Lecture #10:
Continuous-time Markov chains

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Continuous-time Markov Chains

- Suppose $X(t)$ is a continuous-time process with finite state space

- We say $X(t)$ is a **continuous-time Markov chain** if

$$\mathbf{P}(X(t_n) = x_n \,|\, X(t_{n-1}) = x_{n-1}, \ldots, X(t_0) = x_0) = \mathbf{P}(X(t_n) = x_n \,|\, X(t_{n-1}) = x_{n-1})$$

for all $t_0, \ldots, t_n$, and all states $x_0, \ldots, x_n$

Continuous-time Markov Chains (cont.)

- We will only consider time-homogeneous MCs here
- That is,

$$\mathbf{P}(X(t+\tau) = y \,|\, X(t) = x) = \mathbf{P}(X(\tau) = y \,|\, X(0) = x)$$

for all t, τ and states x, y

- Therefore, for any $\delta t > 0$

$$\mathbf{P}(X(t+\delta t) = y) = \sum_{x \in \mathcal{X}} \mathbf{P}(X(\delta t) = y \,|\, X(0) = x)\mathbf{P}(X(t) = x)$$

Example: Simple epidemic model

- Have a population of N individuals, x of whom are infected
- State space is $\mathcal{X} = \{0, 1, 2, \ldots, N\}$
- $X(t)$ gives the number of infected individuals at time t
- Once infected, an individual stays infected
- Infected individual infects a specific susceptible by time t w.p. $1 - e^{-\lambda t}$
- The process $X(t)$ is a continuous-time MC

What we'd like to know

- How do the marginal probabilities evolve over time?
- How can we simulate such a process?
- How do we compute steady-state visitation frequencies?
- How do we compute hitting times?

Marginal probabilities

- $\mathbf{P}(X(t+\delta t)=y) = \sum_{x\in\mathcal{X}} \mathbf{P}(X(\delta t)=y \,|\, X(0)=x)\mathbf{P}(X(t)=x)$
- This implies

$$\frac{\mathbf{P}(X(t+\delta t)=y) - \mathbf{P}(X(t)=y)}{\delta t} = \sum_{x\in\mathcal{X}} \frac{1}{\delta t}\big(\mathbf{P}(X(\delta t)=y \,|\, X(0)=x) - [I]_{yx}\big)\mathbf{P}(X(t)=x)$$

- Suppose the limit exists for each $y, x \in \mathcal{X}$:

$$\lim_{\delta t\to 0} \frac{1}{\delta t}\big(\mathbf{P}(X(\delta t)=y \,|\, X(0)=x) - [I]_{yx}\big)$$

Marginal probabilities (cont.)

- We'll define the matrix Q with elements

$$[Q]_{yx} = \lim_{\delta t \to 0} \frac{1}{\delta t}\big(\mathbf{P}(X(\delta t) = y \mid X(0) = x) - [I]_{yx}\big)$$

- So

$$\lim_{\delta t \to 0} \frac{\mathbf{P}(X(t+\delta t) = y) - \mathbf{P}(X(t) = y)}{\delta t} = \sum_{x \in \mathcal{X}} [Q]_{yx} \mathbf{P}(X(t) = x)$$

- We will let $\rho(t)$ denote the vector with elements

$$[\rho(t)]_x = \mathbf{P}(X(t) = x)$$

Matrix differential equations

- The marginal probabilities satisfy the matrix differential equation

$$\frac{d}{dt}\rho(t) = Q\rho(t)$$

- This means for each $y \in \mathcal{X}$,

$$\frac{d}{dt}[\rho(t)]_y = \sum_{x \in \mathcal{X}} [Q]_{yx}[\rho(t)]_x$$

- System of coupled linear, constant coefficient differential eqns.

- For scalar a, solution to $\frac{d}{dt}y(t) = ay(t)$ is $y(t) = e^{at}y(0)$

Matrix exponentials

- The marginal probabilities satisfy the matrix differential equation

$$\frac{d}{dt}\rho(t) = Q\rho(t)$$

- This differential equation has the solution

$$\rho(t) = (e^{tQ})\rho(0)$$

- The **matrix exponential** is defined as

$$e^{tQ} = \sum_{k=0}^{\infty} \frac{(tQ)^k}{k!}$$

Matrix exponentials (cont.)

