \sigma_{-i}(a_{-i}|s) f(\epsilon_i') d\epsilon_i' ds'. \right\}$$

At each point in time, agents choose an action to maximize expected discounted utility $a_i \in A_i$. The term $\Pi_i(a_i, s) + \epsilon_i(a_i)$ is the current period return from choosing $a_i$. Note that $\Pi_i(a_i, s)$ integrates out agent $i$’ uncertainty about the actions of the other $-i$ agents. The second term captures $i$’s utility from future time periods. In our model, agents choose their actions simultaneously. Therefore, agent $i$’s beliefs about the evolution of the state given his current information will be $\sum_{a_{-i}} \mathbf{g}(s'|s, a_i, a_{-i}) \sigma_{-i}(a_{-i}|s)$. This integrates out agent $i$’s uncertain about the actions of $-i$. By integrating against the evolution of the state $s'$ and the error term $\epsilon_i'$, the second term captures $i$’s beliefs about the value function moving one period ahead.

We shall assume that the observed behavior is a Markov perfect equilibrium.

**Definition** A Markov perfect equilibrium is a collection of policy functions, $\delta^*_i(s, \epsilon_i)$ and corresponding conditional choice probabilities $\sigma^*_i(a_i|s)$ such that for all $i$, all $s$ and all $\epsilon_i$, $\delta^*_i(s, \epsilon_i)$ maximizes
In a Markov perfect equilibrium, an agent’s strategy $\delta^*_i(s, \epsilon_i)$ is restricted to be a function of the state $(s, \epsilon_i)$. This solution concept restricts equilibrium behavior by not allowing for time dependent punishment strategies, such as trigger strategies or tit-for-tat which do not depend on payoff relevant state variables. While Markov perfect equilibrium restricts behavior considerably, it has the advantage that equilibrium behavior can be expressed using familiar techniques from dynamic programming.

If the state space of the model is discrete, existence can be established as a consequence of Brouwer’s fixed point theorem (see Pesendorfer and Schmidt-Dengler (XXXX)). We are also concerned about the case in which the state space is continuous. To the best of our knowledge, this result is not available in the existing literature. We therefore prove the existence of equilibrium in the appendix.

4 Nonparametric identification

Next, we turn to the problem of identification of the model. The strategy for identifying the model will be similar to the static model. We begin with some preliminaries by first defining the choice specific value function and noting some key equations that must hold in our dynamic model. The starting point of our analysis is to define the choice specific value function

\[
V_i(a_i, s) = \Pi_i(a_i, s) + \beta \int \sum_{a_{-i}} W_i(s', e'_i; \sigma^*_{-i}) g(s'|s, a_i, a_{-i}) \sigma_{-i}(a_{-i}|s) f(e'_i)de'ds',
\]  

(15)

Similar to (2), the choice specific value function is the expected utility from choosing the action $a_i$, excluding the current period error term $\epsilon_i(a_i)$. As in the static setting, the term $\Pi_i(a_i, s)$ integrates out player $i$’s expectations about the actions of the other players. In a dynamic setting, however, we have to include the utility from future time periods. We do
this by integrating out the value function \( W_i(s', \epsilon'_i; \sigma) \) with respect to next periods private information, \( \epsilon'_i \), and state \( s' \). Note that \( \overline{G}(s'|s, a_i, a_{-i}) \) holds the term \( a_i \) fixed. In words, we can interpret the choice specific value function as the returns, excluding \( \epsilon_i(a_i) \), from choosing \( a_i \) today and then reverting to the solution to the dynamic programming problem (14) in all future time periods. Next, we define the ex ante value function, or social surplus function, as

\[
V_i(s) = \int W_i(s, \epsilon_i; \sigma) f(\epsilon_i) d\epsilon_i
\]  

The ex ante value function is the expected value of \( W_i \) tomorrow given that the state today is \( s \). In order to compute this expectation, we integrate over the distribution of \( s \) and \( \epsilon_i \) given that the current state is \( s \).

Using equations (15) and (16), the ex ante and choice specific value functions are related to each other through the following equation

\[
V_i(a_i, s) = \Pi_i(a_i, s) + \beta E \left[ V_i(s')|s, a_i \right].
\]  

Analogously to (3), in the dynamic model, if the state is equal to \( s \), the ex ante value function is related to the choice specific value function by:

\[
V_i(s) = E_{\epsilon_i} \max_{a_i} [V_i(a_i, s) + \epsilon_i(a_i)]
\]  

That is, the utility maximizing action maximizes the sum of the choice specific value function and the private information \( \epsilon_i(a_i) \). As in our static model, the equilibrium probabilities and the choice specific value functions are relate through the following equation

\[
\sigma_i(a_i|s) = \frac{\exp(V_i(a_i, s))}{\sum_{a'_i} \exp(V_i(a'_i, s))}
\]  

### 4.1 Constructive Proof of Identification

As in the static model, we prove the identification of our model constructively. Our strategy is to assume that the econometrician has knowledge of the population choice probabilities. We then show that it is possible to uniquely recover \( \Pi_i(a_i, a_{-i}, s) \) after making appropriate
normalizations and checking a rank condition. As in the static model, we begin by taking
the log of both sides of (19). Straightforward algebra implies that

\[
\log(\sigma_i(a_i = k|s)) - \log(\sigma_i(a_i = 0|s)) = V_i(a_i = k, s) - V_i(a_i = 0, s) \tag{20}
\]

This equation demonstrates that it is possible to recover the choice specific value functions
up to a first difference, if we know the population choice probabilities.

Next, we note that by (18) and the well known properties of the extreme value distri-
bution, it follows that:

\[
V_i(s) = E_{\epsilon_i} \max_{a_i} V_i(a_i, s) + \epsilon_i(a_i) \tag{21}
\]

\[
= \log \sum_{k=0}^{K} \exp(V_i(k, s)) \tag{22}
\]

\[
= \log \sum_{k=0}^{K} \exp(V_i(k, s) - V_i(0, s)) + V_i(0, s) \tag{23}
\]

We now combine (21)-(23) with equation (17) to yield:

\[
V_i(0, s) = \Pi_i(a_i = 0, s) + \beta E [V_i(s')|s, a_i = 0].
\]

\[
= \Pi_i(a_i = 0, s) + \beta E \left[ \log \left( \sum_{k=0}^{K} \exp(V_i(k, s') - V_i(0, s')) \right) + V_i(0, s)|s, a_i = 0 \right]
\]

\[
= \Pi_i(a_i = 0, s) + \beta E \left[ \log \left( \sum_{k=0}^{K} \exp(V_i(k, s') - V_i(0, s')) \right)|s, a_i = 0 \right] + \beta E [V_i(0, s')|s, a_i = 0]
\]

Next, suppose that we are willing to make the “outside good” assumption as in equation
(9). The dynamic model strictly generalizes our static model. Therefore, we will need
assumptions that are at least as strong to identify this more general framework. Then
equation (20) implies that:

\[
V_i(0, s) = \beta E \left[ \log \left( \sum_{k=0}^{K} \exp(V_i(k, s') - V_i(0, s')) \right) + V_i(0, s)|s, a_i = 0 \right]
\]

\[
= \beta E \left[ \log \left( \sum_{k=0}^{K} \exp(\log(\sigma_i(a_i = k|s')) - \log(\sigma_i(a_i = 0|s'))) \right)|s, a_i = 0 \right] + \beta E [V_i(0, s)|s, a_i = 0]
\]

(24)
Since the population probabilities $\sigma_i(a_i = k|s)$ are assumed to be known for the purposes of our identification argument, the term

$$\beta E \left[ \log \sum_{k=0}^{K} \exp \left( \log(\sigma_i(a_i = k|s')) - \log(\sigma_i(a_i = 0|s')) \right) | s, a_i = 0 \right]$$

can be treated as a known constant. Then, equation (24) is a functional equation involving the unknown function $V_i(0, s)$. Suppose that we are willing to assume that the choice specific value functions are bounded and continuous functions. The existence proof in the appendix gives primitive conditions under which this will be true. Then Blackwell’s sufficient conditions imply that (24) is a contraction mapping and therefore there is a unique solution for $V_i(0, s)$. As a result, we have shown that $V_i(0, s)$ is identified. Moreover, $V_i(k, s)$ is identified for all $k$ by substituting $V_i(0, s)$ into (20). Finally, we note that the ex ante value functions can be identified by (21)-(23) given that we have identified the $V_i(k, s)$.

