Linear Differential Equations Models of Short and Long Term Dynamics Applied to Longitudinal Aging Data

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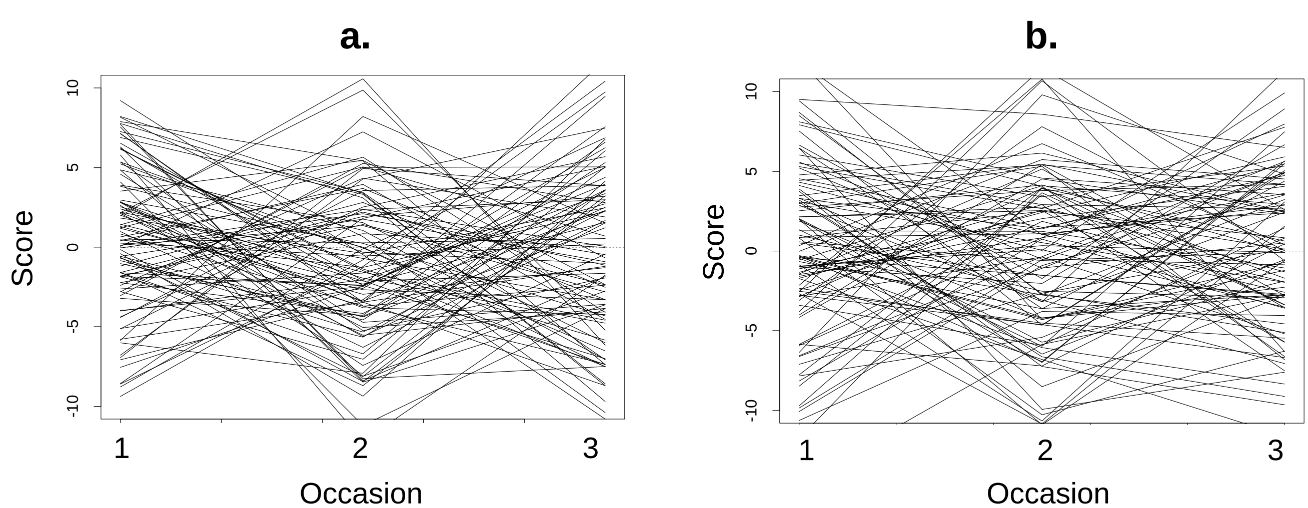
Self-Regulation in Psychological Systems

- Repeated measurements of psychological variables can show patterned variability.
- Consider a psychological process that could influence its own level over time and had some preferred comfort zone in level.
- How would the process adapt its level to remain near its preferred equilibrium?
- How could we measure and model this?

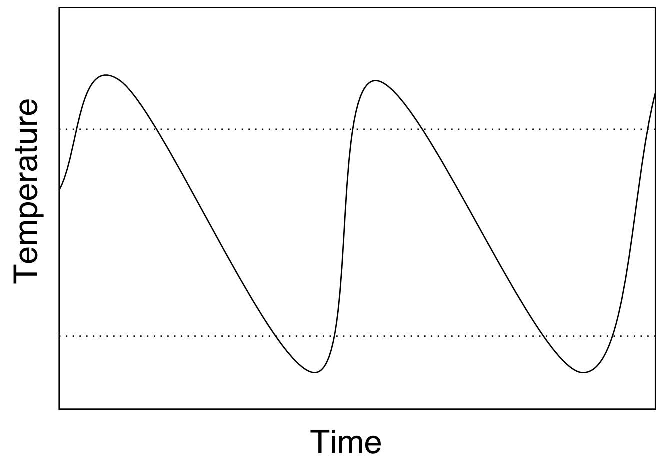
Intraindividual Variability

- Are observed short term fluctuations patterned variability or are they noise?