- Taking the derivative of the matrix exponential gives

$$\begin{aligned}\frac{d}{dt}e^{tQ} &= \sum_{k=0}^{\infty} \frac{Q^k}{k!}\frac{d}{dt}t^k \\ &= \sum_{k=0}^{\infty} \frac{Q^k}{k!} k t^{k-1} \\ &= \sum_{k=1}^{\infty} \frac{Q^k}{(k-1)!} t^{k-1} \\ &= Qe^{tQ}\end{aligned}$$

- Therefore, $\frac{d}{dt}e^{tQ}\rho(0) = Qe^{tQ}\rho(0)$

Matrix exponentials in Matlab

- In Matlab, evaluate the matrix exponential with `expm(t*Q)`
- For example, consider the rate matrix

$$Q = \begin{bmatrix} -2 & 5 \\ 2 & -5 \end{bmatrix}$$

- Can evaluate `expm(Q)` to get

$$e^Q \approx \begin{bmatrix} 0.715 & 0.714 \\ 0.285 & 0.286 \end{bmatrix}$$

- Notice that this matrix is stochastic. Why?

Properties of the matrix exponential

- For matrices Q_1 and Q_2, if $Q_1 Q_2 = Q_2 Q_1$...
- ...then $e^{tQ_1} e^{tQ_2} = e^{t(Q_1+Q_2)}$
- **Proof idea:**
  - If $Q_1 Q_2 = Q_2 Q_1$, then $Q_2 e^{tQ_1} = e^{tQ_1} Q_2$
  - Then

$$\begin{aligned} \frac{d}{dt} e^{tQ_1} e^{tQ_2} &= \left(\frac{d}{dt} e^{tQ_1} \right) e^{tQ_2} + e^{tQ_1} \left(\frac{d}{dt} e^{tQ_2} \right) \\ &= (Q_1 + Q_2) e^{tQ_1} e^{tQ_2} \end{aligned}$$

  - $\rho(t) = e^{tQ_1} e^{tQ_2}$ satisfies $\frac{d}{dt}\rho(t) = (Q_1 + Q_2)\rho(t)$, so

$$\rho(t) = e^{t(Q_1+Q_2)}$$

Properties of Q

- The matrix Q satisfies the properties
  1. $[Q]_{xx} \leq 0$ for all $x \in \mathcal{X}$
  2. $[Q]_{xy} \geq 0$ for all $x \neq y \in \mathcal{X}$
  3. $Q^T \mathbf{1} = 0$

- To show property 1:
  1. For any δt,

$$\mathbf{P}(X(\delta t) = x \,|\, X(0) = x) - [I]_{xx} \leq 0$$

Therefore,

$$\begin{aligned} [Q]_{xx} &= \lim_{\delta t \to 0} \left(\mathbf{P}(X(\delta t) = x \,|\, X(0) = x) - [I]_{xx}\right) \\ &\leq 0 \end{aligned}$$

Properties of Q (cont.)

- To show properties 2 & 3:

  2. For any δt and $y \neq x$, $\mathbf{P}(X(\delta t) = y \,|\, X(0) = x) - [I]_{yx} \geq 0$
  Therefore,

$$[Q]_{yx} = \lim_{\delta t \to 0} \big(\mathbf{P}(X(\delta t) = y \,|\, X(0) = x) - [I]_{yx}\big) \geq 0$$

  3. For any δt, $\sum_{y \in \mathcal{X}} \big(\mathbf{P}(X(\delta t) = y \,|\, X(0) = x) - [I]_{yx}\big) = 0$
  Therefore,

$$\sum_{y \in \mathcal{X}} [Q]_{yx} = \lim_{\delta t \to 0} \sum_{y \in \mathcal{X}} \big(\mathbf{P}(X(\delta t) = y \,|\, X(0) = x) - [I]_{yx}\big) = 0$$

Discretizing the continuous-time MC

- Suppose we know $\mathbf{P}(X(\delta t) = y \,|\, X(0) = x)$ for some δt and all x, y...
- ...then we can easily find

$$\mathbf{P}\big(X\big((\delta t)k\big) = y \,|\, X(0) = x\big)$$

for all x, y and integers k

- **Idea:** The matrix $P_{\delta t} = e^{\delta t Q}$ is stochastic and

$$[P_{\delta t}^k]_{yx} = \mathbf{P}\big(X\big((\delta t)k\big) = y \,|\, X(0) = x\big)$$

for all integers k

Discretizing the CT MC (cont.)

