Event-Based Control Using Quadratic Approximate Value Functions

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Abstract— **In this paper we consider several problems involving control with limited actuation and sampling rates. Event-based control has emerged as an attractive approach for addressing the problems of control system design under rate limitations. In event-based control, a system is actuated or a control signal is changed only when certain events occur. For example, a control signal might be applied only when some measure of deviation of the system state from equilibrium is exceeded. Thus, control action is only applied when it is needed, keeping control performance satisfactory while reducing the rate that the system must be sensed and actuated.**

In principle, the problem of determining how to optimally schedule the sensing or actuation of a system can be cast as a Markov decision process. However, the optimal value function for these Markov decision processes generally does not have a simple structure. So, determining a closed-form expression or simple parametrization of the optimal value function is generally not possible.

In this paper we develop new computational methods for event-based control. Under a given policy, one can obtain an upper bound on control performance using an approximate value function for the associated Markov decision process. We will consider performance bounds that can be obtained using quadratic approximate value functions. We will find the policy that minimizes the upper bound obtainable over all possible quadratic approximate value functions. This policy and the associated performance bound can be obtained by solving a sequence of semidefinite programs indexed by a scalar parameter.

I. Introduction

The recent growth in the development and deployment of data networks has transformed the way we disseminate and acquire information. Further, the commoditization and miniaturization of computing, sensing, and communication devices will provide the technological basis for systems where physical resources, such as mechanical or biological components, are tightly coupled with communication and computational resources.

Proposed and existing examples of such systems are found in a wide variety of technological areas, including power generation systems, transportation networks, health care, aerospace and avionics, energy efficient operation of vehicles, and manufacturing. In each of these systems, physical components are sensed and manipulated. It is often the case that physical components are controlled with the goal of tracking some desired trajectory, or keeping deviation from some operating point small. It initially seems that regulation of physical components could be addressed using the tools of conventional feedback control theory. However, this approach is complicated by the fact that any control system must operate within the limitations of the communication and computational resources present in the system. That is, these computational and communication resources impose limits on the rate that physical quantities can be sampled, the rate that power can be consumed, rates at which samples can be processed, and rates at which sampled measurements can be transmitted among components in the system. One of the fundamental challenges in designing control systems that contain computational and communication resources lies in the design of methods that can offer satisfactory regulation of physical components, all while limiting the sampling, transmission, and actuation rates.

Event-based control has emerged as an attractive approach for addressing the problem of control system design under rate limitations [4], [21], [3], [28], [45], [22], [57], [13], [14], [52], [35], [55], [5]. In event-based control, a system is actuated or a control signal is changed only when certain events occur. For example, a control signal might be applied only when some measure of deviation of the system state from equilibrium is exceeded. Thus, control action is only applied when it is needed, keeping control performance satisfactory while reducing the rate that the system must be sensed and actuated. While this sounds simple in principle, currently little theory exists for the design of optimal event-based control and estimation strategies [5]. Additionally, much of the work on this topic has focused on single-input-single-output (SISO) systems.

The problem of determining how to optimally schedule the sensing or actuation of a system can often be cast as a Markov decision process. However, the optimal value function for these Markov decision processes generally does not have a simple structure. So, determining a closed-form expression or simple parametrization of the optimal value function is generally not possible. This has led researchers to consider numerical approaches that require discretization of state-space of the physical system model [57]. While this works well for models with a one-dimensional state, the number of grid points that must be used in such a discretization scales exponentially with the dimension of the state.

