A Note on the Characterization of Nash Networks

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Introduction

One-way flow model of Bala and Goyal (2000)

Bala and Boyal (BG) assume:

- An agent can create a link with another agent, without the consent of the latter, by incurring the cost of this link.
- An agent gets resources from another agent if there is a directed link or a directed path between the former and the latter.
- More resources an agent gets, the greater the payoffs.
Introduction

This paper generalizes BG’s model by relaxing the assumption concerning the relationship between obtained resources and agents’ payoffs.

- There is no priori relationship between the number of resources obtained by an agent and her payoffs.

Keep the following appealing assumption in BG:

- For a given number of resources, the more numerous an agent’s links are, the lower her payoffs.
Introduction

The authors use two notions of network theory, *line bases* and *minimum line bases*, to establish their results:

- Nash networks are line bases.
- Strict Nash networks are minimum line bases.
- Efficient networks are minimum line bases.
Basic Model

Definitions:

- Let \( N = \{1, \ldots, n\} \) be a finite set of agents and let \( i \) and \( j \) be typical members of this set.

- Directed networks, \( \Gamma(N, E) \), where \( N \) is the set of vertices (or agents) and \( E \subset N \times N \) is the set of directed links.

- Let \( i_1, i_2, \ldots, i_m \) be a sequence of vertices. Then, the graph \( P(N_P, E_P) \) with \( N_P = \{i_1, i_2, \ldots, i_m\} \), \( E_P = \{i_1i_2, i_2i_3, \ldots, i_{m-1}i_m\} \) is called a directed path from \( i_1 \) to \( i_m \). If \( i_1 = i_m \), then \( P \) is a circle.
Basic Model

• Denote a cycle by $C = (N_C, E_C)$.

• A transitive closure of network $\Gamma$ is the minimal transitive network containing $\Gamma$.

• A link $i_x i_{x+1}$, in $\Gamma$ is said to be critical if $i_x i_{x+1}$ is the only directed path from $i_x$ to $i_{x+1}$ in $\Gamma$. 
Basic Model

- A line basis of a network $\Gamma$, is a minimal collection of links which preserves the reachability of $\Gamma$.

A network $\Gamma'(N,E')$ is a line basis of a network $\Gamma$ if:

1. for all $i \in N$ and $j \in N$, if there is a directed path from $i$ to $j$ in $\Gamma'$;
2. there is no network $\Gamma'' = (N,E'')$ such that $E'' \subset E'$ and which preserves the reachability properties of $\Gamma$ (or $\Gamma'$).
Basic Model

- A minimum line basis is a line basis with the smallest possible number of links.

More precisely, a network $\Gamma = (N, E)$ is a minimum line basis if there is no network $\Gamma' = (N, E')$ such that $\Gamma'$ has the same reachability properties as $\Gamma$ and $|E'| < |E|$.
Basic Model

Fig. 1. Line bases and minimum line bases.
Basic Model

Frame work:

• A strategy of agent $i \in N$ is a row vector $s_i = (s_{i1}, ..., s_{ii-1}, s_{ii+1}, ..., s_{ij}, ..., s_{in})$, where $s_{ij} \in \{0,1\}$, for each $j \in N / \{i\}$.

• Each strategy profile $s = (s_1, ..., s_i, ..., s_n)$ correspond s to a network $\Gamma = (N, E)$. We denote this network $\Gamma_s$.

• $s_{-i} = (s_1, ..., s_{i-1}, s_{i+1}, ..., s_n)$ be the strategy profile played by all agents except $i$. 
Basic Model

We define agent $i$'s payoff function as follows:

$$u_i(s) = \phi_i(x, y)$$  \hspace{1cm} (1)

where,

- $x$ is the number of agents from which $i$ gets resources,
- $y$ is the number of links formed by agent $i$.

We suppose that the setting up of a link is costly. Therefore, for all $x \in N$ and for all $y \in \{0, \ldots, n - 2\}$, we have $\phi_i(x, y) > \phi_i(x, y + 1)$.
Equilibrium networks and efficient networks

The strategy \( s_i \) is said to be a best response of agent \( i \) if
\[
u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i}), \forall s_i' \in S.
\]

A network \( \Gamma_s \) is said to be a Nash network if each agent \( i \) plays a best response:
\[
u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i}), \forall s_i' \in S, \forall i \in N.
\]

A strict Nash network is one where each agent gets strictly higher payoffs with her current strategy than she would with any other strategy.
Equilibrium networks and efficient networks

Formally, $\Gamma_s$ is an efficient network if:

$$\sum_{i \in N} u_i(s) \geq \sum_{i \in N} u_i(s'), \forall s' \in S.$$
Nash networks and strict Nash networks

• Proposition 1. Assume the payoff function is defined by Eq.(1). A Nash network is a line basis.

Fig. 1. Line bases and minimum line bases.
Nash networks and strict Nash networks

Example 2.
Suppose \( N = \{1,2,3,4\} \) and the following payoff function for each \( i \in N \):

\[
\begin{align*}
\phi_i(1,0) &= 5 & \phi_i(2,1) &= 6 & \phi_i(3,1) &= 4 & \phi_i(3,2) &= 3 \\
\phi_i(4,1) &= 2 & \phi_i(4,2) &= 1 & \phi_i(4,3) &= 0
\end{align*}
\]
Nash networks and strict Nash networks
Nash networks and strict Nash networks

• Proposition 2. Assume the payoff function is defined by Eq. (1). A strict Nash network is a minimum line basis.
Lemma 1. Suppose $\Gamma$ is a line basis which contains no cycle. Then $\Gamma$ is a minimum line basis.

Lemma 2. Suppose $\Gamma$ is a line basis which contains only one cycle. Then $\Gamma$ is a minimum line basis.

Lemma 3. Suppose $\Gamma$ is a line basis which contains disjoint cycles. Then $\Gamma$ is a minimum line basis.

Lemma 4. Suppose $\Gamma$ is a line basis. If $\Gamma$ is a non-minimum line basis, then $\Gamma$ contains non-disjoint cycles.
Nash networks and strict Nash networks
Efficient Networks

• **Proposition 3.** Assume the payoff function is defined by Eq.(1). An efficient network is a minimum line basis.
Efficient Networks

Fig. 2. Strict Nash networks and efficient networks.

**Example 3.** Let $N = \{1, 2, 3, 4\}$ and $i \in \{1, 2, 3\}$. We assume the following payoff function:

$$
\begin{align*}
\phi_i(4, 1) &= 10, & \phi_i(1, 0) &= 2, & \phi_4(3, 1) &= 3, \\
\phi_i(3, 1) &= 5, & \phi_i(4, 2) &= 1, & \phi_4(4, 2) &= 2, \\
\phi_i(3, 2) &= 4, & \phi_4(1, 0) &= 5, & \phi_4(3, 2) &= 1, \\
\phi_i(2, 1) &= 3, & \phi_4(4, 1) &= 4, & \phi_4(2, 1) &= 0,
\end{align*}
$$

and for all $j \in N$, $\phi_j(4, 3) = -1$. 
Conclusion

• Not assumed any priori relationship between the number of resources that an agent gets and her payoffs.
• Nash networks, the strict Nash networks and the efficient networks can be characterized by the notions of line bases and minimum line bases.
• The static analysis can be extended to a dynamic analysis by introducing a myopic best response dynamics.