- From the definition of Q, we know that

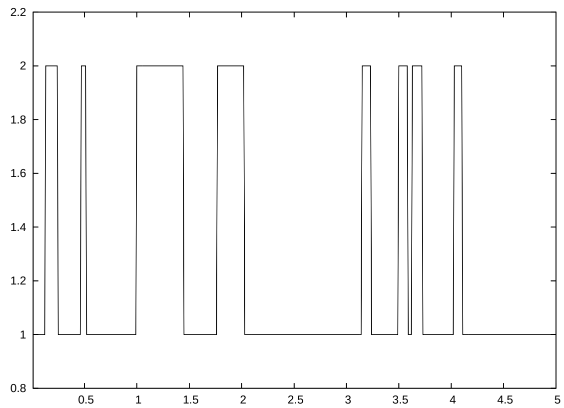
$$\begin{aligned} [P_{\delta t}]_{yx} &= [e^{(\delta t)Q}]_{yx} \\ &= \mathbf{P}\big(X(\delta t) = y \,|\, X(0) = x\big) \end{aligned}$$

- Suppose $P_{\delta t}^k = e^{(\delta t)kQ}$ for some k...
- ...then

$$\begin{aligned} P_{\delta t}^{k+1} &= e^{(\delta t)Q} e^{(\delta t)kQ} \\ &= e^{(\delta t)Q + (\delta t)kQ} \\ &= e^{(\delta t)(k+1)Q} \end{aligned}$$

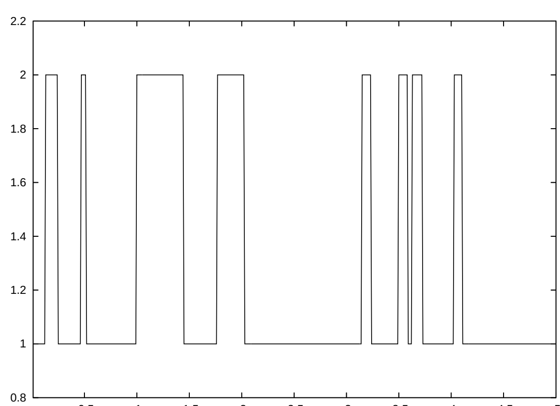
Evolution of the continuous-time MC

- So far, we've seen how $P(X(t) = x)$ evolves over time...
- ...how does $X(t)$ evolve over time?
- For example, consider the system with two states and $Q = \begin{bmatrix} -2 & 5 \\ 2 & -5 \end{bmatrix}$
- A sample path for this system looks like



Evolution of the CT MC (cont.)

- If $X(t) = x$, $X(t+\tau) = x$ for some random amount of time...
- ...then $X(t+\tau)$ will 'jump' to a new state
- When do we jump, and which state do we jump to?



Evolution of the CT MC (cont.)

- There are two ways to interpret the Q matrix
- This first is often most useful when simulating a model
- This second is often most useful when constructing a model

Interpretation I: Holding time and Jump chain

- In the first interpretation:
  - If in state x, wait an exponentially distributed amount of time...
  - ...when we move, transition according to a discrete time MC
- The time we wait is called the 'holding time'
- The discrete time MC dictating jumps is called the 'jump chain'
- This gives a simple method of simulation, as we'll see...