A3 The ex ante value functions $V_i(k, s)$ are bounded and continuous. Therefore, Blackwell’s sufficient conditions are satisfied so that (24) is a contraction mapping and there is a unique $V_i(0, s)$ satisfying this functional equation.

In our appendix establishing equilibrium existence, we provide primitive sufficient conditions for A3 to hold. Next, note that (17) implies that

$$\Pi_i(a_i = k, s) = V_i(a_i = k, s) - \beta E \left[ V_i(s') | s, a_i = k \right].$$

(25)

Note that our identification arguments imply that both terms on the right hand side of (25) are known. This implies that $\Pi_i(a_i = k, s)$ is identified. The rest of identification proof can then follow exactly as in equations (10)-(12). We simply need to construction the $\Pi_i(a_i, a_{-i}, s)$ from the static choice specific value functions $\Pi_i(a_i, s)$ by imposing exclusion restrictions.

Theorem Suppose that A1-A3 hold. Also suppose that for each $s_i$, there exists $(K+1)^{n-1}$ points in the support of the conditional distribution of $s_{-i}$ given $s_i$. Assume that the $(K + 1)^{n-1}$ equations defined by (12) are linearly independent. Then the latent utilities $\Pi_i(a_i, s_{-i}, s_i)$ are indentified.
5 Nonparametric and semiparametric estimators

In the next section, we describe a set of nonparametric and semiparametric estimators for our dynamic game of incomplete information. Our estimators are constructed by using the empirical analogue of our identification strategy. The translation from identification arguments to the nonparametric estimator essentially only requires replacing the appropriate conditional expectations with analog sample projections. While there are many possible local and global nonparametric smoothing techniques to estimate conditional expectations, for the clarity of presentation we describe the nonparametric procedure using series expansions as in most of the recent literature (e.g. Newey (1994) and Chen, Linton, and Van Keilegom (2003)). We will describe a fully nonparametric estimator of our model as well as a semiparametric estimator.

5.1 Nonparametric estimator

In the rest of paper we will maintain the assumption that one has access to a data set from a collection of independent markets \( m = 1, \ldots, M \) with at least two periods of observations each. Players \( i = 1, \ldots, n \) are observed to play the game in each of these markets. There are \( t = 1, \ldots, T(m) \) plays of the game in market \( m \). We will let \( a_{i,m,t} \) denote agent \( i \)'s actions in market \( m \) at time period \( t \) and \( s_{i,m,t} \) the state variable. The set up of our estimator can be changed to allow for different data structures, such as different players in different markets or varying numbers of players across markets. However, this would come at the cost of greater notational complexity. The nonparametric estimator is implemented by using the following four steps. Our goal will be to estimate \( \Pi_i(a_i, a_{-i}, s) \), the nonparametric mean utility parameters at a single point \( s \). While the nonparametric procedure we propose is extremely flexible, it suffers from the standard problem of nonparametric methods such as the curse of dimensionality. We do not envisage this method to be useful in most applied problems, except those with a very large number of observations and a small state space, for this reason. However, this method is very useful to exposit how estimation can be constructed analogously to our nonparametric estimation. We shall propose a somewhat more practical semiparametric estimator in the next subsection. For expositional reasons, it is useful to exposit the nonparametric case first so that the principals of the semiparametric
estimator will be clearer to the reader.

**Step 1: Estimate \( \hat{V}_i(k, s) - \hat{V}_i(0, s) \) using (20)** Suppose that we “flexibly” construct an estimator \( \hat{\sigma}_i(a_i|s) \) of the equilibrium choice probabilities \( \sigma_i(a_i|s) \). Then equation (20) shows that we can estimate \( V_i(k, s) - V_i(0, s) \) as

\[
\hat{V}_i(k, s) - \hat{V}_i(0, s) = \log(\hat{\sigma}_i(k|s)) - \log(\hat{\sigma}_i(0|s))
\]

One method for estimating the choice probabilities flexibly is by using a “sieve logit”. Let \( q_l(s) \) for \( l = 1, 2, \ldots \) denote a sequence of known basis functions that can approximate any square-measurable function of the state variable \( s \) arbitrarily well. It is well known in series estimation that the number of basis terms must go to infinity at a rate that is appropriately slower than the sample size, otherwise the estimator will be inconsistent. How to choose the number of basis terms depends on the data configuration. If the number of observations \( T(m) \) increases to infinity at least as fast as the number of total markets \( M \), then the number of basis terms can be a function \( k \) of the number of observations \( T(m) \) in market \( m \). In this case the model can be estimated market by market to allow for substantial unobserved heterogeneity across markets. On the other hand, if \( T(m) \) is bounded from above, then the number of basis terms should be a function of the total number of markets \( M \).

To simplify notation in the following we shall focus on the case when \( T(m) \) is finite and denote the number of basis terms as \( k(M) \) and in our asymptotic theory section we shall derive how \( k \) depends on the sample size. We will denote the column vector of \( k(M) \) basis terms by

\[
q^{k(M)}(s) = (q_1(s), \ldots, q_{k(M)}(s))^\prime.
\]

The sieve logit estimator is simply the standard multinomial logit where the independent variables are \( q^{k(M)}(s) \). We will estimate the choice probabilities \( \hat{\sigma}_i(k|s) \) separately for each agent. Obviously, pooling observations across agents is possible if we are willing to assume that agents will play the same strategies if they have the same state variables. We opt for a specification in which strategies vary across agents since this approach is more general.
We will let $\gamma_{i,k}$ denote the parameters for agent $i$ for a particular choice $k = 0, 1, \ldots, K$ and $\gamma_i = (\gamma_{i,0}, \gamma_{i,1}, \ldots, \gamma_{i,K})$ a vector which collects all the $\gamma_{i,k}$s. We let $\gamma^{(M)}_{i,k}$ denote the first $k$ $(M)$ parameters corresponding to the basis vector $q^{(M)}(s)'$. We estimate our model parameters as 

$$\hat{\gamma}_i = \arg\max_{\gamma_i} \sum_{m=1}^{M} \sum_{k=0}^{K} \sum_{t=1}^{T(m)} 1(a_{i,m,t} = k) \log \frac{\exp(q^{(M)}(s_{i,m,t})' \gamma^{(M)}_{i,k})}{\sum_{k'=0}^{K} \exp(q^{(M)}(s_{i,m,t})' \gamma^{(M)}_{i,k}')}}. $$

