Lecture #7:
Countable-state Markov Chains

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Countable state-spaces

- So far, we've seen Markov chains with finite state-spaces

- Now we'll look at MCs with **countable state spaces**

- Recall that countable means infinite, one for each integer

Countable vs. finite state MCs

- For finite-state MCs:
  - Poisson's equation gives visitation frequency
  - At least one state is recurrent
  - Single recurrent class $\Longrightarrow$ limiting probability exists
  - If set S is reachable, hitting time to S is finite
- For countable-state MCs:
  - All states can be transient!
  - Countable-state MCs don't always have properties listed above
  - Want to know when similar results hold for countable-state MCs

Example: All states are transient

- Consider the countable state-space $\mathcal{X} = \{0, 1, 2, \ldots\}$
- We have the following Markov chain:
  - When in state x, move to state $x + 1$ with probability 1
- No state is visited more than once, so all states are transient!

Example: Reflecting random walk

- Again, countable state-space $\mathcal{X} = \{0, 1, 2, \ldots\}$
- We have the following Markov chain:
  - $X_{t+1} = x + 1$ with probability q
  - If $X_t > 0$, $X_{t+1} = x - 1$ with probability $1 - q$
  - If $X_t = 0$, $X_{t+1} = 0$ with probability $1 - q$
- The state space of this MC contains a single closed class
- I.e., can reach any state from any other state with positive prob.

Example: Reflecting random walk (cont.)

- We consider the expected value of the first passage time,

$$T_0^{(1)}(\omega) = \inf\{t > 0 \,|\, X_t(\omega) = 0, X_0(\omega) = 0\}$$

- We'll work this out in terms of the hitting times for state 0:

$$T_{x,\{0\}}(\omega) = \inf\{t \geq 0 \,|\, X_t(\omega) = 0, X_0(\omega) = x\}$$

- If $\mathbf{E}[T_{x,\{0\}}] < \infty$ for all $x \in \{0, 1, \ldots\}$, then for $x > 0$

$$\mathbf{E}[T_{x,\{0\}}] = 1 + (1 - q)\mathbf{E}[T_{x-1,\{0\}}] + q\mathbf{E}[T_{x+1,\{0\}}]$$

- Unique solution is

$$\mathbf{E}[T_{x,\{0\}}] = \frac{x}{1 - 2q}$$

Example: Reflecting random walk (cont.)

- Expected first passage time for state $x = 0$ is

$$\begin{aligned}\mathbf{E}[T_0^{(1)}] &= (1-q) + q\big(1 + \mathbf{E}[T_{1,\{0\}}]\big) \\ &= 1 + \frac{q}{1-2q} \\ &= \frac{1-q}{1-2q}\end{aligned}$$

- For all $q < \frac{1}{2}$, $\mathbf{E}[T_0^{(1)}] < \infty$

- However, as $q \to \frac{1}{2}$, $\mathbf{E}[T_0^{(1)}] \to \infty$

Example: Reflecting random walk (cont.)

- Let's look closer at the case of $q = \frac{1}{2}$

- To analyze this, consider a related finite-state MC

- We have the Markov chain with state-space $\mathcal{Y} = \{0, 1, 2, \ldots, n\}$:
  - If $Y_t = 0$, $Y_{t+1} = 0$
  - If $Y_t = n$, $Y_{t+1} = n$
  - Otherwise,

$$Y_{t+1} = \begin{cases} y-1 & \text{with probability } \frac{1}{2} \\ y+1 & \text{with probability } \frac{1}{2} \end{cases}$$

Example: Reflecting random walk (cont.)

- For the Markov chain $Y_0, Y_1, \ldots$,

$$\mathbf{P}_Y(T_{y,\{0\}} < T_{y,\{n\}}) = \frac{n-y}{n}$$

- Can prove with a Martingale argument; we'll see this later in the course

- For all n, the first passage time for $X_0, X_1, \ldots$ satisfies

$$\mathbf{P}_X(T_0^{(1)} < \infty) \geq (1-q) + q\mathbf{P}_Y(T_{1,\{0\}} < T_{1,\{n\}})$$

- Therefore, $\mathbf{P}_X(T_0^{(1)} < \infty) = 1$

Positive recurrence

- In the example, if $q \leq \frac{1}{2}$, then $\mathbf{P}(T_0^{(1)} < \infty) = 1$