A Simple Thermostat

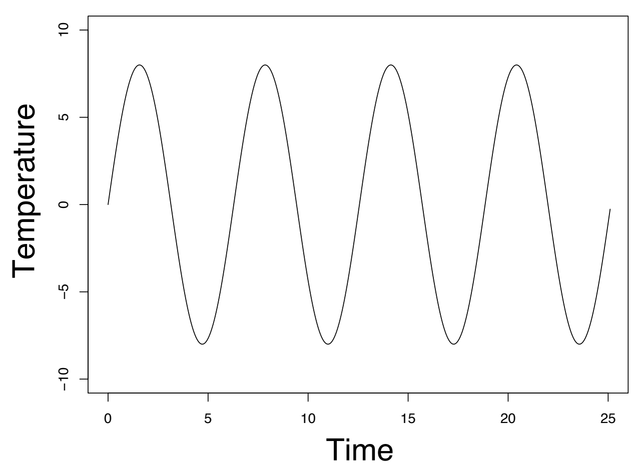
- Consider a thermostat that acts to adapt the temperature in a room to be near some equilibrium value.
- If it is colder outside the room than inside, the room will have a negative rate of change in temperature.
- Suppose the thermostat activates a furnace when the room is colder than some lower threshold and turns off when above some upper threshold.



A Simple Thermostat

A Better Thermostat

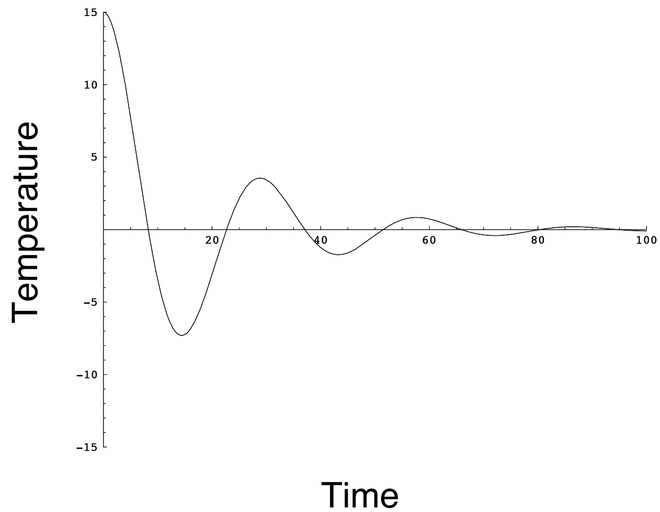
- Suppose the thermostat can control both heating and airconditioning.
- Suppose the amount of heating or air conditioning is proportional to the difference between the current temperature and the desired equilibrium.



A Better Thermostat

An Even Better Thermostat

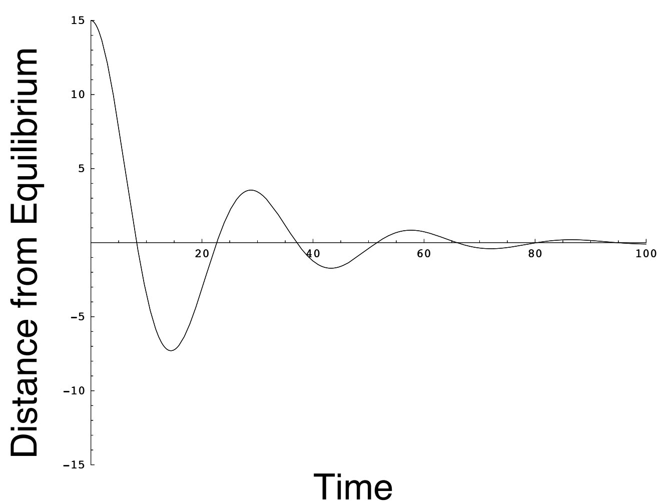
- Suppose that the thermostat can also sense the rate of change in the temperature.
- Suppose that the thermostat can control the furnace and airconditioning to be proportional to the rate of change in the temperature.



An Even Better Thermostat

Active or Passive Regulation?