Holding times

- What is the probability that $X(t+s) = x$ for all $s \le \tau$?
- Let

$$z(t) = \mathbf{P}(X((\delta t)k) = x \text{ for all integers } k \text{ with } (\delta t)k \le t)$$

- Then

$$z(t+\delta t) = [e^{\delta t Q}]_{xx} z(t)$$

- This implies

$$\frac{z(t+\delta t) - z(t)}{\delta t} = \frac{1}{\delta t}\big([e^{\delta t Q}]_{xx} - 1\big) z(t)$$

Holding times (cont.)

- The matrix exponential satisfies

$$\begin{aligned} [e^{\delta t Q}]_{xx} &= \left[\sum_{k=0}^{\infty} \frac{(\delta t Q)^k}{k!}\right]_{xx} \\ &= 1 + (\delta t)[Q]_{xx} + O((\delta t)^2) \end{aligned}$$

- Therefore,

$$\begin{aligned} \lim_{\delta t \to 0} \frac{1}{\delta t}\big([e^{\delta t Q}]_{xx} - 1\big) &= \lim_{\delta t \to 0} \big([Q]_{xx} + \frac{1}{\delta t} O((\delta t)^2)\big) \\ &= [Q]_{xx} \end{aligned}$$

- We'll look at the following differential equation for the holding times

$$\frac{d}{dt} z(t) = [Q]_{xx} z(t)$$

Holding times (cont.)

- Consider the differential equation

$$\frac{d}{dt}z(t) = [Q]_{xx}z(t)$$

- This equation has the solution $z(t) = e^{t[Q]_{xx}}z(0)$
- Since $X(0) = x$, by definition of z we have $z(0) = 1$
- Also, $[Q]_{xx} \leq 0$, so

$$\mathbf{P}(X(s) = x \text{ for all } s \leq t) = e^{t[Q]_{xx}}$$

is exponentially distributed

The Jump Chain

- We're in each state for an exponentially distributed amount of time
- When we do switch states, where do we go?
- Consider

$$\mathbf{P}(X(t+\delta t) = y \,|\, X(t+\delta t) \neq x, X(t) = x)$$

- What happens as $\delta t \to 0$?

The Jump Chain (cont.)

- For each $x \neq y$, $\mathbf{P}(X(t+\delta t) = y \,|\, X(t) = x) = [e^{(\delta t)Q}]_{yx}$

- Therefore,

$$\mathbf{P}(X(t+\delta t) = y \,|\, X(t+\delta t) \neq x, X(t) = x) = \frac{[e^{(\delta t)Q}]_{yx}}{\sum_{z \neq x}[e^{(\delta t)Q}]_{zx}}$$

- For $z \neq x$, $[e^{(\delta t)Q}]_{zx} = (\delta t)[Q]_{zx} + O\big((\delta t)^2\big)$

The Jump Chain (cont.)

- Since $[e^{(\delta t)Q}]_{zx} = (\delta t)[Q]_{zx} + O\big((\delta t)^2\big)$,

$$\begin{aligned} \mathbf{P}(X(t+\delta t) = y \,|\, X(t+\delta t) \neq x, X(t) = x) &= \frac{[e^{(\delta t)Q}]_{yx}}{\sum_{z \neq x}[e^{(\delta t)Q}]_{zx}} \\ &= \frac{[Q]_{yx} + O\big(\delta t\big)}{\sum_{z \neq x}[Q]_{zx} + O\big(\delta t\big)} \end{aligned}$$

- Therefore,

$$\lim_{\delta t \to 0} \mathbf{P}(X(t+\delta t) = y \,|\, X(t+\delta t) \neq x, X(t) = x) = \frac{[Q]_{yx}}{\sum_{z \neq x}[Q]_{zx}}$$

The Jump Chain (cont.)