- Therefore, $y = 0$ is recurrent when $q \leq \frac{1}{2}$

- However, we only have $\mathbf{E}[T_0^{(1)}] < \infty$ when $q < \frac{1}{2}$

- We say that a state $x \in \mathcal{X}$ is **positive recurrent** if both

$$\mathbf{P}(T_x^{(1)} < \infty) = 1 \quad \text{and} \quad \mathbf{E}[T_x^{(1)}] < \infty$$

Null recurrence

- We say that a state $x \in \mathcal{X}$ is **null recurrent** if

$$\mathbf{P}(T_x^{(1)} < \infty) = 1 \quad \text{but} \quad \mathbf{E}[T_x^{(1)}] = \infty$$

- In our new terminology, results for the reflecting random walk are:
  - If $q < \frac{1}{2}$, state $x = 0$ is positive recurrent
  - If $q = \frac{1}{2}$, state $x = 0$ is null recurrent
  - If $q > \frac{1}{2}$, state $x = 0$ is transient

Foster's criterion

- Suppose $x \leftrightarrow y$ for all $x, y \in \mathcal{X}$
- All states are either transient, positive recurrent, or null recurrent
- How can we test if all states are positive recurrent?
- Can use **Foster's criterion**

Foster's criterion

- **Theorem:**
  - Suppose $x \leftrightarrow y$ for all $x, y \in \mathcal{X}$
  - Let $\mathcal{S} \subseteq \mathcal{X}$ be a finite subset of the state-space
  - Suppose there is a nonnegative function $V : \mathcal{X} \to \mathbb{R}_+$ with:
    - $\sum_{y \in \mathcal{X}} p(y \,|\, x) V(y) < \infty$ for all $x \in \mathcal{X}$
    - $\sum_{y \in \mathcal{X}} p(y \,|\, x) V(y) \leq V(x) - \epsilon$ for all $x \notin \mathcal{S}$ (where $\epsilon > 0$)
  - Then the MC is positive recurrent

Foster's criterion (cont.)

- **Proof:**
  - **Idea:** Each time we're in some state $x \notin \mathcal{S}$...
  - ...on average, we drift back toward $\mathcal{S}$
  - Let $T_{x,\mathcal{S}}(\omega) = \inf\{t \geq 0 \,|\, X_t(\omega) \in \mathcal{S}, X_0(\omega) = x\}$
  - Define the 'stopped process'

$$Y_t(\omega) = \begin{cases} X_t(\omega) & \text{if } t \leq T_{x,\mathcal{S}}(\omega) \\ Y_{t-1}(\omega) & \text{if } t > T_{x,\mathcal{S}}(\omega) \end{cases}$$

  - $\mathbf{E}[T_{x,\mathcal{S}} \,|\, X_0 = x] = \sum_{t=0}^{\infty} \mathbf{P}(Y_t \notin \mathcal{S} \,|\, Y_0 = x)$
  - Continued on next slide...

Foster's criterion (cont.)

- **Proof (cont.):**
  - We can denote $\sum_{y \in \mathcal{X}} p(y \mid x) V(y) = \mathbf{E}[V(X_{t+1}) \mid X_t = x]$
  - It's true that

$$\begin{aligned}
&\mathbf{E}[V(Y_t) \mid Y_0 = x] - V(x) \\
&\qquad = \sum_{k=0}^{t-1} (\mathbf{E}[V(Y_{k+1}) \mid Y_0 = x] - \mathbf{E}[V(Y_k) \mid Y_0 = x]) \\
&\qquad = \sum_{k=0}^{t-1} \mathbf{E}\big[\mathbf{E}[V(Y_{k+1}) \mid Y_k] - V(Y_k) \,\big|\, Y_0 = x\big] \\
&\qquad \leq -\epsilon \sum_{k=0}^{t-1} \mathbf{P}(Y_k \notin \mathcal{S} \mid Y_0 = x) \quad ; \quad \text{continued...}
\end{aligned}$$

Foster's criterion (cont.)

- **Proof (cont.):**
  - Therefore, for all t we have

$$\begin{aligned}
\sum_{k=0}^{t-1} \mathbf{P}(Y_k \notin \mathcal{S} \mid Y_0 = x) &\leq \frac{V(x) - \mathbf{E}[V(Y_t) \mid Z_0 = x]}{\epsilon} \\
&\leq \frac{V(x)}{\epsilon}
\end{aligned}$$

  - In other words, $\mathbf{E}[T_{x,\mathcal{S}} \mid X_0 = x] \leq \frac{V(x)}{\epsilon}$
  - Finally, from any $x \in \mathcal{S}$, expected first passage time is less than

$$\begin{aligned}
1 + \sum_{z \in \mathcal{X}} p(z \mid x) \mathbf{E}[T_{z,\mathcal{S}} \mid X_0 = z] &\leq 1 + \frac{1}{\epsilon} \sum_{z \in \mathcal{X}} p(z \mid x) V(z) \\
&< \infty
\end{aligned}$$

Example: Foster's Criterion

- Recall the reflecting random walk example
- We've already seen that $x \leftrightarrow y$ for all $x, y \in \mathcal{X}$
- Consider the case with $q < \frac{1}{2}$
- We'll show that we have positive recurrence in this case

Example: Foster's criterion (cont.)