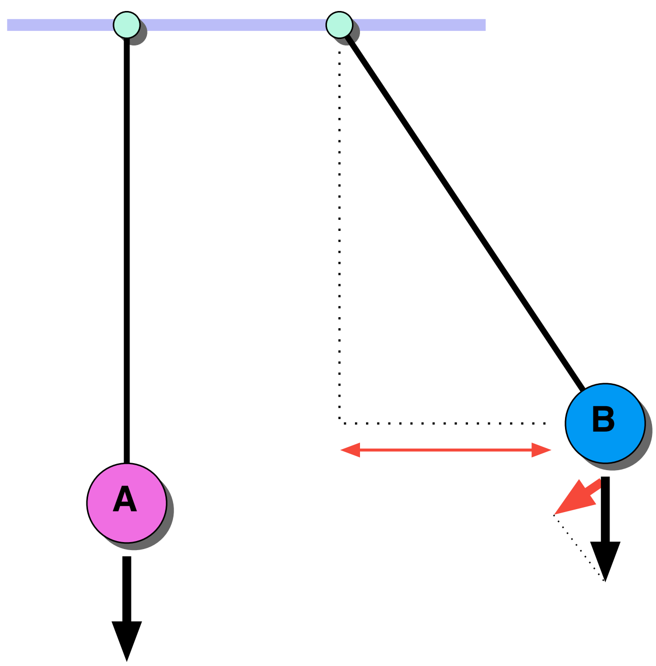
- Must an *active* mechanism exist if we are to observe behavior appearing to be self–regulatory?
- Consider the motion of a damped pendulum in a gravitational field.
- The trajectory of a pendulum with friction looks like the third type of thermostat.



A Pendulum with Friction in a Gravitional Field

Passive Self–Regulation

- There is no "thermostat" mechanism controlling the motion of the pendulum.
- The relationship between the displacement, first and second derivatives of the pendulum allows us to infer a gravitational field.
- The force acting on the pendulum is constant, but the direction of motion is more aligned with the direction of force the more the pendulum is displaced from equilibrium.



Acceleration due to gravitation is proportional to displacement

Differential Equations

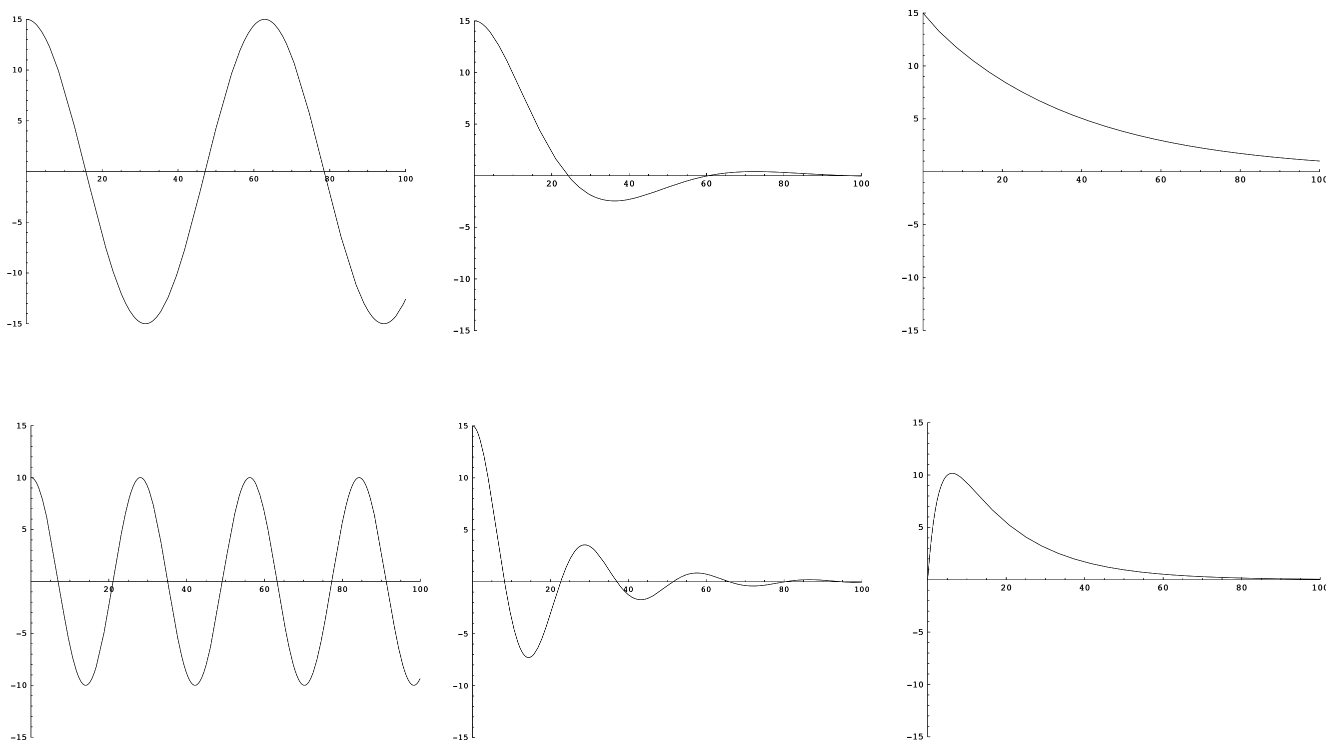
- The pendulum and thermostat are similar in that there is a linear relationship between their derivatives.
- This can be expressed as a differential equation.

