Lecture #11: Martingales

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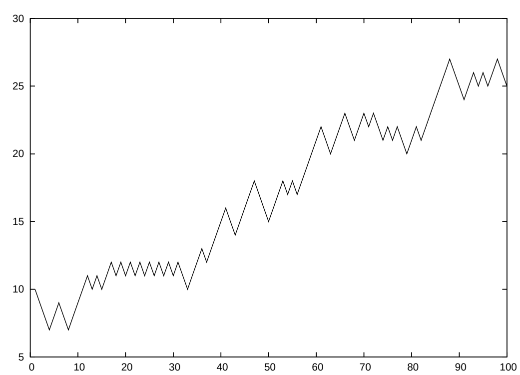
Example: Symmetric random walk

- Consider the symmetric random walk X_t, defined as

$$X_{t+1} = \begin{cases} X_t + 1 & \text{with probability } \frac{1}{2} \\ X_t - 1 & \text{with probability } \frac{1}{2} \end{cases}$$



- What is $\mathbf{E}[X_t \,|\, X_0 = x_0]$?

Example: Symmetric random walk

- For this process, $\mathbf{E}[X_t \,|\, X_0 = x] = x_0$

- To show this, we use the fact that

$$\mathbf{E}[X_{t+1} \,|\, X_t = x_t, \ldots, X_0 = x_0] = x_t \tag{1}$$

  for all $x_0, \ldots, x_t$

- A random process with the property (1) is called a **Martingale**

Continuous random variables

- In the previous example, for each t

$$\mathbf{Im}(X_t) = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$$

- Now we may encounter processes where $\mathbf{Im}(X_t)$ is uncountable

- We'll review some basic results for continuous random variables...

Continuous random variables (cont.)

- Suppose, we have a probability model where Ω is uncountable
- Let $X : \Omega \longrightarrow \mathbb{R}$ be an RV taking uncountably many values
- We can define the **cumulative distribution function** (CDF) as

$$F(x) = \mathbf{P}(X \le x)$$

  for all $x \in \mathbb{R}$

- From the CDF, we have

$$\mathbf{P}(x_1 < X \le x_2) = F(x_2) - F(x_1)$$

Continuous random variables (cont.)

- If the CDF is a continuous function of x, we can define

$$\begin{aligned} f(x) &= \lim_{\epsilon \longrightarrow 0} \frac{1}{\epsilon}\big(F(x+\epsilon) - F(x)\big) \\ &= \frac{d}{dx}F(x) \end{aligned}$$

- f is the **probability density function** (PDF) of X
- In terms of the PDF, the expected value of X is

$$\mathbf{E}[X] = \int_{-\infty}^{\infty} x f(x) dx$$

Continuous random variables (cont.)

- For a pair of RVs X and Y, we have the joint CDF

$$F(x, y) = \mathbf{P}(X \leq x, Y \leq y)$$

- We also have the joint PDF,

$$f(x, y) = \frac{\partial^2}{\partial x \partial y} F(x, y)$$

- We can construct the PDF of Y from the joint PDF by

$$f_Y(y) = \int_{-\infty}^{\infty} f(x, y) dx$$

Conditional expectation

- For discrete RVs X and Y, we defined

$$\begin{aligned} \mathbf{E}[X \mid Y = y] &= \sum_{x \in \mathbf{Im}(X)} x\, \mathbf{P}(X = x \mid Y = y) \\ &= \frac{\sum_{x \in \mathbf{Im}(X)} x\, \mathbf{P}(X = x, Y = y)}{\mathbf{P}(Y = y)} \end{aligned}$$

- For continuous RVs X and Y, how do we define $\mathbf{E}[X \mid Y = y]$?

- A problem arises since, in general, $\mathbf{P}(Y = y) = 0$

Conditional expectation (cont.)

- When f is continuous, define conditional expectation as

$$\mathbf{E}[X \mid Y = y] = \lim_{\epsilon \to 0} \mathbf{E}\big[X \mid Y \in [y, y+\epsilon]\big]$$

- For $\epsilon > 0$,

$$\mathbf{E}\big[X \mid Y \in [y, y+\epsilon]\big] = \frac{\int_{-\infty}^{\infty} x \int_{y}^{y+\epsilon} f(x,y) dy dx}{\int_{-\infty}^{\infty} \int_{y}^{y+\epsilon} f(x,y) dy dx}$$

- So for any $\epsilon > 0$, the conditional expectation is defined

Conditional expectation (cont.)