- Use $V(x) = x$; clearly $V(x) \geq 0$ for all $x \in \mathcal{X}$
- For all $x > 0$,

$$\begin{aligned} E[V(X_{t+1}) \,|\, X_t = x] - V(x) &= q(x+1) + (1-q)(x-1) - x \\ &= 2q - 1 \\ &< 0 \end{aligned}$$

- Also,

$$\begin{aligned} E[V(X_{t+1}) \,|\, X_t = x] &= x + 2q - 1 < \infty \qquad \text{for } x > 0 \\ E[V(X_{t+1}) \,|\, X_t = x] &= q < \infty \qquad \text{for } x = 0 \end{aligned}$$

- Therefore, all states are positive recurrent

Example: Foster's criterion (cont.)

- As a byproduct, this gives us bounds on expected hitting times
- That is, consider the expected hitting time to $\{0\}$ from state $x \in \mathcal{X}$
- In this case, $\epsilon = 1 - 2q$
- Therefore,

$$\begin{aligned}\mathbf{E}[T_{x,\{0\}}] &\le \frac{V(x)}{\epsilon} \\ &= \frac{x}{1-2q}\end{aligned}$$

Poisson's equation

- Suppose there are functions $\beta : \mathcal{X} \to \mathbb{R}$ and $h : \mathcal{X} \to \mathbb{R}$ satisfying
  - $\beta(x) = \sum_{y \in \mathcal{X}} p(y \mid x) \beta(y)$ for all $x \in \mathcal{X}$
  - $h(x) = r(x) + \sum_{y \in \mathcal{X}} p(y \mid x) h(y) - \beta(x)$ for all $x \in \mathcal{X}$
  - $\lim_{t \to \infty} \frac{1}{t} \mathbf{E}[h(X_t) \mid X_0 = x] = 0$ for all $x \in \mathcal{X}$
- Then for all $x \in \mathcal{X}$,

$$\beta(x) = \lim_{t \to \infty} \frac{1}{t} \sum_{k=0}^{t-1} E[r(X_k) \mid X_0 = x]$$

Example: Reflecting random walk

- Consider Poisson's equation for 'reflecting random walk' example
- Let $r(0) = 1$, $r(x) = 0$ everywhere else
- Consider the solution:
  - $\beta(x) = \frac{1-2q}{1-q}$ for all $x \in \mathcal{X}$
  - $h(x) = -\frac{x}{1-q}$ for all $x \in \mathcal{X}$

Example: Reflecting random walk (cont.)

- Since β is constant, $\beta(x) = \sum_{y \in \mathcal{X}} p(y \mid x)\beta(y)$ for all $x \in \mathcal{X}$
- For $x = 0$

$$\begin{aligned} r(0) + \sum_{y \in \mathcal{X}} p(y \mid 0)h(y) - \beta(0) &= 1 - q\left(\frac{1}{1-q}\right) - \frac{1-2q}{1-q} \\ &= \frac{(1-q) - q - (1-2q)}{1-q} \\ &= 0 \\ &= h(0) \end{aligned}$$

Example: Reflecting random walk (cont.)

- For $x > 0$

$$
\begin{aligned}
r(x) + \sum_{y \in \mathcal{X}} p(y \,|\, x) h(y) - \beta(x) \\
&= q \left(\frac{-(x+1)}{1-q} \right) + (1-q) \left(\frac{-(x-1)}{1-q} \right) - \frac{1-2q}{1-q} \\
&= \frac{-x - q + (1-q) - (1-2q)}{1-q} \\
&= \frac{-x}{1-q} \\
&= h(x)
\end{aligned}
$$

Example: Reflecting random walk (cont.)

- This choice of β and h satisfy Poisson's equation

- However, it's not clear that for all $x \in \mathcal{X}$,

$$\lim_{t \to \infty} \frac{1}{t} \mathbf{E}[h(X_t) \,|\, X_0 = x] = 0$$

- Next, we'll look at a condition for checking this...

Poisson's equation (cont.)