The Damped Linear Oscillator

- The differential equation can be expressed as

$$\ddot{x} = \eta x + \zeta \dot{x}$$

- The two parameters have meaning.
  1. η is the square of the frequency.
  2. ζ is proportional to the "friction" of the pendulum.



Example linear oscillator trajectories

Steps in a Dynamical Analysis

1. In general we want to remove the overall mean and linear trend for each individual.
2. The residuals are input to the dynamical analysis.
3. Derivatives are estimated.
4. Relationships between derivatives are estimated.
5. Second level relationships are estimated.

Fitting a Linear Oscillator Model

- Can be expressed as a manifest variable model.
- Derivatives estimated by linear approximation.
- Then the univariate linear oscillator model becomes a simple multiple regression.

$$\ddot{x}_i = b_1 x_i + b_2 \dot{x}_i + e_i$$

Two Methods

1. Local Linear Approximation (LLA)
   - Best at intervals $\approx 1/4$ period.
   - Has problem with measurement error and short measurement intervals.
2. Latent Differential Equations (LDE)
   - Can use multivariate factor data.
   - Breaks down if the measurement interval $> 1/8$ period

Local Linear Approximation

1. Remove the linear trend or growth curve and use the residuals.
2. Estimate the derivatives. For 3 sequential observations (x_1, x_2, x_3) separated by an interval τ, the derivatives at x_2 are

$$\begin{aligned} \dot{x_2} &= (x_3 - x_1)/2\tau \\ \ddot{x_2} &= (x_3 + x_1 - 2x_2)/\tau^2 \end{aligned}$$

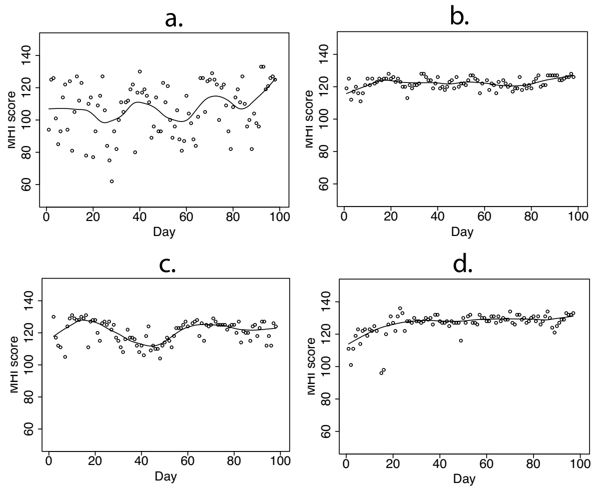
3. Estimate linear relations between derivatives using GLM, SEM or Mixed Models.

