Lecture #9:
Poisson Processes

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Discrete time vs. Continuous time

- So far, our processes evolve in discrete time
- Now we will look at an important class of continuous-time processes
- These model 'arrivals', and are called **Poisson processes**
- A first step toward general continuous-time Markov processes

An arrival process

- Suppose we model the arrival of email in our inbox...
- We start counting at 8 AM:
  - Between 8 AM and 9 AM, one email arrives
  - Between 9 AM and 10 AM, three emails arrive
  - Between 10 AM and 11 AM, two emails arrive
- We could also count arrivals cumulatively:
  - Between 8 AM and 9 AM, one email arrives
  - Between 8 AM and 10 AM, four emails arrive
  - Between 8 AM and 11 AM, six emails arrive

An arrival process (cont.)

- How might we model this process?
- Suppose the hourly arrival rate is fixed and known...
- ...i.e., on average three emails arrive per hour...
- ...denote the rate of arrivals per unit time as λ
- An email can arrive at any point in time...

An arrival process (cont.)

- We could break up an hour into 60 one-minute blocks

- In each minute, one email arrives with probability $p = \frac{\lambda}{60}$...
- ...no emails arrive with probability $1 - p$

- For this model, expected number of arrivals per hour is λ

- Not quite accurate, since only one email can arrive per minute...

An arrival process (cont.)

- The model just described is a countable-state Markov chain

- The state-space is $\mathcal{X} = \{0, 1, 2, \ldots\}$

- X_n gives the total number of arrivals up to minute n

- We start with $X_0 = 0$

- If $X_n = x$,
  - $X_{n+1} = x + 1$ with probability p
  - $X_{n+1} = x$ with probability $1 - p$

An arrival process (cont.)

- All states of this Markov chain are transient

- It doesn't make sense to talk about steady-state behavior of this MC

- We are mainly interested in computing probabilities of the form

$$\mathbf{P}(X_{n_1} = x_1, X_{n_2} = x_2, \ldots, X_{n_k} = x_k)$$

Properties of the arrival process

- This MC is time-homogeneous, so

$$\mathbf{P}(X_{t+\tau} = x \,|\, X_t = 0) = \mathbf{P}(X_\tau = x \,|\, X_0 = 0)$$

- Also, this MC has the property that for all $k, j \le t$,

$$\mathbf{P}(X_{t+\tau} = x + k \,|\, X_t = k) = \mathbf{P}(X_{t+\tau} = x + j \,|\, X_t = j)$$

- Can translate into statements about the **increments** of the process

Stationary increments

- For this MC, increments are **stationary**...
- ...that is, for all t, τ and k, $\mathbf{P}(X_{t+\tau} - X_t = x) = \mathbf{P}(X_\tau = x)$
- To show this,

$$\begin{aligned}
\mathbf{P}(X_{t+\tau} - X_t = x) &= \sum_{k=0}^{t} \mathbf{P}(X_{t+\tau} = x + k, X_t = k) \\
&= \sum_{k=0}^{t} \mathbf{P}(X_{t+\tau} = x + k \,|\, X_t = k)\mathbf{P}(X_t = k) \\
&= \sum_{k=0}^{t} \mathbf{P}(X_\tau = x)\mathbf{P}(X_t = k) \\
&= \mathbf{P}(X_\tau = x)
\end{aligned}$$

Independent increments

- Also, disjoint increments are independent...
- ...that is, for $n \geq m + j$

$$\begin{aligned}
&\mathbf{P}(X_{n+k} - X_n = x, X_{m+j} - X_m = y) \\
&= \mathbf{P}(X_{n+k} - X_n = x)\mathbf{P}(X_{m+j} - X_m = y) \quad \text{(in HW)} \\
&= \mathbf{P}(X_k = x)\mathbf{P}(X_j = y)
\end{aligned}$$

- A similar result holds for arbitrary number of disjoint intervals
- This means we can compute any joint probability via the marginals...

Independent increments (cont.)