- **Theorem:** For a function $h : \mathcal{X} \to \mathbb{R}$,
  - Suppose
$$\sup_{x \in \mathcal{X}} \left\{ \sum_{y \in \mathcal{X}} p(y \,|\, x) h(y)^2 - h(x)^2 \right\} < \infty$$
  - Then
$$\lim_{t \to \infty} \frac{1}{t} \mathbf{E}[h(X_t) \,|\, X_0 = x] = 0$$
- We can use this in the reflecting random walk example...

Poisson's equation (cont.)

- **Proof:**
  - Denote $\mathbf{E}[h(X_{t+1})^2 \,|\, X_t = x] = \sum_{y \in \mathcal{X}} p(y \,|\, x) h(y)^2$
  - Let
$$\sup_{x \in \mathcal{X}} \left\{ \mathbf{E}[h^2(X_{t+1}) \,|\, X_t = x] - h(x)^2 \right\} = M < \infty$$
  - Then,
$$\begin{aligned} tM \ &\geq \ \mathbf{E}\left[\sum_{k=0}^{t-1} \left(\mathbf{E}[h(X_{k+1})^2 \,|\, X_k] - h(X_k)^2 \right) \middle| \, X_0 = x \right] \\ &= \mathbf{E}[h(X_t)^2 \,|\, X_0 = x] - h(x)^2 \end{aligned}$$
  - Continued on next slide...

Poisson's equation (cont.)

- **Proof (cont.):**
  - Then,

$$\begin{aligned}\frac{1}{t}\mathbf{E}\left[|h(X_t)| \,\middle|\, X_0 = x\right] &\leq \frac{1}{t}\sqrt{\mathbf{E}[h(X_t)^2 \,|\, X_0 = x]} \\ &\leq \frac{1}{t}\sqrt{\left(h(x)^2 + tM\right)}\end{aligned}$$

  - Hence,

$$\begin{aligned}\limsup_{t\to\infty} \frac{1}{t}\mathbf{E}\left[|h(X_t)| \,\middle|\, X_0 = x\right] &\leq \lim_{t\to\infty} \frac{1}{t}\sqrt{\left(h(x)^2 + tM\right)} \\ &= 0\end{aligned}$$

Example: Reflecting random walk (cont.)

- Finally, we'll verify that when $q \leq \frac{1}{2}$,

$$\lim_{t\to\infty} \frac{1}{t}\mathbf{E}[h(X_t) \,|\, X_0 = x] = 0$$

- For $x = 0$,

$$\begin{aligned}\sum_{y\in\mathcal{X}} p(y \,|\, x)h(y)^2 - h(x)^2 &= \frac{q}{(1-q)^2} \\ &< \infty\end{aligned}$$

Example: Reflecting random walk (cont.)

- For $x > 0$,

$$\begin{aligned}\sum_{y \in \mathcal{X}} p(y \,|\, x) h(y)^2 - h(x)^2 &= q\frac{(x+1)^2}{(1-q)^2} + (1-q)\frac{(x-1)^2}{(1-q)^2} - \frac{x^2}{(1-q)^2} \\ &= \frac{2qx - 2(1-q)x + 1}{(1-q)^2} \\ &= \frac{(4q-2)x+1}{(1-q)^2}\end{aligned}$$

- If $q \le \frac{1}{2}$,

$$\sup_{x \in \mathcal{X}} \left\{ \frac{(4q-2)x+1}{(1-q)^2} \right\} < \infty$$

Example: Reflecting random walk (cont.)

- Let

$$r(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases}$$

- All of this shows that for $q \le \frac{1}{2}$,

$$\lim_{n\to\infty} \sum_{k=0}^{t-1} \mathbf{E}[r(X_k) \,|\, X_0 = x] = \frac{1-2q}{1-q}$$

- Expected frequency of visits to state $x = 0$ is 0 when $q = \frac{1}{2}$

Invariant probabilities

- For a countable-state MC, suppose $x \leftrightarrow y$ for all $x, y \in \mathcal{X}$
- Also suppose that all states are aperiodic
- **Theorem:** If there is a positive recurrent state:
  1. There is an invariant probability ; that is, a probability π satisfying
  $$\pi(x) = \sum_{y \in \mathcal{X}} p(x \,|\, y)\pi(y) \qquad \text{for all } x \in \mathcal{X}$$
  2. For all initial states $z \in \mathcal{X}$,
  $$\pi(x) = \lim_{t \to \infty} \mathbf{P}(X_t = x \,|\, X_0 = z) \qquad \text{for all } x \in \mathcal{X}$$

Invariant probabilities (cont.)