Self–Reported Mental Health in Recent Widows

- Recent widows reported a Mental Health Inventory (**MHI**) daily for 90 days.
- Other variables were collected in pre- and post-interviews, including two social support variables: Emotion Focused Coping and Problem Focused Coping.

Mental Health Inventory

- Mental Health Inventory (MHI) was selected in part due to its relatively high internal consistency (in the range .83 to .91) but relatively low test-retest correlation (in the range (.56 to .64).
- Considerable intraindividual variability was observed in self reported MHI scores over the course of 90 days.



Example MHI scores with Loess smooth

Two–Step Estimation of Dynamics

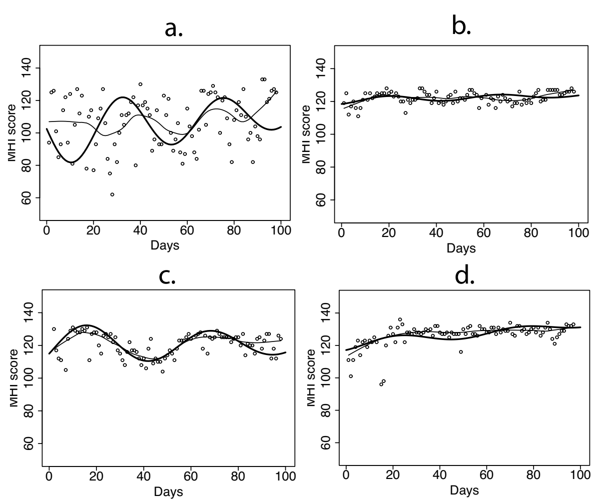
1. The linear trend for each widow was estimated and the residuals from that trend were taken.
2. Derivatives were estimated using local linear approximation.
3. The coefficients were estimated using a random coefficients model.

Interindividual Differences in Intraindividual Dynamics

- The random coefficients model.

$$\begin{aligned} \ddot{x}_{ij} &= b_{1i}x_{ij} + b_{2i}\dot{x}_{ij} + e_{ij} \\ b_{1i} &= c_{00} + c_{01}y_i + c_{02}z_i + u_{0i} \\ b_{2i} &= c_{10} + c_{11}y_i + c_{12}z_i + u_{1i} \end{aligned}$$

where x_{ij} is the ith persons MHI score at the jth occasion, y_i is Problem Focused Coping and z_i is Emotion Focused Coping for person i.



Example MHI scores with model trajectories

Fixed Effects of Predicting Second Derivative of MHI

	Value	SE	t	p
mhi	-0.0152	0.0005	-28.435	$< .0001$
PRmhi	0.0001	0.0003	0.286	0.7751
EMmhi	-0.0001	0.0002	-0.581	0.5613
dmhi	-0.0189	0.0056	-3.386	0.0007
PRdmhi	0.0062	0.0029	2.128	0.0337
EMdmhi	-0.0035	0.0019	-1.827	0.0680

Overall mean $R^2 = .766$

Conclusions from Widows' Data

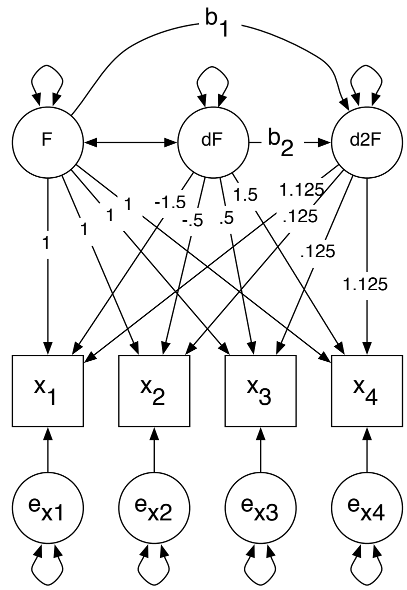
- The residual variability in the MHI was unlikely to be only uncorrelated measurement error.
- Problem focused coping predicted slower damping of oscillations in MHI.
- Emotion focused coping was near threshold for predicting faster damping of oscillations in MHI.