- We transition from x to y with probability,

$$\frac{[Q]_{yx}}{\sum_{z \neq x} [Q]_{zx}}$$

- Transition probabilities are only defined when $\sum_{z \neq x} [Q]_{zx} > 0$
- What happens when $\sum_{z \neq x} [Q]_{zx} = 0$?
- In this case, $[Q]_{xx} = 0$, so holding time at x is infinite

Simulating a Continuous-time MC

- The previous discussion tells us how to simulate a continuous-time MC
- Simulation algorithm:
  1. To start, pick a random state according to $\rho(0)$
  2. If we moved to state x at time t, sample T according to

$$\mathbf{P}(T \geq \tau) = e^{\tau [Q]_{xx}}$$

  3. Sample state y with probability

$$\frac{[Q]_{yx}}{\sum_{z \neq x} [Q]_{zx}}$$

  4. Move from state x to state y at time $t + T$, then repeat.

Sampling an exponential RV

- In the algorithm above, how do we sample T according to

$$\mathbf{P}(T \geq \tau) = e^{\tau [Q]_{xx}} \,?$$

- Suppose you can sample a uniform random variable U on $[0, 1]$...
- ..for example, using `rand(1)` in Matlab
- If we let $T = \frac{\ln(U)}{[Q]_{xx}}$, then

$$\begin{aligned} \mathbf{P}(T \geq \tau) &= \mathbf{P}\left(\frac{\ln(U)}{[Q]_{xx}} \geq \tau\right) \\ &= \mathbf{P}(U \leq e^{\tau [Q]_{xx}}) \\ &= e^{\tau [Q]_{xx}} \end{aligned}$$

Example: Simple epidemic model

- At time t, say $X(t) = x$ individuals are infected
- Time T_{ij} for individual i to infect a individual j satisfies

$$\mathbf{P}(T_{ij} \geq \tau) = e^{-\lambda \tau}$$

- Time for susceptible individual j to be infected satisfies

$$\mathbf{P}\left(\min_i \{T_{ij}\} \geq \tau\right) = e^{-\lambda x \tau}$$

- Time when next susceptible individual is infected satisfies

$$\mathbf{P}\left(\min_j \left\{\min_i \{T_{ij}\}\right\} \geq \tau\right) = e^{-\lambda (N-x) x \tau}$$

Example: Simple epidemic model

- When in state $X(t) = x$, holding time satisfies

$$\mathbf{P}(X(t+s) = x \text{ for all } 0 \leq s \leq \tau) = e^{-\lambda(N-x)x\tau}$$

- When we transition, we move to state $x + 1$

- So, the Q matrix is

$$[Q]_{yx} = \begin{cases} -\lambda(N-x)x & \text{if } y = x \\ \lambda(N-x)x & \text{if } y = x+1 \\ 0 & \text{otherwise} \end{cases}$$

Interpretation II: Poisson race

- In the second interpretation:
  - We have independent Poisson processes for each state
  - We transition to the state with the first arrival

- Often useful for constructing a model...

- ...we'll see an example of this

Poisson race

- Consider m independent Poisson processes $Y_1(t), \ldots, Y_m(t)$
- These processes have rates $\lambda_1, \ldots, \lambda_m$
- How long do we wait for the first arrival?
- The probability of no arrival from process i by time t is

$$\mathbf{P}(Y_i(t) = 0) = e^{-\lambda_i t}$$

- The probability of no arrival from any process by time t is

$$\begin{aligned} \mathbf{P}(Y_1(t) = 0, \ldots, Y_m(t) = 0) &= e^{-\lambda_1 t} \cdots e^{-\lambda_m t} \\ &= e^{-(\lambda_1 + \cdots \lambda_m)t} \end{aligned}$$

Poisson race (cont.)

- Which Poisson process has the first arrival?
- Process i has first arrival with probability

$$\begin{aligned} \mathbf{P}\left(Y_i(t) = 1 \,\middle|\, \sum_{j=1}^m Y_j(t) = 1\right) &= \frac{\mathbf{P}(Y_i(t) = 1, Y_j(t) = 0 \text{ for } j \neq i)}{\mathbf{P}\left(\sum_{j=1}^m Y_j(t) = 1\right)} \\ &= \frac{\left(\frac{t\lambda_i}{1!} e^{-\lambda_i t}\right) \prod_{j \neq i} \left(\frac{1}{0!} e^{-\lambda_j t}\right)}{\frac{t(\lambda_1 + \cdots + \lambda_m)}{1!} e^{-(\lambda_1 + \cdots + \lambda_m)t}} \\ &= \frac{t\lambda_i e^{-(\lambda_1 + \cdots + \lambda_m)t}}{t(\lambda_1 + \cdots + \lambda_m) e^{-(\lambda_1 + \cdots + \lambda_m)t}} \\ &= \frac{\lambda_i}{\lambda_1 + \cdots + \lambda_m} \end{aligned}$$