Our estimate of $\hat{\sigma}_i(k|s)$ and $\hat{\tilde{V}}_i(k, s) - \hat{\tilde{V}}_i(0, s)$ are then 

$$\hat{\sigma}_i(k|s) = \frac{\exp(q^{(M)}(s_{i,m,t})' \gamma^{(M)}_{i,k})}{\sum_{k'=0}^{K} \exp(q^{(M)}(s_{i,m,t})' \gamma^{(M)}_{i,k}')}},$$

$$\hat{\tilde{V}}_i(k, s) - \hat{\tilde{V}}_i(0, s) = \log(\hat{\sigma}_i(k|s)) - \log(\hat{\sigma}_i(0|s)).$$

We also need to construct an estimate of $g(s'|s, a_i, a_{-i})$. The details of estimating $g(s'|s, a_i, a_{-i})$ will vary with the application. In many problems, the law of motion for the state variable is deterministic and therefore does not need to be directly estimated. Another common case is when $g(s'|s, a_i, a_{-i})$ is defined by a density. Let $g(s'|s, a_i, a_{-i}, \alpha)$ be a flexible parametric density with parameter $\alpha$. In this case, one could use maximum likelihood or other appropriate methods to form an estimate $\hat{\alpha}$ of $\alpha$. Making a parametric distributional assumption about $g$ is used here only for expositional convenience. One can also nonparametrically estimate $g$, in which case we only need to make a slight modification to the estimator.

**Step 2: Estimate the choice specific value function for $k=0$, $\hat{\tilde{V}}_i(0, s)$**

Step number 1 only identifies the choice specific value functions up to a first difference. As in our identification arguments, we next construct an estimate of $\hat{\tilde{V}}_i(0, s)$ by iterating on the empirical analogue of equation (24). In order to do this, we first need to construct an estimate the density of next periods state $s'$ given that the current periods state is $s$ and the action chosen by player $i$ is $a_i = 0$. We will denote this density as $\hat{g}(s'|s, a_i = 0)$. Using the results from step 1, we can construct this density as:

$$\hat{g}(s'|s, a_i = 0) = \sum_{a_{-i}} g(s'|s, a_i = 0, a_{-i}, \hat{\alpha})\hat{\sigma}_{-i}(k|s)$$
The empirical analogue of (24) is then

\[ \hat{V}_i(0, s) = \beta \int_{ds'} \log \left( \sum_{k=0}^{K} \exp(\hat{V}_i(k, s') - \hat{V}_i(0, s')) \right) \hat{g}(s'|s, a_i = 0) + \beta \int_{ds'} \hat{V}_i(0, s) \hat{g}(s'|s, a_i = 0) \]

(26)

The term \( \beta \int \log \left( \sum_{k=0}^{K} \exp(\hat{V}_i(k, s') - \hat{V}_i(0, s')) \right) \hat{g}(s'|s, a_i = 0)ds' \) is known from our previous step. In practice, we imagine computing this integral using standard methods for numerical integration. Given that we know this term, the equation (26) can be viewed as a functional equation in \( \hat{V}_i(0, s) \). Define the operation \( T \) by:

\[ T \circ \hat{V}_i(0, s) = \beta \int_{ds'} \log \left( \sum_{k=0}^{K} \exp(\hat{V}_i(k, s') - \hat{V}_i(0, s')) \right) \hat{g}(s'|s, a_i = 0) + \beta \int_{ds'} \hat{V}_i(0, s) \hat{g}(s'|s, a_i = 0) \]

(27)

As in our identification section, it is easy to verify that (27) satisfies Blackwell’s sufficient conditions for a contraction and therefore has a unique fixed point. There is a large literature on solving functional equations defined by contraction mappings and in applied work we imagine using standard numerical methods to solve for \( \hat{V}_i(0, s) \).

An alternative method based on series expansion can also be used to estimate \( V_i(0, s) \) without the need of explicitly estimating \( \hat{g}(s'|s, a_i = 0) \) nonparametrically and calculating the fixed point to the above contraction mapping. Consider a linear series approximation of the value function \( V_i(0, s) \): \( V_i(0, s) = q^{k(M)}(s)' \theta_i^{k(M)} \). Then by premultiplying equation (26) by \( q^{k(M)}(s) \), by the law of iterated expectation we can write

\[ E1(a_i = 0) q^{k(M)}(s) q^{k(M)}(s)' \theta_i^{k(M)} = \beta E1(a_i = 0) q^{k(M)}(s) \log \left( \sum_{k=0}^{K} \exp(\hat{V}_i(k, s') - \hat{V}_i(0, s')) \right) \]

\[ + \beta E1(a_i = 0) q^{k(M)}(s) q^{k(M)}(s)' \theta_i^{k(M)}. \]

Therefore \( \theta_i^{k(M)} \) can be estimated by an empirical analog: \( \hat{\theta}_i^{k(M)} = \left( \hat{X} - \beta \hat{Z} \right)^{-1} \hat{Y} \), where

\[ \hat{X} = \sum_{m=1}^{M} \sum_{t=1}^{T} 1(a_{i,m,t} = 0) q^{k(M)}(s_{m,t}) q^{k(M)}(s_{m,t})', \]

\[ \hat{Z} = \sum_{m=1}^{M} \sum_{t=1}^{T-1} 1(a_{i,m,t} = 0) q^{k(M)}(s_{m,t}) q^{k(M)}(s_{m,t+1})'. \]
and
\[ \hat{Y} = \sum_{m=1}^{M} \sum_{t=1}^{T-1} 1(a_{i,m,t} = 0) q^{k(M)}(s_{m,t}) \log \left( \sum_{k=0}^{K} \exp(\hat{V}_i(k, s_{m,t+1}) - \hat{V}_i(0, s_{m,t+1})) \right). \]

The baseline choice specific value function is then estimated by the “fitted value” from the “linear regression”: \( \hat{V}_i(0, s) = q^{k(M)}(s) \hat{\theta}_i(k) \).

**Step 3: Estimate the static choice specific value function \( \hat{\Pi}_i(k, s) \)** We evaluate the empirical analogue of (25) to estimate the static choice specific value function which we denote as \( \hat{\Pi}_i(k, s) \). From the previous step, we have constructed an estimate of \( \hat{V}_i(0, s) \) and from step 1 we have constructed an estimate of \( \hat{V}_i(k, s) - \hat{V}_i(0, s) \). Putting these two steps together implies that we have an estimate of \( \hat{V}_i(k, s) \) for all \( i, k, s \).

The empirical analogue of equation (25) is then
\[ \hat{\Pi}_i(a_i = k, s) = \hat{V}_i(a_i = k, s) - \beta \int \hat{V}_i(s') \hat{g}(s'|s, a_i = k) ds'. \] (28)

As a practical matter, in order to evaluate the above expression, it is useful to use the empirical analogue fo equation (22), that is,
\[ \hat{V}_i(s) = \log \sum_{k=0}^{K} \exp(\hat{V}_i(k, s))) \] (29)

Substituting (29) into (28) yields:
\[ \hat{\Pi}_i(a_i = k, s) = \hat{V}_i(a_i = k, s) - \beta \int \left( \log \sum_{k=0}^{K} \exp(\hat{V}_i(k, s))) \right) \hat{g}(s'|s, a_i = k) ds'. \] (30)

As in the previous steps, we imagine evaluating \( \hat{\Pi}_i(a_i = k, s) \) using standard methods from numerical integration.