Problems with Linear Approximation Method

- Parameter values may be biased when sampling interval isn't optimal.
- Noise can masquerade as fast frequency signal.
- No measurement model.
- Not multivariate.

Recent Developments

- A confirmatory factor model can be constrained in such a way to estimate the parameters of a univariate linear oscillator.
- There is a straightforward multivariate extension.
- Simulations have been very promising.



Univariate Latent Differential Equation Structural Model

Model Matrices

- Factor Loadings.

$$\mathbf{L} = \begin{bmatrix} 1 & -1.5t & (-1.5t)^2/2 \\ 1 & -0.5t & (-0.5t)^2/2 \\ 1 & 0.5t & (0.5t)^2/2 \\ 1 & 1.5t & (1.5t)^2/2 \end{bmatrix}$$

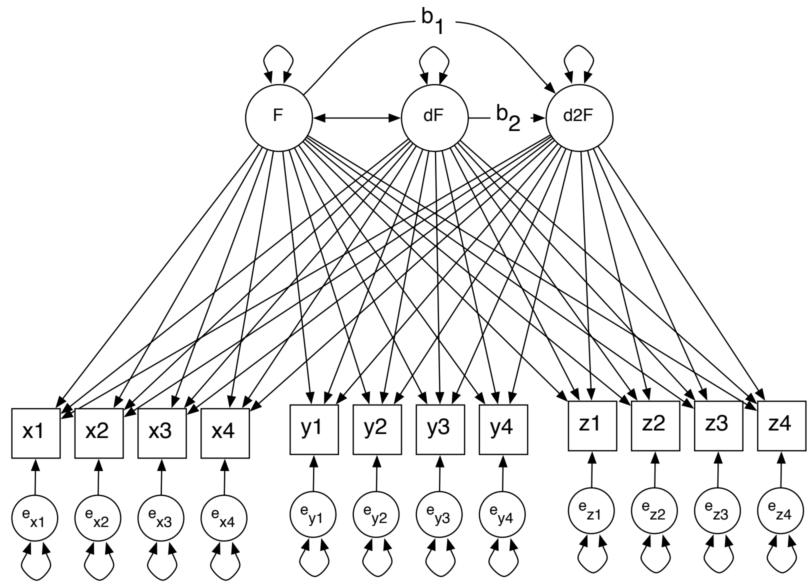
Model Matrices

- Latent Structure.

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ b_1 & b_2 & 0 \end{bmatrix}$$

$$\mathbf{S} = \begin{bmatrix} V_F & C_{FdF} & 0 \\ C_{FdF} & V_{dF} & 0 \\ 0 & 0 & V_{d2F} \end{bmatrix}$$

$$\mathbf{R} = \mathbf{L}(\mathbf{I} - \mathbf{A})^{-1}\mathbf{S}(\mathbf{I} - \mathbf{A})^{-1'}\mathbf{L}' + \mathbf{U}$$



Multivariate Latent Differential Equation Structural Model

$$\mathbf{L} = \begin{bmatrix} 1 & -1.5t & (-1.5t)^2/2 \\ 1 & -0.5t & (-0.5t)^2/2 \\ 1 & 0.5t & (0.5t)^2/2 \\ 1 & 1.5t & (1.5t)^2/2 \\ a & -1.5at & (-1.5at)^2/2 \\ a & -0.5at & (-0.5at)^2/2 \\ a & 0.5at & (0.5at)^2/2 \\ a & 1.5at & (1.5at)^2/2 \\ b & -1.5bt & (-1.5bt)^2/2 \\ b & -0.5bt & (-0.5bt)^2/2 \\ b & 0.5bt & (0.5bt)^2/2 \\ b & 1.5bt & (1.5bt)^2/2 \end{bmatrix}$$