- We are assuming f is continuous, so for any $\delta > 0$ we have

$$(f(x,y) - \delta)\epsilon \le \int_{y}^{y+\epsilon} f(x,y) \le (f(x,y) + \delta)\epsilon$$

for all sufficiently small ϵ...

- This gives,

$$\begin{aligned} \mathbf{E}[X \mid Y = y] &= \lim_{\epsilon \to 0} \mathbf{E}\big[X \mid Y \in [y, y+\epsilon]\big] \\ &= \frac{\int_{-\infty}^{\infty} x f(x,y) dx}{f_Y(y)} \end{aligned}$$

Iterated expectation

- Suppose $X_0, \ldots, X_t$ are correlated random variables
- We want to find $\mathbf{E}[f(X_0, \ldots, X_t)]$
- We can first take the expectation over X_t, then over $X_0, \ldots, X_{t-1}$
- That is, let

$$g(x_0, \ldots, x_{t-1}) = \mathbf{E}[f(X_0, \ldots, X_t) \,|\, X_0 = x_0, \ldots, X_{t-1} = x_{t-1}]$$

- Then

$$\begin{aligned} \mathbf{E}[g(X_0, \ldots, X_{t-1})] &= \mathbf{E}\big[\mathbf{E}[f(X_0, \ldots, X_t) \,|\, X_0, \ldots, X_{t-1}]\big] \\ &= \mathbf{E}[f(X_0, \ldots, X_t)] \end{aligned}$$

Example: Iterated expectation

- Suppose X and Y are RVs both with $\mathbf{Im}(X) = \mathbf{Im}(Y) = \{1, 2\}$
- Let

$$\mathbf{P}(X = x, Y = y) = \begin{cases} 0.1 & \text{if } x = 1, y = 1 \\ 0.2 & \text{if } x = 2, y = 1 \\ 0.3 & \text{if } x = 1, y = 2 \\ 0.4 & \text{if } x = 2, y = 2 \end{cases}$$

- Here we can evaluate $\mathbf{E}[XY]$ directly as

$$\begin{aligned} \mathbf{E}[XY] &= 0.1(1) + 0.2(2) + 0.3(2) + 0.4(4) \\ &= 2.7 \end{aligned}$$

Example: Iterated expectation (cont.)

- We could also first compute $\mathbf{E}[XY \,|\, Y = y]$,

$$\mathbf{E}[XY \,|\, Y = y] = \begin{cases} \frac{1}{3}(1) + \frac{2}{3}(2) = \frac{5}{3} & \text{if } y = 1 \\ \frac{3}{7}(2) + \frac{4}{7}(4) = \frac{22}{7} & \text{if } y = 2 \end{cases}$$

- We then find

$$\begin{aligned} \mathbf{E}[XY] &= \mathbf{E}\big[\mathbf{E}[XY \,|\, Y]\big] \\ &= 0.3\left(\frac{5}{3}\right) + 0.7\left(\frac{22}{7}\right) \\ &= 2.7 \end{aligned}$$

Martingales

- A **Martingale** is a stochastic process $X_0, X_1, X_2, \ldots$ with

$$\mathbf{E}[X_{t+1} \,|\, X_t = x_t, \ldots, X_0 = x_0] = x_t$$

  for all t and $x_0, \ldots, x_t$

- More general definitions exist, but this one will be sufficient for us

- The next theorem shows $\mathbf{E}[X_t \,|\, X_0 = x_0] = x_0$...

Martingales (cont.)

- **Theorem:** $\mathbf{E}[X_t \mid X_0 = x_0] = x_0$ for all t and x_0

- **Proof:**
  - Suppose $\mathbf{E}[X_{t-1} \mid X_0 = x_0] = x_0$. Then

$$\begin{aligned}\mathbf{E}[X_t \mid X_0 = x_0] &= \mathbf{E}\big[\mathbf{E}[X_t \mid X_{t-1}, X_{t-2}, \ldots, X_0] \bigm| X_0 = x_0\big] \\ &= \mathbf{E}[X_{t-1} \mid X_0 = x_0] \\ &= x_0\end{aligned}$$

  - Since $\mathbf{E}[X_1 \mid X_0 = x_0] = x_0$, by induction for all t

$$\mathbf{E}[X_t \mid X_0 = x_0] = x_0$$

Example: Growth of a stock

- Suppose at time t the price of a stock is X_t

- Let $X_{t+1} = R_t X_t$, where $R_0, R_1 \ldots$ are independent...
- ...and $\mathbf{E}[R_t] = r$ for all t

- The R_i are not necessarily identically distributed

- In general we could have $\mathbf{Im}(R_i) = [0, \infty)$

- The process $X_0, X_1, \ldots$ is not a Martingale...