**Step 4: Estimate the nonparametric mean utilities \( \hat{\Pi}_i(a_i, a_{-i}, s_i) \)** The final step of our analysis is to perform the empirical analogue of inverting the linear system (12). Recall that in order to identify the system we needed to make an exclusion restriction. That is,
the state has to be partitioned as \( s = (s_i, s_{-i}) \) and the variables \( s_{-i} \) are assumed not to enter into \( i \)'s mean utilities. This allows us to write \( i \)'s utility as \( \Pi_i(a_i, a_{-i}, s_i) \).

Our approach to inverting this system will be to run a local linear regression (see Fan and Gijbels XXXX). Local linear regression is essentially a weighted least squares regressions where the weights are defined using a kernel distance between the observations. Formally, our estimator for \( \Pi_i(a_i, a_{-i}, s_i) \) is defined as the solution to the following minimization problem:

\[
\hat{\Pi}_i(a_i, a_{-i}, s_i) = \arg \min_{\Pi_i(a_i, a_{-i}, s_i)} \sum_{m=1}^{M} \sum_{t=1}^{T(m)} (\hat{\Pi}_i(a_i, s_{m,t}) - \sum_{a_{-i}} \hat{\sigma}_{-i}(a_{-i}|s_{m,t})\Pi_i(a_i, a_{-i}, s_i))^2 w(m, t),
\]

\[(31)\]

\[
w(m, t) = K\left(\frac{s_{int} - s_i}{h}\right)
\]

\[(32)\]

In the above, \( s_{mt} \) is the state variable in market \( m \) at time \( t \), and \( s_{int} \) is the component of \( s_{mt} \) that enters \( i \)'s mean utilities. The term \( K\left(\frac{s_{int} - s_i}{h}\right) \) is the distance, as measured by the kernel function \( K \), between \( s_{int} \) and \( s_{mt} \). Our weighting scheme overweights observations near \( s_i \) and underweights observations that are farther away. The term \( h \) is the bandwidth. The minimization problem (31) can be interpreted as a regression in which the static choice specific value function, \( \hat{\Pi}_i(a_i, s_{m,t}) \), is the dependent variable and \( \hat{\sigma}_{-i}(a_{-i}|s_{m,t}) \) are the regressors. The regression coefficients are \( \Pi_i(a_i, a_{-i}, s_i) \), our structural mean utility parameters. Our exclusion restrictions guarantee that standard the rank condition from the theory of regression is satisfied.

Choosing the bandwidth involves a variance-bias trade off. A smaller \( h \) reduces the bias by increasing the weight on nearby observations, but increases the variance of our estimator. In practice, choosing the bandwidth would follow standard practice. Cross validation, rules of thumb or simply "eyeballing" the bandwidth are commonly used in applied work. The theory of local linear regression establishes that if we shrink the bandwidth \( h \) at an appropriate rate, we will have a consistent estimate of \( \Pi_i(a_i, a_{-i}, s_i) \).
5.2 Semiparametric payoff function

While the nonparametric estimator in the previous section is very flexible, it is not very practical for most applied problems. With standard sample sizes, nonparametric estimators suffer from a curse of dimensionality and may be poorly estimated with standard sample sizes. Also, the final estimates may be quite sensitive to ad hoc assumptions about the bandwidth or choice of the kernel. Therefore, it is desirable to have a semiparametric approach to the problem. We will specify Π_i(a, s_i, θ) to depend on a finite number of parameters. Parametric specifications are almost universal in the empirical literature. Frequently, applied researchers will assume that utility is linear in the structural parameters, see for example the model in (5). In what follows, we shall assume that the mean utilities take the form

\[ \Pi_i(a, s_i) = \Phi_i(a, s_i)'\theta_{i,a} \]  

(33)

This is slightly more general than assuming that utility is linear, as in (5). Here, \( \Phi_i(a, s_i) \) is a vector valued basis function and \( \theta \) is used to weight the elements of the basis function.

In our semiparametric model, steps 1-3 of the nonparametric section are left unchanged. In step one, we estimate the choice probabilities \( \hat{\sigma}_i(k|s) \) flexibly using a sieve multinomial logit. We then apply the Hotz-Miller inversion to learn \( \hat{V}_i(k, s) - \hat{V}_i(0, s) \). Steps 2 and 3 allow us to estimate \( \hat{\Pi}_i(a_i, s_{mt}) \), the static choice specific value function given that the action is \( a_i \) and the state is \( s_{mt} \). Note that all of these steps are nonparametric and do not impose ad hoc functional form restrictions.

In our semiparametric estimator, we simply modify (31) in step 4 to include the parametric restrictions in (33).

\[
\hat{\theta}_{i,a} = \arg \min_{\Pi_i(a_i, a_{-i}, s_i)} \sum_{m=1}^{M} \sum_{t=1}^{T(m)} (\hat{\Pi}_i(a_i, s_{m,t}) - \sum_{a_{-i}} \hat{\sigma}_{-i}(a_{-i}|s_{m,t})\Phi_i(a, s_i)'\theta_{i,a})^2
\]

An advantage of our semiparametric estimator is that it can be shown that \( \hat{\theta} \) converges to the true parameter value at a rate proportional to the square root of the sample size and has a normal asymptotic distribution. This is a common result in semiparametric estimation. Even though the nonparametric part of our model, \( \hat{\sigma}_{-i}(a_{-i}|s_{m,t}) \) and \( \hat{\Pi}_i(a_i, s_{m,t}) \), the payoff parameters \( \theta \) converge at the standard rate.
5.3 Semiparametric efficient estimation

In the previous section we described the multi-stage semiparametric procedure which allows us to estimate the finite-dimensional parameters of the profit function. This procedure is very intuitive because it follows directly from the identification argument. However, this approach inherits the disadvantages of many multi-stage estimation techniques. First of all, the standard errors are hard to compute and propagation of errors from the previous steps of the procedure will depend on the degree of smoothness of unknown functions of the model. Second, the design and implementation of the efficient estimation procedure will be problematic. In particular it is hard to design a multistage estimation procedure that at a given step will need to compensate the estimation errors that will arise only at subsequent estimation steps. In the procedure that we outlined in the previous sections we effectively do not use the assumption that we observe a stationary controlled Markov process (except in the computation of the conditional choice probabilities). As we will see below, stationarity assumption allows us to avoid any functional iterations during the estimation procedure and, more importantly, we do not have to estimate the transition density of the state variable. This makes the procedure much less fragile with respect to the non-smoothness in the state transition density and, therefore, we can avoid the use of excessively strong regularity conditions.