Poisson transitions

- Suppose $X(t) = x$
- For all $y \neq x$ we have a Poisson process with rate $[Q]_{yx}$
- The time until the first arrival, T, satisfies

$$\begin{aligned} \mathbf{P}(T \geq \tau) &= e^{-\tau\left(\sum_{y \neq x}[Q]_{yx}\right)} \\ &= e^{\tau [Q]_{xx}} \end{aligned}$$

- The probability that state y has first arrival is

$$\frac{[Q]_{yx}}{\sum_{z \neq x}[Q]_{zx}}$$

Poisson transitions (cont.)

- Suppose we jump to the state with the first arrival
- The time until first arrival is the holding time
- Probability of jumping to y is the same as the jump chain

Example: Netflix

- Online DVD rental service has N copies of a movie
- Requests for that movie arrive as a Poisson process with rate λ
- A DVD is checked out an exponentially distributed time with rate μ
- Check out times are independent
- If the movie is out of stock, requests are denied

Example: Netflix (cont.)

- Let $X(t)$ denote the number of copies rented at time t
- The state space is $\mathcal{X} = \{0, \ldots, N\}$
- In state $x = 0$, we can only transition to $x + 1$
- In state $x = N$, we can only transition to $x - 1$
- In any state $x \in \{1, \ldots, N - 1\}$, we move to either $x + 1$ or $x - 1$
- What is the Q matrix for this process?

Example: Netflix (cont.)

- Suppose we are in state $x = 0$
- We move to $x = 1$ when next request arrives
- Requests arrive with rate λ, so $[Q]_{1,0} = \lambda$

Example: Netflix (cont.)

- Now suppose we are in state $x = N$
- We move to $x = N - 1$ when the first checked-out DVD is returned
- Check-out times are independent and exponential with rate μ
- Waiting time for first return is exponentially distributed with rate μN
- So, $[Q]_{N-1,N} = \mu N$

Example: Netflix (cont.)

- Suppose we are in state $x \in \{1, \dots, N-1\}$
- We have Poisson processes for states $x+1$ or $x-1$
- Arrivals occur with rate λ, so $[Q]_{x+1,x} = \lambda$
- First return time is exponentially distributed with rate μx
- ...therefore, $[Q]_{x-1,x} = \mu x$

Steady-state visitation frequency

- What is the fraction of time spent in a particular state?
- As before, define the indicator for state y as

$$r(z) = \begin{cases} 1 & \text{if } z = y \\ 0 & \text{otherwise} \end{cases}$$

- Over t time units, the fraction of time spent in state y is

$$\frac{1}{t}\int_0^t r(X(\tau))d\tau$$

Steady-state visitation frequency (cont.)

- Over t time units, expected fraction of time in y starting from x is

$$\mathbf{E}\left[\frac{1}{t}\int_0^t r(X(\tau))d\tau \,\middle|\, X(0)=x\right] = \frac{1}{t}\int_0^t \mathbf{E}[r(X(\tau))\,|\,X(0)=x]d\tau$$

- Steady-state expected fraction of time in y starting x is

$$[\beta]_x = \lim_{t\to\infty}\frac{1}{t}\int_0^t \mathbf{E}[r(X(\tau))\,|\,X(0)=x]d\tau$$

- As before, write this in matrix-vector notation as

$$\beta = \lim_{t\to\infty}\frac{1}{t}\int_0^t \left(e^{\tau Q}\right)^T r d\tau$$

Steady-state visitation frequency (cont.)

- We also have a continuous-time version of Poisson's equation

- We can compute β by solving

$$\begin{bmatrix} Q^T & I \\ \mathbf{0} & Q^T \end{bmatrix}\begin{bmatrix} h \\ \beta \end{bmatrix} = \begin{bmatrix} r \\ \mathbf{0} \end{bmatrix}$$

- Can solve in Matlab using `pinv`

Steady-state visitation frequency (cont.)