Example: Growth of a stock (cont.)

- ...however, the process $Y_t = \left(\frac{1}{r^t}\right) X_t$ is a Martingale
- To verify this,

$$\begin{aligned}
&\mathbf{E}[Y_{t+1} \,|\, Y_t = y_t, \ldots, Y_0 = y_0] \\
&\qquad = \frac{1}{r}\mathbf{E}[R_t Y_t \,|\, Y_t = y_t, \ldots, Y_0 = y_0] \\
&\qquad = \frac{y_t}{r}\mathbf{E}[R_t] \\
&\qquad = y_t
\end{aligned}$$

- Therefore, $\mathbf{E}[Y_t \,|\, Y_0 = y_0] = y_0$, or

$$\mathbf{E}[X_t \,|\, X_0 = x_0] = r^t x_0$$

since $y_0 = x_0$

Example: Random walk

- Now consider the random walk X_t, defined as

$$X_{t+1} = \begin{cases} X_t + 1 & \text{with probability } p \\ X_t - 1 & \text{with probability } 1-p \end{cases}$$

- This process is not a Martingale...

- However, the process $Y_t = X_t - (2p-1)t$ is a Martingale

Example: Random walk

- To verify that Y_t is a martingale, note that

$$Y_{t+1} = \begin{cases} Y_t + 1 - (2p-1) & \text{with probability } p \\ Y_t - 1 - (2p-1) & \text{with probability } 1-p \end{cases}$$

- Therefore,

$$\begin{aligned} &\mathbf{E}[Y_{t+1} \,|\, Y_t = y_t, \ldots, Y_0 = y_0] \\ &= \big(y_t + 1 - (2p-1)\big)p + \big(y_t - 1 - (2p-1)\big)(1-p) \\ &= y_t \end{aligned}$$

- This implies $\mathbf{E}[Y_t \,|\, Y_0 = y_0] = y_0$, or

$$\mathbf{E}[X_t \,|\, X_0 = x_0] = x_0 + (2p-1)t$$

Example: Absorbing random walk

- If $X_t > 0$, let

$$X_{t+1} = \begin{cases} X_t + 1 & \text{with probability } p \\ X_t - 1 & \text{with probability } 1-p \end{cases}$$

- But now, if $X_t = 0$, then $X_{t+1} = 0$

- That is, 0 is an absorbing boundary for this random walk

- Again, X_t is not a Martingale...

Example: Absorbing random walk

- For a sample path $X_0(\omega), X_1(\omega), \ldots$, let

$$T_0(\omega) = \inf\{t \geq 0 \mid X_t(\omega) = 0\}$$

be the hitting time to 0

- The process $Y_0, Y_1, \ldots$ defined as

$$Y_t(\omega) = X_t(\omega) - (2p-1)\min\{t, T_0(\omega)\}$$

is a Martingale

Example: Absorbing random walk

- Suppose that $p < \frac{1}{2}$, $\mathbf{E}[T_0 \mid Y_0 = y_0] < \infty$, and $\lim_{t\to\infty} \mathbf{E}[X_t \mid X_0 = 0] = 0$

- Then

$$\begin{aligned} y_0 &= \lim_{t\to\infty} \mathbf{E}[Y_t \mid Y_0 = y_0] \\ &= \lim_{t\to\infty} \mathbf{E}[X_t - (2p-1)\min\{t, T_0\} \mid Y_0 = y_0] \\ &= (1-2p)\mathbf{E}[T_0 \mid Y_0 = y_0] \end{aligned}$$

- That is, $\mathbf{E}[T_0 \mid X_0 = x_0] = \frac{x_0}{1-2p}$

- Now we'll generalize this example...