The model of the game produces a system of moment equations that represent the Bellman equations for individual players. Solving the Bellman equation accompanied by the expression for the action probability in terms of the expected value of the player allows us to estimate the parameter of the player’s payoff. The system of equations of interest can be laid out as

\[ V_i(k, s) = \Pi_i(k, s; \beta) + \beta \int \sum_{a_{-i} \in A_{-i}} \sigma_{-i}(a_{-i}|s) \log \left[ \sum_{l=0}^{K} \exp \left( V_i(l, s') \right) \right] g(s'|s, a_i = k, a_{-i}) \, ds', \]

and

\[ \sigma_i(a_i = k|s) = \frac{\exp (V_i(k, s))}{\sum_{l=0}^{K} \exp (V_i(l, s))}, \]
for $i = 1, \ldots, n$ and $k = 0, \ldots, K$. Denote $d^i_l$ the dummy for choice $l$ by player $i$. We can use the second equation to substitute it into the first one, which leads to $(T - 1) \times m \times n \times K$ conditional moment equations

$$E \left[ V_i (a_i, s_{m,t}) - \beta V_i (a_i, s_{m,t+1}) - \left( 1 - d^i_{m,t} \right) [\Pi_i (a_i, a_{-i}, s_{m,t}; \gamma) - \beta \log \sigma_i (a_i | s_{m,t+1})] \right. \left. + d^i_{m,t} \beta \log \left( 1 - \sum_{j=1}^{K} \sigma_i (j | s_{m,t+1}) \right) \right] = 0. \quad (34)$$

If we are not concerned with the efficient estimation of coefficients $\gamma$ in the payoff function, then the construction of a simple estimation procedure from this equation is straightforward. We can form an instrument $z_{m,t}$ using the state variables $s_{m,t}$, its lags $s_{m,t-\tau}$ and polynomial powers $s_{m,t}$ and its lags. Equation (34) implies the following $n K m \dim (z_{m,t}) \times 1$ moment vector with elements

$$E \left[ d^i_{m,t} z_{m,t} \left( V_i (a_i, s_{m,t}) - \beta V_i (a_i, s_{m,t+1}) - \left( 1 - d^i_{m,t} \right) [\Pi_i (a_i, a_{-i}, s_{m,t}; \gamma) - \beta \log \sigma_i (a_i | s_{m,t+1})] \right. \left. + d^i_{m,t} \beta \log \left( 1 - \sum_{j=1}^{K} \sigma_i (j | s_{m,t+1}) \right) \right] = 0,$$

for $p = 0, \ldots, K$, $i = 1, \ldots, n$, $t = 1, \ldots, T - 1$, and $m = 1, \ldots, M$. To estimate $\gamma$ we can follow two steps.

**Step 1**

Estimate the conditional choice probabilities non-parametrically using the orthogonal series representation:

$$\tilde{\sigma}_i (a_i = p | s) = q^{k(M)} (s) \left( \sum_{m,t} q^{k(M)} (s_{m,t}) q^{k(M)} (s_{m,t}) \right)^{-1} \sum_{m,t} q^{k(M)} (s_{m,t}) d^i_{m,t}.$$

**Step 2**

We consider a series approximation for the value function

$$V_i (a_i = p, s) = q^{k(M)} (s) b^{i,p} + \Delta^{k(M)},$$

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where $\Delta_k(M)$ is a numerical approximation error. Using the Weierstrass’ theorem we can guarantee that this error is infinitesimal as $k(M) \to \infty$ on every bounded subset of the state space $S$. Next we form an instrument $z_{m,t}$ by stacking the state variables $s_{m,t}$ across the markets forming vectors $s_t$, and then choosing the linearly independent subset of vectors from the collection

$$(q_0(s_{m,t-\tau}), \ldots, q_k(M)(s_{m,t-\tau})),$$

for all $0 \leq \tau \leq t - 1$. Additional instruments come from other functions of $a_{m,t}$ and $s_{m,t}$, including those functions that enter the definition of $\Pi_i(a_i, a_{-i}, s_{m,t}; \gamma)$ and the estimated choice probabilities $\hat{\sigma}_i(j|s_{m,t+1})$. This produces an over-identified empirical moment vector with the elements

$$\hat{\phi}_{i,p} (\gamma, b) = \frac{1}{T} \sum_{t=1}^{T-1} a_{t}^{0:p} z_t \left( b^{0:p} \left( q^{k(M)}(s_{m,t}) - \beta q^{k(M)}(s_{m,t+1}) \right) \right. - \left. \left( 1 - d_t^{a_i,0} \right) \left[ \Pi_i(a_i, a_{-i}, s_{m,t}; \gamma) - \beta \log \hat{\sigma}_i(a_i|s_{m,t+1}) \right] + d_t^{a_i,0} \beta \log \left( 1 - \sum_{j=1}^{K} \hat{\sigma}_i(j|s_{m,t+1}) \right) \right).$$

Then we introduce a weighting matrix $W$ with dimensions $n K m \dim(z_t) \times n K m \dim(z_t)$. In the simplest case we can use the identity matrix in lieu of $W$. For this weighting matrix we form a GMM objective and minimize it with respect to parameters of interest $\gamma$ as well as the parameters of the expansion of the value function

$$\min_{\gamma, b} \hat{\varphi} (\gamma, b)' W \hat{\varphi} (\gamma, b).$$

In case where the weighting matrix is equal to the identity matrix and the profit function is linear in parameters:

$$\Pi_i(a_i, a_{-i}; \gamma) = \Phi_i(a_i, a_{-i})' \gamma,$$

the estimation procedure is equivalent to linear two-stage least squares estimation. We can implement it by computing the dependent variable $Y_t$ with elements

$$Y_{i,p,t} = \left( 1 - d_t^{a_i,0} \right) \beta \log \hat{\sigma}_i(a_i|s_{m,t+1}) + d_t^{a_i,0} \beta \log \left( 1 - \sum_{j=1}^{K} \hat{\sigma}_i(j|s_{m,t+1}) \right),$$

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the vector of independent variables $X_t$ with elements

$$X_{i,p,t} = -\left(1 - d_{i,t}^{0,a} \right) \Phi_i \left(a_i, a_{-i} \right), \quad q^{k(M)} \left(s_{m,t} \right) - \beta q^{k(M)} \left(s_{m,t+1} \right),$$

and instrument matrix $Z_t$ with elements

$$Z_{i,p,t} = d_{i,t}^{a_i,p} z_t.$$

Then parameters of interest $\gamma$ and $b$ can be estimated by running an IV regression of $Y_t$ on $X_t$ using $Z_t$ as an instrument. Explicitly we can write that

$$\begin{pmatrix} \gamma \\ b \end{pmatrix} = \left( \sum_{t=1}^{T} X_t' Z_t \left( Z_t Z_t' \right)^{-1} Z_t' X_t \right)^{-1} \sum_{t=1}^{T} X_t' Z_t \left( Z_t Z_t' \right)^{-1} Z_t' Y_t.$$

Even though this procedure produces consistent parameter estimates for $\gamma$ and $\beta$, which follows from standard GMM arguments, we might be interested in obtaining asymptotically efficient estimates.

Conditional moment equation (34) contains an unknown value function which needs to be estimated simultaneously with the parameters of interest $\gamma$. A theory for efficient estimation of parameters in conditional moment models with unknown functions is developed in ?. However, a big difference between our model and the model in ? is that in equation (34) the unknown function $V(\cdot)$ enters non-linearly as a function of the state variable in the period $t$ and in the period $t + 1$. This means that the theory in ? needs to be adapted to our case of a dynamic game.