- Poisson's equation is equivalent to $Q^T\beta = \mathbf{0}$ and $\beta = r - Q^T h$

- If $Q^T\beta = \mathbf{0}$, then for any t,

$$\begin{aligned}\frac{1}{t}\int_0^t (e^{\tau Q})^T \beta d\tau &= \frac{1}{t}\int_0^t \sum_{k=0} \frac{\tau^k}{k!}(Q^T)^k \beta d\tau \\ &= \frac{1}{t}\int_0^t \beta d\tau \\ &= \beta\end{aligned}$$

Steady-state visitation frequency (cont.)

- If $Q^T\beta = \mathbf{0}$ and $\beta = r - Q^T h$, then for all τ,

$$(e^{\tau Q})^T \beta = (e^{\tau Q})^T r - (e^{\tau Q})^T Q^T h$$

and for all t,

$$\begin{aligned}\beta &= \frac{1}{t}\int_0^t (e^{\tau Q})^T \beta d\tau \\ &= \frac{1}{t}\int_0^t (e^{\tau Q})^T r d\tau - \frac{1}{t}\int_0^t (e^{\tau Q})^T Q^T h d\tau\end{aligned}$$

- Therefore,

$$\beta = \lim_{t\to\infty} \frac{1}{t}\int_0^t (e^{\tau Q})^T r d\tau - \lim_{t\to\infty} \frac{1}{t}\int_0^t (e^{\tau Q})^T Q^T h d\tau$$

Steady-state visitation frequency (cont.)

- Note that $\frac{d}{d\tau}\left(e^{\tau Q}\right)^T = \left(e^{\tau Q}\right)^T Q^T$
- Therefore,

$$\begin{aligned}\frac{1}{t}\int_0^t \left(e^{\tau Q}\right)^T Q^T h d\tau &= \frac{1}{t}\int_0^t \frac{d}{d\tau}\left(e^{\tau Q}\right)^T h d\tau \\ &= \frac{1}{t}\left(\left(e^{tQ}\right)^T - I\right)h\end{aligned}$$

- Since $\lim_{t\to\infty} \frac{1}{t}\left(\left(e^{tQ}\right)^T - I\right)h = \mathbf{0}$,

$$\beta = \lim_{t\to\infty}\frac{1}{t}\int_0^t \left(e^{\tau Q}\right)^T r d\tau$$

Example: Poisson's equation

- Find the fraction of time spent in state 1 when

$$Q = \begin{bmatrix} -2 & 5 \\ 2 & -5 \end{bmatrix}$$

- We can solve

$$\begin{bmatrix} -2 & 2 & 1 & 0 \\ 5 & -5 & 0 & 1 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & 5 & -5 \end{bmatrix} \begin{bmatrix} [h]_1 \\ [h]_2 \\ [\beta]_1 \\ [\beta]_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- This gives

$$h = \begin{bmatrix} 0 \\ \frac{1}{7} \end{bmatrix} \qquad \beta = \begin{bmatrix} \frac{5}{7} \\ \frac{5}{7} \end{bmatrix}$$

Communicating states

- As before, state x **leads to** state y if we can reach y from x
- We denote this as $x \longrightarrow y$
- For $x \neq y$, $x \longrightarrow y$ if there are $x_1, \ldots, x_{k-1}$ with

$$[Q]_{y,x_{k-1}}[Q]_{x_{k-1},x_{k-2}} \cdots [Q]_{x_1,x} > 0$$

- x and y **communicate**, denoted $x \leftrightarrow y$, if $x \longrightarrow y$ and $y \longrightarrow x$

Communicating states (cont.)