- **Proof (for part 1):**
  - For some state $z \in \mathcal{X}$, define
  $$q(x) = \sum_{t=1}^{\infty} \mathbf{P}(X_t = x, t \leq T_z^{(1)} \,|\, X_0 = z)$$
  - Since $E[T_z^{(1)}] < \infty$,
  $$\begin{aligned} q(x) &\leq \sum_{y \in \mathcal{X}} q(y) \\ &= E[T_z^{(1)}] \\ &< \infty \end{aligned}$$
  - Continued...

Invariant probabilities (cont.)

- **Proof (for part 1, cont.):**

$$\begin{aligned} q(x) &= \sum_{t=1}^{\infty} \mathbf{P}(X_t = x, t \le T_z^{(1)}) \\ &= \sum_{y \in \mathcal{X}} \sum_{t=1}^{\infty} \mathbf{P}(X_{t-1} = y, X_t = x, t \le T_z^{(1)}) \\ &= \sum_{y \in \mathcal{X}} p(x \,|\, y) \sum_{t=1}^{\infty} \mathbf{P}(X_{t-1} = y, t \le T_z^{(1)}) \\ &= \sum_{y \in \mathcal{X}} p(x \,|\, y) q(y) \end{aligned}$$

  - So we have the invariant probability $\pi(x) = \dfrac{q(x)}{\sum_{y \in \mathcal{X}} q(y)}$

Invariant probabilities (cont.)

- **Proof idea (for part 2):**
  - Consider two independent, identical instances of the MC...
  - ...denote them as $X_0, X_1, \ldots$ and $Y_0, Y_1, \ldots$
  - Define T as

$$T(\omega) = \inf\{t \ge 0 \,|\, X_t(\omega) = Y_t(\omega) = x, X_0(\omega) = x_0, Y_0(\omega) = y_0\}$$

  - $X_0, \ldots$ and $Y_0, \ldots$ are irreducible, recurrent, and aperiodic, so

$$\mathbf{P}\big(T < \infty\big) = 1$$

  - Continued...

Invariant probabilities (cont.)

- **Proof idea (for part 2, cont.):**
  - By the strong Markov property,

$$\mathbf{P}(X_t = x \mid t \geq T, X_0 = x_0) = \mathbf{P}(Y_t = x \mid t \geq T, Y_0 = y_0)$$

  - We can use this to show

$$\begin{aligned}
&\mathbf{P}(X_t = x \mid X_0 = x_0) \\
&= \mathbf{P}(X_t = x, t \geq T \mid X_0 = x_0) + \mathbf{P}(X_t = x, t < T \mid X_0 = x_0) \\
&\leq \mathbf{P}(X_t = x, t \geq T \mid X_0 = x_0) + \big(1 - \mathbf{P}(t \geq T)\big) \\
&= \mathbf{P}(Y_t = x, t \geq T \mid Y_0 = y_0) + \big(1 - \mathbf{P}(t \geq T)\big) \\
&\leq \mathbf{P}(Y_t = x \mid Y_0 = y_0) + \big(1 - \mathbf{P}(t \geq T)\big)
\end{aligned}$$

  - Can also show

$$\mathbf{P}(Y_t = x \mid Y_0 = y_0) \leq \mathbf{P}(X_t = x \mid X_0 = x_0) + \big(1 - \mathbf{P}(t \geq T)\big)$$

Invariant probabilities (cont.)

- **Proof idea (for part 2, cont.):**
  - We know that

$$\lim_{t \to \infty} \mathbf{P}(T \leq t) = \mathbf{P}(T < \infty) = 1$$

  - Therefore,

$$\lim_{t \to \infty} \mathbf{P}(X_t = x \mid X_0 = x_0) = \lim_{t \to \infty} \mathbf{P}(Y_t = x \mid Y_0 = y_0)$$

  - If we choose y_0 with probability $\pi(y_0)$, then for all t and $x \in \mathcal{X}$,

$$\mathbf{P}(Y_t = x) = \pi(x)$$

  - So for all $x_0, x \in \mathcal{X}$, $\lim_{t \to \infty} \mathbf{P}(X_t = x \mid X_0 = x_0) = \pi(x)$

Summary

- All states can be transient when $\mathcal{X}$ is infinite
- Recurrent states can be null recurrent or positive recurrent
- If there is a single positive recurrent class:
  - There is an invariant probability
  - Invariant probability $=$ expected steady-state visitation frequency
- We can test positive recurrence using Foster's criterion
- To use Poisson's equation, we need an extra condition on h