- We can compute any joint probability via the marginals...

- In other words,

$$\mathbf{P}(X_{n_1} = x_1, X_{n_2} = x_2, \ldots, X_{n_k} = x_k)$$
$$= \mathbf{P}(X_{n_1} = x_1) \prod_{i=2}^{k} \mathbf{P}(X_{n_i - n_{i-1}} = x_i - x_{i-1})$$

Evolution of the marginals

- Let $\rho_n(k) = \mathbf{P}(X_n = k)$

- For all $k > 0$,

$$\rho_n(k) = (1-p)\rho_{n-1}(k) + p\rho_{n-1}(k-1)$$

- For $k = 0$,

$$\rho_n(0) = (1-p)\rho_{n-1}(0)$$

- It turns out that the solution to these equations is

$$\rho_n(k) = \binom{n}{k}(1-p)^{n-k}p^k$$

Discretization of time

- In out current model, at most one email arrives per minute
- We can use a finer discretization of time...
- ...say, at most one email arrives per second
- Now let X_n give the total number of arrivals up to **second** n
- As before, if $X_n = x$,
  - $X_{n+1} = x + 1$ with some probability p
  - $X_{n+1} = x$ with some probability $1 - p$
- To preserve the rate of λ arrivals per hour, we now use $p = \frac{\lambda}{3600}$

Discretization of time (cont.)

- In general, we can divide an hour into m units of equal length
- Let t be the time in hours, and let $\delta t = \frac{1}{m}$
- After n time steps, $t = (\delta t)n$ hours have passed
- Now let's index X_n in terms of t...
- ...that is, we write $X(t)$ to mean X_n where $t = (\delta t)n$
- Similarly, we'll write $\rho_n(k)$ in terms of t as $\rho(t, k)$

Discretization of time (cont.)

- If we make δt smaller, want to preserve the hourly rate λ...
- ...if $X(t) = x$, then $X(t + \delta t) = x + 1$ with probability $p = (\delta t)\lambda$
- In terms of t, $\mathbf{P}(X(t) = k)$ evolves as
  - $\rho(t + \delta t, k) = (1 - (\delta t)\lambda)\rho(t, k) + (\delta t)\lambda\rho(t, k - 1)$ for $k > 0$
  - $\rho(t + \delta t, 0) = (1 - (\delta t)\lambda)\rho(t, 0)$

Discretization of time (cont.)

- The evolution equations can be rewritten as
  - For $k > 0$,
    $$\frac{\rho(t + \delta t, k) - \rho(t)}{\delta t} = -\lambda\big(\rho(t, k) - \rho(t, k - 1)\big)$$
  - For $k = 0$,
    $$\frac{\rho(t + \delta t, 0) - \rho(t, 0)}{\delta t} = -\lambda\rho(t, 0)$$
- Letting $\delta t \to 0$ gives a set of differential equations
- Now we're ready to look at a continuous-time version of this process...

Poisson process

- **Continuous-time stochastic process:**

  A family of random variables $X(t)$, one for each $t \in \mathbb{R}$

- Define the **Poisson process** to be the cont.-time process with:

  1. Marginals $\rho(t,k) = \mathbf{P}(X(t) = k)$ satisfying:
     - $\dfrac{d}{dt}\rho(t,k) = -\lambda\big(\rho(t,k) - \rho(t,k-1)\big)$ for $k > 0$
     - $\dfrac{d}{dt}\rho(t,0) = -\lambda\rho(t,0)$
  2. Stationary and independent increments:

$$\mathbf{P}(X(t+k)-X(t) = x, X(\tau+j)-X(\tau) = y) = \mathbf{P}(X(k) = x)\mathbf{P}(X(j) = y)$$