Note that we can transform this system of conditional moments for the unknown Euclidean parameter $\gamma$ to an unconditional moment. Introduce $\dim (\gamma) \times nK$ matrix $M(s)$ and $nK \times 1$ vector

$$\zeta \left(s, d^a \right) = \left( \frac{d^{a_1,0}}{\sigma_1 \left(0|s \right)}, \ldots, \frac{d^{a_n,K}}{\sigma_n \left(K|s \right)} \right),$$

and then form $A \left(s, d^a \right) = M(s) \zeta \left(s, d^a \right)$. Our inference problem has reduced to estimating parameters $\gamma$ in the conditional moment equation with $nK \times 1$ moment vector with entry $(i, a_i)$

$$\varphi \left(s, s', a; \gamma, V_i, \sigma_i \right) = V_i \left(a_i, s \right) - \beta V_i \left(a_i, s' \right)$$

$$+ \left(1 - d_{i,t}^{0,a} \right) \left[-\Pi_i \left(a_i, a_{-i}, s; \gamma \right) + \beta \log \sigma_i \left(a_i|s' \right)\right] + d_{i,t}^{0,0} \beta \log \left(1 - \sum_{j=1}^{K} \sigma_i \left(j|s' \right)\right).$$
This moment function is similar in its structure to the partially linear regression model. The particular feature of this model is that the unknown function $V_i$ is present in the moment in two places, making the moment non-linear in $V_i$. We transform the original conditional moment equation into an unconditional moment equation

$$E \left[ A(s, d^a) \varphi(s, s', a; \gamma, V_i, \sigma_i) \right] = 0.$$  

We follow the approach in ? to characterize the efficient structure of the instrument matrix $M(s)$ as well as the semiparametric efficiency bound for the model.

We present the results of analysis of the semiparametric efficiency in the following theorem. The procedure for computing the semiparametric efficiency bound of the model, which will allow us to obtain simultaneously the structure of the efficient instrument can be found in the proof of the theorem.

**Theorem 1** Consider the linear expectation operator

$$\mathcal{P}_i \circ f = E \left[ f(s_{m,t+1}) \mid s_{m,t} = s, a_{i} \right],$$

where expectation is defined for the conditional density

$$\sum_{a_{i}} g(s_{m,t+1} \mid s_{m,t} = s, a_{i} = k, a_{-i} = a_{-i} \mid s_{m,t} = s),$$

which has a discrete spectrum with eigenfunctions $\left\{ \Theta_{j}^{i,k}(s) \right\}_{j=0}^{\infty}$ and eigenvalues $\left\{ \lambda_{j}^{i,k} \right\}_{j=0}^{\infty}$ different from zero. Also let

$$\left( \frac{\partial \varphi_i}{\partial V_i} \right)_h = \sum_{j=0}^{\infty} h_j \left( 1 - \beta \lambda_{j}^{i,k} \right) \Theta_{j}^{i,k}(s),$$

for all $h$ with $\sum_{j=0}^{\infty} |h_j| \left\| \Theta_{j}^{i,k}(s) \right\| < \infty$. The semiparametric efficiency bound for estimation of $\gamma$ in equation (34) can be found as

$$V(\gamma) = \left( \left( \pi(s) \left( 1 - d^{a,0} \right) - \left( \frac{\partial \varphi}{\partial V} \right)_h \right) \zeta(d^a, s)' \Omega(s, a)^{-1} \zeta(d^a, s) \left( \pi(s) \left( 1 - d^{a,0} \right) - \left( \frac{\partial \varphi}{\partial V} \right)_h \right) \right)^{-1} \zeta(d^a, s) \left( \pi(s) \left( 1 - d^{a,0} \right) - \left( \frac{\partial \varphi}{\partial V} \right)_h \right),$$

where $h^{*}$ solves

$$\inf_{h} \left( \pi(s) \left( 1 - d^{a,0} \right) - \left( \frac{\partial \varphi}{\partial V} \right)_h \right) \zeta(d^a, s)' \Omega(s, a)^{-1} \zeta(d^a, s) \left( \pi(s) \left( 1 - d^{a,0} \right) - \left( \frac{\partial \varphi}{\partial V} \right)_h \right).$$
In this expression we defined
\[
\Omega(s, a) = \text{Var} \left( \varphi + \frac{\partial a \neq 0}{\sigma (a | s_{m,t+1})} - \frac{\partial a, 0}{\sigma (0 | s_{m,t+1})} \right | s_{m,t} = s, a),
\]
where \( \varphi \) is a stacked vector of moments \( \varphi (s, s', a ; \gamma, V_i, \sigma_i) \).

**Proof:**

First we need to characterize the tangent set of the model. The likelihood of the model will be determined by the choice probabilities and the transition density for the state variable. Given that choices of players are observed by the econometrician, the log-likelihood of the model can be written as
\[
\mathcal{L} (s, s', d) = \sum_{i=1}^{n} \sum_{k=0}^{K} d_{i,k} \log \sigma_i (a_i = k | s) + \sum_{a \in A} d^a \log g (s | s', a) + \log p (s'),
\]
where \( g(\cdot | s', a) \) is the transition density of the state variable, \( d^a \) is the indicator of the action profile \( a \), and \( p(\cdot) \) is the stationary density of the state variable. We choose a particular parametrization path \( \theta \) for the model and compute the score by differentiating the model along the path:
\[
S_\theta (s, s', d) = \sum_{a \in A} d^a s_{1\theta} (s | s', a) + s_{2\theta} (s') + \sum_{i=1}^{n} \sum_{k=0}^{K-1} \left( d_{i,k} \sigma_i (k | s) - \frac{d_{i,K}}{\sigma_i (K | s)} \right) \hat{\sigma}_i (k | s),
\]
where \( E [s_{1\theta} (s | s', a) | s', a] = 0 \), \( E [s_{2\theta} (s')] = 0 \), \( E \left[ |s_{1\theta} (s | s', a)|^2 | s', a \right] < \infty \), \( E |s_{2\theta} (s')|^2 < \infty \), and \( E |\sigma_i (k | s)|^2 < \infty \). Then we characterize the tangent set as
\[
T = \left\{ \sum_{a \in A} d^a \eta_1 (s | s', a) + \eta_2 (s') + \sum_{i=1}^{n} \sum_{k=0}^{K-1} \eta_3 (s) \left( \frac{d_{i,k}}{\sigma_i (k | s)} - \frac{d_{i,K}}{\sigma_i (K | s)} \right) \right\},
\]
with \( E [\eta_1 (s | s', a) | s', a] = 0 \), \( E [\eta_2 (s')] = 0 \), \( E \left[ |\eta_1 (s | s', a)|^2 | s', a \right] < \infty \), \( E |\eta_2 (s')|^2 < \infty \), and \( E |\eta_3 (s)|^2 < \infty \). We will derive the semiparametric efficiency bound for this model under the absence of parametric restrictions on the state transition density and the choice probabilities estimated in the first stage. To derive the bound we find the parametric and the non-parametric parts of the score of the model using a particular parametrization path for the non-parametric component. For the chosen parametric path \( \theta \) we denote
\[
\frac{\partial V_i (k, s)}{\partial \theta} = \zeta_i (k, s) \quad \text{and} \quad \frac{\partial V_i (k, s)}{\partial \gamma'} = \tilde{\zeta}_i (k, s).
\]
Also denote \( \pi_i(k, s) = \frac{\partial \Pi(k, s; \beta)}{\partial \gamma} \). We form vectors \( V^i = (V_i(1, s), \ldots, V_i(K, s))^\top, V = (V^1, \ldots, V^n)^\top, \sigma^i = (\sigma_i(1|s), \ldots, \sigma_i(K|s))^\top \) and \( \zeta = (\zeta_1(1, s), \ldots, \zeta_1(K, s), \ldots, \zeta_n(K, s))^\top \). First of all, we note that we can transform the original moment equation. Consider the operator

\[
P_i \circ f = E \left[ f(s') | s, a_i \right],
\]

where expectation is defined for the conditional density \( \sum_{a_{-i}} g(s'| s, a_i = k; a_{-i}) \sigma_{-i}(a_{-i}|s). \) This operator has a discrete spectrum with eigenfunctions \( \{ \Theta_{i,k}^j(s) \}_{j=0}^\infty \) and eigenvalues \( \{ \lambda_{i,k}^j \}_{j=0}^\infty \) different from zero. This follows directly from the properties of the Hilbert-Schmidt operators which can be found in \( ? \). Then we can represent the value function as

\[
V_i(k, s) = \sum_{j=0}^\infty a_{j,k} \Theta_{i,k}^j(s).
\]