- If $x \leftrightarrow y$ for all $x, y \in \mathcal{X}$...
- ...then Q has a unique eigenvalue equal to zero
- For any α, we know that $Q^T(\alpha\mathbf{1}) = \mathbf{0}$
- If $x \leftrightarrow y$ for all $x, y \in \mathcal{X}$, Poisson's equation has $\beta = \alpha\mathbf{1}$

Absorbing states

- If $[Q]_{xx} < 0$, we eventually leave x with probability 1...
- ...that is

$$\begin{aligned}\mathbf{P}(X(t) = x \text{ for all } t) &= \lim_{\tau\to\infty} e^{\tau[Q]_{xx}} \\ &= 0\end{aligned}$$

- If $[Q]_{xx} = 0$, we stay in x with probability 1...
- ...that is

$$\begin{aligned}\mathbf{P}(X(t) = x \text{ for all } t) &= \lim_{\tau\to\infty} e^{\tau[Q]_{xx}} \\ &= 1\end{aligned}$$

- A state x is **absorbing** if and only if $[Q]_{xx} = 0$

Hitting times

- How long does it take to reach a group of absorbing states $\mathcal{S}$?
- As before, define the function

$$g(x) = \begin{cases} 1 \text{ if } z \notin \mathcal{S} \\ 0 \text{ otherwise} \end{cases}$$

- The hitting time to the set $\mathcal{S}$ is

$$\int_0^\infty g(X(\tau))d\tau$$

- The hitting time to the set $\mathcal{S}$ from state x is

$$[\gamma]_x = \int_0^\infty \mathbf{E}[g(X(\tau)) \mid X(0) = x]d\tau$$

Hitting times (cont.)

- In terms of g, expected hitting time from each state is

$$\gamma = \int_0^\infty \left(e^{\tau Q}\right)^T g d\tau$$

- We have

$$\begin{aligned} Q^T \int_0^t \left(e^{\tau Q}\right)^T g d\tau &= \int_0^t \frac{d}{d\tau}\left(e^{\tau Q}\right)^T g d\tau \\ &= \left(e^{tQ}\right)^T g - g \end{aligned}$$

- As long as $\lim_{t\to\infty}\left(e^{tQ}\right)^T g = 0$, we have $g + Q^T\gamma = \mathbf{0}$.

Hitting times (cont.)

- As before, we have $[\gamma]_x = 0$ for all $x \in \mathcal{S}$
- Want to find γ with $[\gamma]_x = 0$ for all $x \in \mathcal{S}$ satisfying

$$g + Q^T\gamma = \mathbf{0}$$

- Suppose we arrange $\mathcal{X}$ so that $\mathcal{S}$ is the first m states of $\mathcal{X}$
- Let Q_{T2} be the submatrix of Q corresponding to states not in $\mathcal{S}$
- If Q_{T2}^{-1} exists, then $\gamma = \begin{bmatrix} \mathbf{0} \\ -\left(Q_{T2}^{-1}\right)^T \mathbf{1} \end{bmatrix}$

Example: Hitting time

- Suppose we use $\mathcal{S} = \{1, 2\}$ and

$$Q = \begin{bmatrix} 0 & 0 & 0.1 & 0.4 \\ 0 & 0 & 0.2 & 0.1 \\ 0 & 0 & -0.6 & 0.3 \\ 0 & 0 & 0.3 & -0.8 \end{bmatrix}$$

- The submatrix of transient states is $Q_{T2} = \begin{bmatrix} -0.6 & 0.3 \\ 0.3 & -0.8 \end{bmatrix}$
- We can compute

$$\gamma = \begin{bmatrix} \mathbf{0} \\ -(Q_{T2}^{-1})^T \mathbf{1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2.82 \\ 2.31 \end{bmatrix}$$

Example: Simple epidemic model

- Starting with x infected individuals...
- ...find expected time until all individuals are infected

- To start, we know that $[\gamma]_N = 0$

- Remaining equations for $x \geq 1$ are

$$1 + \lambda(N - x)x([\gamma]_{x+1} - [\gamma]_x) = 0$$

Example: Simple epidemic model

- We can rewrite equations as

$$[\gamma]_x = [\gamma]_{x+1} + \frac{1}{\lambda(N-x)x}$$

- We can easily solve these equations to get

$$[\gamma]_x = \frac{1}{\lambda}\sum_{i=1}^{N-x}\frac{1}{(N-i)i}$$

for all $x \geq 1$