Poisson distribution

- The solution to the set of equations
  - $\dfrac{d}{dt}\rho(t,k) = -\lambda\big(\rho(t,k) - \rho(t,k-1)\big)$ for $k > 0$
  - $\dfrac{d}{dt}\rho(t,0) = -\lambda\rho(t,0)$

  with $\rho(0,0) = 1$ is

$$\rho(t,k) = \frac{(\lambda t)^k}{k!}e^{-\lambda t}$$

- This $\rho(t,k)$ is called the **Poisson distribution**

- $\rho(t,k)$ gives probability of exactly k arrivals in the interval $[0,t]$

Example: Arrival of email

- The rate of email to an inbox is $\lambda = 3$ emails per hour

- If we start counting at 8 AM what is the probability that:
- ...exactly 3 have arrived by 9 AM...
- ...exactly 6 have arrived by 10 AM...
- ...and exactly 9 have arrived by 11 AM?

Example: Arrival of email (cont.)

- We want to find

$$\begin{aligned}
&\mathbf{P}(X(1) = 3, X(2) = 6, X(3) = 9) \\
&\qquad = \mathbf{P}(X(1) = 3) \\
&\qquad\quad \times \mathbf{P}(X(2) - X(1) = 3) \\
&\qquad\quad \times \mathbf{P}(X(3) - X(2) = 3) \\
&\qquad = \Big(\mathbf{P}(X(1) = 3)\Big)^3 \\
&\qquad = \left(\frac{(3)^3}{3!} e^{-3}\right)^3 \\
&\qquad = 0.0112
\end{aligned}$$

Waiting times

- For a Poisson process, how long do we wait between arrivals?
- Or, what is probability we wait at least t time units for first arrival?
- Same as looking for the probability of no arrivals in t time units:

$$\mathbf{P}(X(t) = 0) = e^{-\lambda t}$$

- Waiting times as **exponentially distributed**

Waiting times (cont.)

- Consider the conditional waiting time for $\tau \leq t$:

$$\mathbf{P}(X(t) = 0 \,|\, X(\tau) = 0)$$

- This satisfies

$$\begin{aligned} \mathbf{P}(X(t) = 0 \,|\, X(\tau) = 0) &= \frac{\mathbf{P}(X(t) = 0, X(\tau) = 0)}{\mathbf{P}(X(\tau) = 0)} \\ &= \frac{e^{-\lambda t}}{e^{-\lambda \tau}} \\ &= e^{-\lambda(t-\tau)} \\ &= \mathbf{P}(X(t-\tau) = 0) \end{aligned}$$

- Exponentially distributed random variables are **memoryless**

Poisson race

- $X_1(t)$ and $X_2(t)$ are independent & Poisson with rates λ_1 and λ_2
- What is waiting time until first arrival?
- Same as looking for the probability of no arrivals in t time units:

$$\begin{aligned}\mathbf{P}(X_1(t) = 0 \text{ and } X_2(t) = 0) &= \mathbf{P}(X_1(t) = 0)\mathbf{P}(X_2(t) = 0) \\ &= e^{-\lambda_1 t} e^{-\lambda_2 t} \\ &= e^{-(\lambda_1+\lambda_2)t}\end{aligned}$$

- Waiting time is exponential with rate $\lambda_1 + \lambda_2$

Merging and splitting

- Poisson processes have two important properties:
- **Merging of Poisson processes:**
  - $X(t)$ and $Y(t)$ are independent & Poisson with rates λ_1 and λ_2
  - Then $Z(t) = X(t) + Y(t)$ is a Poisson process with rate $\lambda_1 + \lambda_2$
- **Splitting of Poisson processes:**
  - Suppose $Z(t)$ is a Poisson process
  - We create two new processes $X(t)$ and $Y(t)$
  - Assign each arrival of Z independently to X with probability q
  - Assign each arrival of Z independently to Y with probability $1-q$
  - $X(t)$, $Y(t)$ are Poisson & independent with rates $q\lambda$, $(1-q)\lambda$

Merging (cont.)