Then we can transform the moment equation to

\[
\tilde{\varphi}(s, s', a; \gamma, V_i, \sigma_i) = \sum_{j=0}^\infty a_{j,k} \left( 1 - \beta \lambda_{i,k}^j \right) \Theta_{i,k}^j(s)
\]

for all \( h \) with \( \sum_{j=0}^\infty |h_j| \left\| \Theta_{i,k}^j(s) \right\| < \infty \). Differentiating the unconditional moment equation with respect to the parametrization path we obtain

\[
E \left[ A(s, d^a) \pi(s) (1 - d^{a,0}) \right] \dot{\gamma} - E \left[ A(s, d^a) \left( \frac{\partial \varphi}{\partial \gamma} \right)_{\gamma} \right] \dot{h}
\]

for all \( h \) with \( \sum_{j=0}^\infty |h_j| \left\| \Theta_{i,k}^j(s) \right\| < \infty \).
We consider the right-hand side and try to find a function $\tilde{\Psi}$ such that the expression on the right-hand side can be represented as $\langle \Psi, S_\theta \rangle$. This function can be obtained as

$$\tilde{\Psi} = A(s, d^a) \left\{ (\varphi - E[\varphi | s, a]) + \frac{d^a \neq 0 - \sigma(a|s)}{\sigma(a|s')} - \frac{d^a.0 - \sigma(0|s)}{\sigma(0|s')} \right\}.$$ 

We note that conditional moment equation (34) holds and we can differentiate it with respect to the parametrization path. Then we can substitute the expression for the derivative into the expression for the unconditional moment. This allows us to express the directional derivative of $\gamma$ and, consequently, the efficient influence function for a fixed instrument matrix:

$$\Psi = E \left[ A(s, d^a) \left( \pi(s) (1 - d^a.0) - \left( \frac{\partial \varphi}{\partial V} \right)_h \right) \right]^{-1} \tilde{\Psi}.$$ 

The semiparametric efficiency bound as a minimum variance of the influence function. Denoting

$$\Omega(s, a) = \text{Var} \left( \varphi + \frac{d^a \neq 0}{\sigma(a|s')} - \frac{d^a.0}{\sigma(0|s')} \bigg| s, a \right).$$

Using standard GMM arguments, we can express the efficiency bound for fixed instrument as

$$V_h(\beta) = \left( \left( \pi(s) (1 - d^a.0) - \left( \frac{\partial \varphi}{\partial V} \right)_h \right)^{\prime} \Omega(s, a)^{-1} \left( \pi(s) (1 - d^a.0) - \left( \frac{\partial \varphi}{\partial V} \right)_h \right) \right)^{-1}. $$

The efficiency bound overall can be found as $V_{h^*}(\beta)$ for $h^*$ solving

$$\inf_h \left( \pi(s) (1 - d^a.0) - \left( \frac{\partial \varphi}{\partial V} \right)_h \right)^{\prime} \Omega(s, a)^{-1} \left( \pi(s) (1 - d^a.0) - \left( \frac{\partial \varphi}{\partial V} \right)_h \right).$$

The optimal instrument matrix can be explicitly written as

$$M(s) = E \left[ \left( \pi(s) (1 - d^a.0) - \sum_{j=0}^\infty h_j^* (1 - \beta \lambda_j) \Theta_j (s) \right)^{\prime} \Omega(s, a)^{-1} \bigg| s \right].$$

Q.E.D.
6 Implimentation Issues

In this section, we discuss several practical implimentation issues. In the previous section we present two alternative estimation procedures. The first one follows the identification strategy closely while the second method is formulated in a framework of conditional moment models that makes it transparent to obtain an efficient estimator. In this section we will focus first on the first method because it follows straightforwardly from the identification arguments, and does not require the empirical researchers to specify the instruments and the weighting matrices that are used in forming the efficient estimator. The asymptotic distribution has an explicit analytical structure. On the other hand, the second approach has the advantage that given the choice of instrument functions and the weighting matrix, practical inference can be performed using standard parametric methods as if a finite dimensional linear parametric model of \( V_i(k, s) \) and \( \Pi(a, s) \) is estimated, as long as the estimation noise in the estimation of \( \sigma_i(k|s) \) is appropriately accounted for.

Given the ease of the estimation procedure discussed in the previous sections, we suggest that the most practical method of inference is bootstrapping. However, understanding the derivation of the asymptotic distribution is still important for ensuring the theoretical validity of resampling method such as bootstrap or subsampling. Section 6.1 gives high level conditions and describes the form of the asymptotic variance using a powerful set of results developed in Newey (1994). The next two subsections develop the asymptotic distribution under suitable regularity conditions, for both the nonparametric estimator and the semiparametric estimator. The results for the nonparametric estimator are a necessary first step for the semiparametric estimator. Section 6.4 discusses how to incorporate models of unobserved heterogeneity. On the other hand, given the practical advantage of the efficient estimation method described in section

6.1 Semiparametric limit variance

We will not study functional dependence among different \( \theta_{i,a} \) components of the parameter vector \( \theta \). To simplify notation in the following we will use \( \theta \) to denote a particular component \( \theta_{i,a} \) of equation (33) in section 5.2, where \( a_i = k \). In particular, \( \theta \) (short-handed for \( \theta_{i,a} \)
where \( a_i = k \) is identified by the relation 
\[
\sigma_i (k | s) \Pi_i (k, s) = E \left[ d^k_i \Phi (a_i, s_i) | s \right]' \theta, \]
where
\[
\Pi_i (k, s) = V_i (k, s) + \beta E \left[ \log \sigma_i (0 | s') | s, a_i = k \right] - \beta E \left[ V_i (0, s') | s, a_i = k \right].
\]
This suggests that a class of instrument variable estimators for \( \theta \), which includes the least square estimator in section 5.2 as a special case, takes the form of an empirical analog of
\[
Ed^k_i A (s) \Phi (a, s_i)' \theta = Ed^k_i A (s) V_i (k, s) + \beta Ed^k_i A (s) \log \sigma_i (0 | s') - \beta Ed^k_i A (s) V_i (0, s').
\]
Then we can write \( \hat{\theta} = \hat{X}^{-1} \hat{Y} \), where (\( \sum \tau \) is used to denote \( \sum_{t=1}^{T-1} \sum_{m=1}^{M} \)),
\[
\hat{X} = \sum_{\tau} d^k_i \Phi (a_{\tau}, s_{i\tau}) / M (T - 1),
\]
and
\[
\hat{Y} = \frac{1}{M (T - 1)} \sum_{\tau} d^k_i A (s_{\tau}) \left[ \log \sigma_i (k | s_{\tau}) / \sigma_i (0 | s_{\tau}) + \beta \log \sigma_i (0 | s_{m,t+1}) + \hat{V}_i (0, s_{\tau}) - \beta \hat{V}_i (0, s_{m,t+1}) \right].
\]
The instrument matrix itself can be estimated nonparametrically as \( \hat{A} (s) \). The least square estimator in section 5.2 effectively uses \( \tilde{E} \left[ \phi (a, s_i) | s, a_i = k \right] \) as the instrument matrix \( \hat{A} (s) \).
It is well known, however, that estimation of the instruments has no effect on the asymptotic distribution under suitable regularity conditions. Therefore with no loss of generality we treat the instruments \( A (s) \) as known in deriving the form of the asymptotic distribution. Furthermore, by a standard law of large number \( \hat{X} \) can be replaced by its population limit
\[
X = Ed^k_i A (s) \Phi (a, s_i).
\]
Therefore the asymptotic distribution of \( \hat{\theta} \) will solely be determined by the convergence of
\[
\sqrt{M (T - 1)} (\hat{Y} - Y),
\]
where
\[
Y = \frac{1}{M (T - 1)} \sum_{\tau} d^k_i A (s_{\tau}) \left[ \log \sigma_i (k | s_{\tau}) / \sigma_i (0 | s_{\tau}) + \beta \log \sigma_i (0 | s_{m,t+1}) + V_i (0, s_{\tau}) - \beta V_i (0, s_{m,t+1}) \right].
\]
This asymptotic distribution is given in the following theorem.