Random times

- Let T be an integer-valued random variable
- Suppose $\{\omega \in \Omega \,|\, T(\omega) = t\}$ is determined by $X_0, \ldots, X_t$...
- ...then we say that T is a **random time**
- For example, consider the hitting time

$$T_{\mathcal{S}}(\omega) = \inf\{t \geq 0 \,|\, X_t(\omega) \in \mathcal{S}\}$$

- This is a random time since

$$\{\omega \in \Omega \,|\, T_{\mathcal{S}}(\omega) = t\} = \{\omega \in \Omega \,|\, X_t(\omega) \in \mathcal{S}, X_\tau(\omega) \notin \mathcal{S} \text{ for } \tau < t\}$$

Stopping times

- If $\mathbf{P}(T < \infty) = 1$, then T is called a **stopping time**
- Suppose $X_0, X_1, \ldots$ is a Martingale and T is a stopping time
- We can define the **stopped process** $Z_t(\omega) = X_{\min\{t, T(\omega)\}}(\omega)$
- The process $Z_0, Z_1, \ldots$ is a Martingale

Stopping times (cont.)

- To show that $Z_0, Z_1, \ldots$ is a Martingale...
- If T is a stopping time and $T \leq t$, then $Z_{t+1} = Z_t$...
- ...therefore,
$$\mathbf{E}[Z_{t+1} \,|\, Z_t = z_t, \ldots, Z_0 = z_0] = z_t$$
- If $t < T$, then $X_k = Y_k$ for $k \leq t+1$...
- ...therefore,
$$\begin{aligned}\mathbf{E}[Z_{t+1} \,|\, Z_t = z_t, \ldots, Z_0 = z_0] &= \mathbf{E}[X_{t+1} \,|\, X_t = z_t, \ldots, X_0 = z_0] \\ &= z_t\end{aligned}$$

Martingale Stopping Theorem

- **Theorem:** Consider the following conditions:

  (i) There is $K < \infty$ with $|Z_t(\omega)| \leq K$ for all t and ω

  (ii) There is $K < \infty$ with $T(\omega) \leq K$ for all ω

  (iii) $\mathbf{E}[T \,|\, X_0 = x_0] < \infty$ and there is $K < \infty$ with
  $$\mathbf{E}\big[|Z_{t+1} - Z_t| \,\big|\, Z_t = z_t, \ldots, Z_0 = z_0\big] \leq K$$
  for all t and $z_0, \ldots, z_t$.

  If either (i), (ii), (iii) holds, then $\mathbf{E}[X_T \,|\, X_0 = x_0] = x_0$.

Martingale Stopping Theorem (cont.)

- **Proof idea:** (we'll only show (i) and (ii))
  - We know that $\mathbf{E}[Z_t \mid Z_0 = z_0] = z_0$ for all t
  - If condition (i) holds, then

    $$\begin{aligned} \left|\mathbf{E}[X_T \mid X_0 = x_0] - x_0\right| &= \left|\mathbf{E}[X_T \mid X_0 = x_0] - \mathbf{E}[Z_t \mid Z_0 = x_0]\right| \\ &\leq 2K\mathbf{P}(t \leq T) \end{aligned}$$

    As $t \to \infty$, we have $\mathbf{P}(t \leq T) \to 0$
  - If condition (ii) holds, then

    $$\lim_{t\to\infty} Z_t(\omega) = X_T(\omega)$$

Examples presented in class

- Some examples presented in class:
  - Wald's equation
  - Expected time for a pattern of coin flips

Example: Absorbing random walk II

- Asymmetric random walk with absorbing boundaries a and $b > a$
- If $a < X_t < b$, let

$$X_{t+1} = \begin{cases} X_t + 1 & \text{with probability } p \\ X_t - 1 & \text{with probability } 1 - p \end{cases}$$

- Let $T(\omega)$ be the time that $X_t(\omega)$ reaches an absorbing boundary
- Using Martingales, we will compute $\mathbf{P}(X_T = a)$

Example: Absorbing RW II (cont.)

- Consider the process Y_t, defined as

$$Y_t(\omega) = \left(\frac{1-p}{p}\right)^{X_t(\omega)}$$

- Y_t is a Martingale. To show this

$$\begin{aligned} \mathbf{E}[Y_{t+1} \,|\, Y_t = y_t, \ldots] &= \left(p \left(\frac{1-p}{p}\right) + (1-p) \left(\frac{1-p}{p}\right)^{-1} \right) y_t \\ &= y_t \end{aligned}$$

Example: Absorbing RW II (cont.)