- To prove the merging property, we need to show:

1. The marginal probabilities of $Z(t)$ satisfy

$$\mathbf{P}(Z(t)=k)=\frac{((\lambda_1+\lambda_2)t)^k}{k!}e^{-(\lambda_1+\lambda_2)t}$$

2. The increments of $Z(t)$ are stationary and independent:

$$\mathbf{P}(Z(t+k)-Z(t)=x, Z(\tau+j)-Z(\tau)=y)=\mathbf{P}(Z(k)=x)\mathbf{P}(Z(j)=y)$$

Merging (cont.)

- The probability $\mathbf{P}(Z(t)=k)$ is given by

$$\begin{aligned}
&\mathbf{P}(Z(t)=k)\\
&\qquad=\sum_{j=0}^{k}\mathbf{P}(X(t)=j, Y(t)=k-j)\\
&\qquad=\sum_{j=0}^{k}\mathbf{P}(X(t)=j)\mathbf{P}(Y(t)=k-j)\\
&\qquad=\sum_{j=0}^{k}\left(\frac{(\lambda_1 t)^j}{j!}e^{-\lambda_1 t}\right)\left(\frac{(\lambda_2 t)^{k-j}}{(k-j)!}e^{-\lambda_2 t}\right)\\
&\qquad=t^k e^{-(\lambda_1+\lambda_2)t}\sum_{j=0}^{k}\frac{\lambda_1^j\lambda_2^{k-j}}{j!(k-j)!}=\frac{((\lambda_1+\lambda_2)t)^k}{k!}e^{-(\lambda_1+\lambda_2)t}
\end{aligned}$$

Merging (cont.)

- For $t_1 \le t_2 \le t_3 \le t_4$,

$$\begin{aligned}
&\mathbf{P}(Z(t_4) - Z(t_3) = k_1, Z(t_2) - Z(t_1) = k_2) \\
&= \sum_{j_1=0}^{k_1} \sum_{j_2=0}^{k_2} \mathbf{P}(X(t_4) - X(t_3) = j_1, \ldots, Y(t_2) - Y(t_1) = k_2 - j_2) \\
&= \sum_{j_1=0}^{k_1} \sum_{j_2=0}^{k_2} \mathbf{P}(X(t_4) - X(t_3) = j_1) \cdots \mathbf{P}(Y(t_2) - Y(t_1) = k_2 - j_2) \\
&= \mathbf{P}(Z(t_4) - Z(t_3) = k_1)\mathbf{P}(Z(t_2) - Z(t_1) = k_2)
\end{aligned}$$

- Increments of X and Y are stationary and independent...
- ...so increments of Z are stationary and independent

Merging (cont.)

- Why is this property surprising?
- Consider two discrete-time arrival processes X_n and Y_n
- It is possible to have $X_n(\omega) = 1$ and $Y_n(\omega) = 1$
- In this case, $Z_n(\omega) = X_n(\omega) + Y_n(\omega) = 2$...
- ...so Z_n is not the same type of arrival process as X_n and Y_n
- ...things can get complicated if we merge and split many of these

Splitting

- Need to introduce some notation to make splitting property precise
- Let $S_1, S_2, \ldots$ be a sequence of IID random variables...
- ...each $S_i = 1$ with probability q and $S_i = 0$ with probability $1 - q$
- If $Z(t)$ gives a Poisson process with rate λ, we define $X(t)$ as

$$X(t) = \sum_{k=0}^{Z(t)} S_k$$

- Similarly, we define $Y(t) = \sum_{k=0}^{Z(t)} (1 - S_k)$

Splitting (cont.)

- To prove the splitting property, we need to show:

1. The marginal probabilities of $X(t)$ and $Y(t)$ satisfy

$$\begin{aligned} \mathbf{P}(X(t) = k) &= \frac{(q\lambda t)^k}{k!} e^{-q\lambda t} \\ \mathbf{P}(Y(t) = k) &= \frac{((1-q)\lambda t)^k}{k!} e^{-(1-q)\lambda t} \end{aligned}$$

- Continued...

Splitting (cont.)