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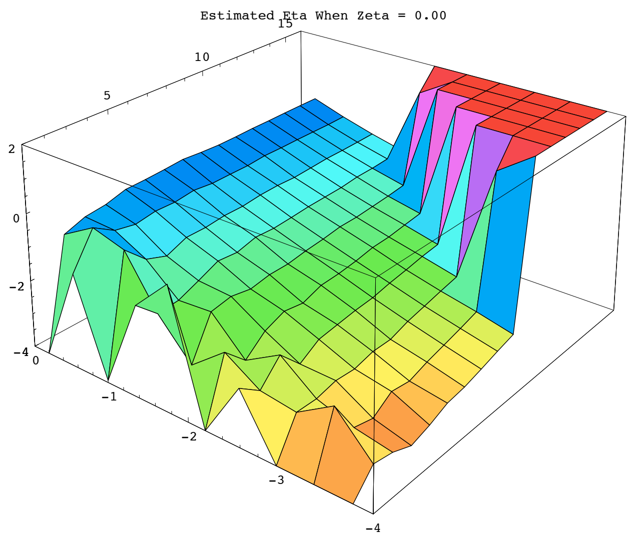
$$\mathbf{R} = \mathbf{L}(\mathbf{I} - \mathbf{A})^{-1}\mathbf{S}(\mathbf{I} - \mathbf{A})^{-1'}\mathbf{L}' + \mathbf{U}$$

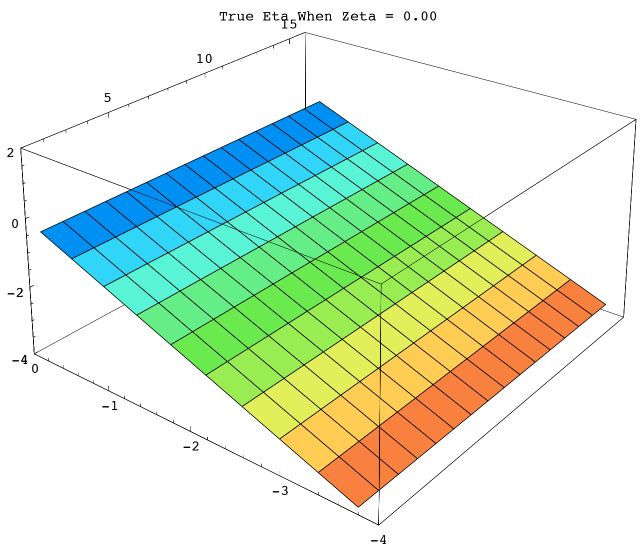
Simulation of Linear Oscillators

- Linear oscillator trajectories were simulated in Mathematica using Runge–Kuttah numerical integration.
- Frequency varied over 10 conditions.
- Damping varied over 9 conditions.
- Each trajectory included 10,000 occasions.
- Four indicators simulated for each true score (communalities in the .5 range).

Simulation Results

- Estimated η and true η for a range of ηs and measurement intervals when $\zeta = 0$.





LDE Conclusions

- Provides low–bias estimates of relationships between derivatives.
- Requires short measurement intervals in comparison to dynamics.
- Can use multiple indicators and factors.
- Can be extended to the multilevel and multigroup cases.
- Can be used for behavioral genetics data.

Designs for Dynamic Data

- Measure as often as you can afford.
- Try to have at least 10 occasions of measurement for a full cycle, leaving the door open to several forms of analysis.
- Try to measure at least 1/4 of a full cycle.
- Use indicators that are internally consistent but with intraindividual variability.

Conclusions

- Self–regulating processes can be modeled with differential equations.
- These models can be fit using standard statistical techniques.
- It is likely that the use of these methods will assist in understanding psychological processes that have heretofore remained unexamined.