- Let $K = \max\left\{\left(\frac{1-p}{p}\right)^a, \left(\frac{1-p}{p}\right)^b\right\}$.

- For this Martingale, $|Z_t(\omega)| \leq K$ for all t and ω

- Condition (i) of Stopping theorem satisfied, so

$$\mathbf{E}[Y_T \,|\, Y_0 = y_0] = y_0$$

Example: Absorbing RW II (cont.)

- We know that

$$\begin{aligned}\mathbf{E}[Y_T \,|\, Y_0 = y_0] &= \left(\frac{1-p}{p}\right)^a \mathbf{P}(X_T = a) \\ &\quad + \left(\frac{1-p}{p}\right)^b \big(1 - \mathbf{P}(X_T = a)\big) \\ &= y_0 \\ &= \left(\frac{1-p}{p}\right)^{x_0}\end{aligned}$$

- We can use this to solve for $\mathbf{P}(X_T = a)$ and $\mathbf{P}(X_T = b)$

Example: Absorbing RW II (cont.)

- We can solve for $\mathbf{P}(X_T = a)$ to find

$$\mathbf{P}(X_T = a) = \frac{\left(\frac{1-p}{p}\right)^b - \left(\frac{1-p}{p}\right)^{x_0}}{\left(\frac{1-p}{p}\right)^b - \left(\frac{1-p}{p}\right)^a}$$

- Also,

$$\mathbf{P}(X_T = b) = 1 - \mathbf{P}(X_T = a) = \frac{\left(\frac{1-p}{p}\right)^{x_0} - \left(\frac{1-p}{p}\right)^a}{\left(\frac{1-p}{p}\right)^b - \left(\frac{1-p}{p}\right)^a}$$

Example: You can't beat a fair game

- Martingales get their name from a well-known betting strategy
- **Idea:** Double your previous bet after each loss
- Popular in 18th century France
- Many people believed that this strategy was a 'sure thing'

Example: You can't beat a fair game

- Here's how it works: Start by betting \$1...
- ...if we lose, bet \$2...
- ...if we lose n times in a row, bet \$$2^n$ next...
- ...stop when we run out of money or win once

- In many plays, it's very likely that we win once...
- ...if we start with enough money, it seems sure that we'll win \$1

- However, $\mathbf{E}[X_T \mid x_0] = x_0$...
- ...on average, we won't make a profit

Submartingales and Supermartingales

- A **Submartingale** is a stochastic process $X_0, X_1, X_2, \ldots$ with

$$\mathbf{E}[X_{t+1} \mid X_t = x_t, \ldots, X_0 = x_0] \geq x_t$$

for all t and $x_0, \ldots, x_t$

- Similarly, a **Supermartingale** is a process with

$$\mathbf{E}[X_{t+1} \mid X_t = x_t, \ldots, X_0 = x_0] \leq x_t$$

for all t and $x_0, \ldots, x_t$

Sub/Supermartingales (cont.)

- Suppose $X_0, X_1, \ldots$ is a Submartingale
- Similar to Martingales, for all t we have

$$\mathbf{E}[X_t \mid X_0 = x_0] \geq x_0$$

- Suppose $X_0, X_1, \ldots$ is a Supermartingale
- Similar to Martingales, for all t we have

$$\mathbf{E}[X_t \mid X_0 = x_0] \leq x_0$$

Stopping Sub/Supermartingales

- Suppose $X_0, X_1, \ldots$ is a Submartingale and T is a stopping time
- Similar to Martingales, if (i), (ii), or (iii) on slide 26,

$$\mathbf{E}[X_T \mid X_0 = x_0] \geq x_0$$

- Suppose $X_0, X_1, \ldots$ is a Supermartingale and T is a stopping time
- Similar to Martingales, if (i), (ii), or (iii) on slide 26,

$$\mathbf{E}[X_T \mid X_0 = x_0] \leq x_0$$

Conclusion

- This is the last topic of the course

- Martingales often provide a quick method of analyzing a process...

- However, some ingenuity is often required to 'find' the Martingale