- We also need to show:

  2. Increments are stationary and independent (also in the HW):

$$\mathbf{P}(X(t+k)-X(t)=x, X(\tau+j)-X(\tau)=y) \\ = \mathbf{P}(X(k)=x)\mathbf{P}(X(j)=y)$$

$$\mathbf{P}(Y(t+k)-Y(t)=x, Y(\tau+j)-Y(\tau)=y) \\ = \mathbf{P}(Y(k)=x)\mathbf{P}(Y(j)=y)$$

  3. $X(t)$ and $Y(\tau)$ are independent for all t and τ:

$$\mathbf{P}(X(t)=x, Y(\tau)=y) = \mathbf{P}(X(t)=x)\mathbf{P}(Y(\tau)=y)$$

Splitting (cont.)

- The probability $\mathbf{P}(X(t)=k)$ is given by

$$\begin{aligned}
\mathbf{P}(X(t)=k) &= \sum_{j=k}^{\infty} \mathbf{P}\left(Z(t)=j, \sum_{i=1}^{j} S_i = k\right) \\
&= \sum_{j=k}^{\infty} \mathbf{P}(Z(t)=j)\mathbf{P}\left(\sum_{i=1}^{j} S_i = k\right) \\
&= \sum_{j=k}^{\infty} \left(\frac{(\lambda t)^j}{j!} e^{-\lambda t}\right) \binom{j}{k} (1-q)^{j-k} q^k \\
&= \frac{(q\lambda t)^k}{k!} e^{-q\lambda t}
\end{aligned}$$

Splitting (cont.)

- The probability $\mathbf{P}(X(t) = k, Y(t) = j)$ is given by

$$\mathbf{P}(X(t) = j, Y(t) = k) = \mathbf{P}\left(Z(t) = j + k, \sum_{i=1}^{j+k} S_i = j\right)$$

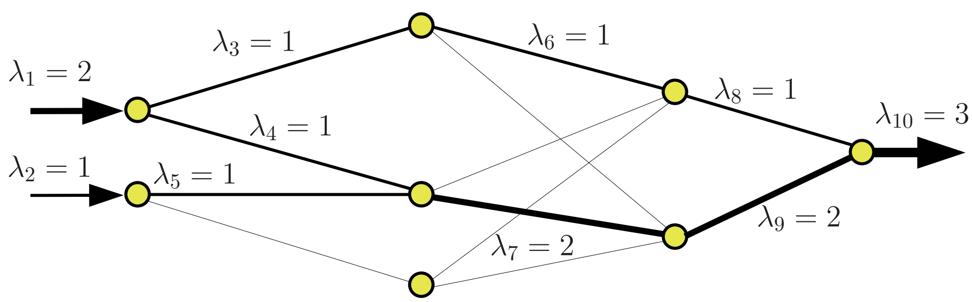
$$= \mathbf{P}(Z(t) = j + k)\mathbf{P}\left(\sum_{i=1}^{j+k} S_i = j\right)$$

$$= \left(\frac{(\lambda t)^{j+k}}{(j+k)!}e^{-\lambda t}\right)\binom{j+k}{k}(1-q)^k q^j$$

$$= \left(\frac{(q\lambda t)^j}{j!}e^{-q\lambda t}\right)\left(\frac{((1-q)\lambda t)^k}{k!}e^{-(1-q)\lambda t}\right)$$

Networks

- Merging and splitting are very useful for analyzing networks...



- Assume incoming arrivals are independent and Poisson...
- ...flows are split by random sampling...
- ...then arrivals on all links are independent and Poisson!

Summary

- The Poisson Process is a type of continuous-time Markov process
- Poisson processes model arrivals at a given rate
- Waiting times are exponentially distributed
- Poisson processes are preserved under merging and splitting

Where to next?

- Next we'll look at general continuous-time Markov processes
- Have several important properties in common with Poisson process...
- ...waiting times between state transitions are exponentially distributed
- Also, have much in common with discrete-time Markov chains...
- ...can analyze as a discrete-time MC at the 'jump times'