**Theorem 2** Under suitable regularity conditions,
\[
\sqrt{M (T - 1)} (\hat{Y} - Y) = \frac{1}{\sqrt{M (T - 1)}} \sum_{\tau} \sum_{j=1}^{4} \phi_j (a_{i\tau}, s_{\tau}) + o_p (1).
\]

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where

\[
\begin{align*}
\phi_1(a_{i\tau}, s_\tau) &= A(s_\tau) \left( d_{i\tau}^k - \sigma_1(k|s_\tau) \right) - \frac{\sigma_1(k|s_\tau)}{\sigma_1(0|s_\tau)} A(s_\tau) \left( d_{i\tau}^0 - \sigma_1(0|s_\tau) \right), \\
\phi_2(a_{i\tau}, s_\tau) &= E \left[ d_{i(m,t-1)}^k A(s_{m,t-1}) | s_\tau \right] \frac{1}{\sigma_1(0|s_\tau)} \left( d_{i\tau}^0 - \sigma_1(0|s_\tau) \right), \\
\phi_3(a_{i\tau}, s_\tau) &= \beta \delta(s_\tau) \left[ - \left( d_{i\tau}^0 \log \sigma_1(0|s_{m,t+1}) - E \left( d_{i\tau}^0 \log \sigma_1(0|s_{m,t+1}) | s_{m,t} \right) \right) \\
& \quad + \left( d_{i\tau}^0 V_1(0, s_{m,t+1}) - E \left( d_{i\tau}^0 V_1(0, s_{m,t+1}) | s_{m,t} \right) \right) \\
& \quad + \frac{d_{i\tau}^0 d_{i,m,t+1}^k}{\sigma_1(0|s_{m,t+1})} - E \left( \frac{d_{i\tau}^0 d_{i,m,t+1}^k}{\sigma_1(0|s_{m,t+1})} | s_{m,t} \right) + d_{i\tau}^0 - \sigma_1(0|s_\tau) \right].
\end{align*}
\]

In the above, \( \delta(s_\tau) \) is defined as the unique solution to the following functional relation which is a contraction mapping:

\[
\delta(s_t) \sigma_i(0|s_t) - \beta E \left( \delta(s_{t-1}) \sigma_i(0|s_{t-1}) | s_t \right) = \sigma_i(k|s_t) A(s_t).
\]

\( \phi_4(a_{i\tau}, s_\tau) \) is defined similarly to \( \phi_3(a_{i\tau}, s_\tau) \), with \( \beta \delta(s_\tau) \) replaced by \( \beta^2 \delta(s_\tau) \), where \( \delta(s_\tau) \) is now defined as the unique solution to the functional relation of

\[
\delta(s_t) \sigma_i(0|s_t) - \beta E \left( \delta(s_{t-1}) \sigma_i(0|s_{t-1}) | s_t \right) = E \left( \frac{d_{i\tau}^k A(s_{t-1}) | s_t} \right).
\]

A few remarks are in order. First, because the second stage moment equations are exact identities when evaluated at the true parameter values, if the first stage nonparametric functions are exactly known and do not need to be estimated, the second stage parameters will have zero asymptotic variance. In other words, all the variations in the asymptotic variance are generated from the first stage nonparametric estimation of the conditional choice probabilities and the transition probabilities. Secondly, as shown in Newey (1994), the form of the asymptotic variance given in theorem 2 is obtained from a pathwise derivative calculation and does not depend on the exact form of the nonparametric methods that are being used to estimated, as long as suitable regularity conditions are met. Therefore, Theorem 2 is stated without regard to the particular nonparametric method and its regularity conditions used in the first stage analysis. Both of these will be discussed in the following two sections. While resampling method is a clear preferrable method of inference, it is in principle possible to estimate nonparametrically each component of the asymptotic variance in theorem 2. On the other hand, given the practical advantage of the effective estimation
method described in 5.3, the explicit expression of the asymptotic influence functions given in theorem 2 is mostly of theoretical value.

6.2 Asymptotic distribution for nonparametric estimator

6.3 Asymptotic distribution for semiparametric estimator

6.4 Unobserved heterogeneity

Unobserved heterogeneity is an important concern for dynamic discrete choice models. (Need to give a long list of citations). A recent insight from this literature is that it is sufficient to estimate a reduced form model of conditional choice probabilities and transition probabilities that account for the presence of the unobserved heterogeneity. A variety of such methods are available in the recent literature, some allowing for a fixed number of support points in the distribution of unobserved state variables while others allowing for a continuous of unobserved state variables. For each of the discrete and continuous support cases of the unobserved state variables, some methods limited to only non time varying unobserved state variables while other methods might allow for serially correlated unobserved state variables.

In the following, we will take as given the ability of estimate a first stage model of conditional choice probabilities and conditional transition probabilities that incorporate the presence of general (discrete and continuous, time invariant and serially correlated) state variables. Therefore, we will that it is possible to use one of the methods available in the existing literature to estimate a reduced form model of \( \hat{\sigma}_i(k|s), \forall i, k \) and \( \hat{g}(s'|s, a) \), where now \( s' \) and \( s \) include both observed and unobserved state variables that can be either discrete or continuous, either time-invariant or serially correlated. In fact, for the purpose of identifying and estimating the structure mean utility functions, it is sufficient to estimate a model with unobserved heterogeneity the single agent dynamic state transition process \( \hat{g}(s'|s, a_i) \) where the conditioning event is \( a_i \) instead of the entire equilibrium action profile \( a \). The transition process \( \hat{g}(s'|s, a) \) is still important for simulation of the dynamic equilibrium outcome process.

We now note that the entire nonparametric identification process in section 4 and the
entire estimation procedure, both nonparametric and semiparametric, described in section 5, depend only on the first stage $\hat{\sigma}_i (k|s), \forall i, k$ and $\hat{g} (s'|s, a_i)$. Therefore, we can follow exactly the same procedures outlined in sections 4 and 5 to estimate the primitive mean utility functions $\Pi_i (a, s_i)$ and $\Phi_i (a, s_i)' \theta$. Perhaps the best way to understand this argument is through simulations. Given knowledge of $\hat{\sigma}_i (k|s), \forall i, k$ and $\hat{g} (s'|s, a_i)$, a research can generate a data set with as many markets and as many time periods as desired, and apply the estimation procedures described in 5 to the simulated data set.

References