In this paper we establish new computational methods for event-based control and estimation. Our approach is based on the use of approximate value functions for the underlying Markov decision processes. Using simple parametrizations of approximate value functions, we will be able to obtain event-based sampling and control strategies that minimize

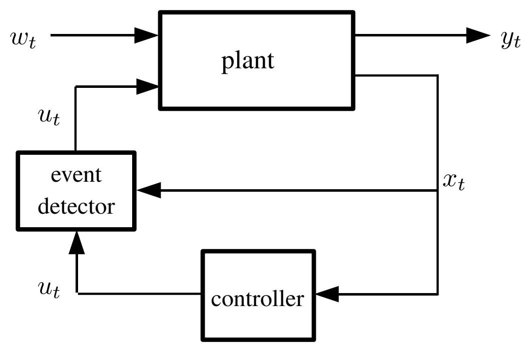

Fig. 1. Event-based control

a class of upper bounds on system performance. Also, most importantly, these methods can be used to efficiently compute strategies for systems with state-spaces of high dimension.

II. Problem Statement

In the first part of this paper we will consider a simple architecture for regulating the state of a system. Consider a system model with state $x_t \in \mathbb{R}^n$ and control actions $u_t \in \mathbb{R}^m$ and $a_t \in \{0, 1\}$. The state evolves as

$$x_{t+1} = Ax_t + a_t B u_t + w_t.$$

Here, w_t is an IID zero mean Gaussian disturbance signal with covariance matrix Σ_w. Our goal is to apply limited actuation to the system while maintaining satisfactory control performance. At each time step, we make a decision of whether to actuate the system. The decision of whether to actuate is represented by the control variable $a_t \in \{0, 1\}$. When the system is actuated, a control vector u_t is applied.

Our goal of balancing a trade-off between control performance and actuation rate is modeled by a cost of

$$x_t^T Q x_t + \lambda a_t$$

incurred in each time step. Here $x_t^T Q x_t$ is a quadratic penalty on the size of the state and λa_t is a penalty for actuation. The goal is to determine a policy for selecting a_t and u_t that minimizes

$$\limsup_{t\to\infty} \frac{1}{t} \sum_{k=0}^{t-1} \mathbf{E}[x_k^T Q x_k + \lambda a_k] \tag{1}$$

A block diagram representing the architecture considered in this problem is shown in Figure 1. In this system, an event detector monitors the state of the plant. When a state-dependent event occurs, a control signal is received from a controller and is applied to the plant. Using a given state feedback control gain K, the problem of determining when to actuate the system reduces to design of an optimal rule for choosing a_t to minimize (1) subject to the dynamics

$$x_{t+1} = (A + a_t BK)x_t + w_t.$$

In our upcoming discussion we will use $A_1 = A$ to denote the open-loop system matrix and use $A_2 = A + BK$ to denote the closed loop system matrix. Using this notation, the state evolves as

$$x_{t+1} = \big((1 - a_t)A_1 + a_t A_2\big)x_t + w_t$$

Note that the problem of choosing a_t to minimize (1) is a Markov decision process with an average cost criterion [2]. That is, for any state-feedback policy $\mu : \mathbb{R}^n \to \{0, 1\}$ that chooses a_t given x_t, the state x_t evolves as a Markov chain. Also (1) gives a measure of the average cost incurred in each time period. Since the event-based control problem can be viewed as an average cost Markov decision process, in principle one can find an optimal value function by solving Bellman's equation and extract an optimal policy from the value function. However, even in what appear to be very simple cases, the optimal value function for this Markov decision process does not exhibit any special structure. To approach the problem of computing an optimal actuation policy, numerical approaches that discretize the state-space can be used. Unfortunately, this approach can only be applied to problems where the state space has low dimension since the number of grid points required to discretize a region of the state space scales exponentially with the dimension of the state space.

In this paper we take a different approach. Under a given control policy, one can obtain an upper bound on control performance using an approximate value function for this Markov decision process. We will consider performance bounds that can be obtained when using quadratic approximate value functions. We will find the policy that minimizes the upper bound obtainable over all possible quadratic approximate value functions. This policy and the associated performance bound can be obtained by solving a sequence of semidefinite programs indexed by a scalar parameter.

III. Main Results

In this section we show how to compute an event-based control strategy that minimizes a particular upper bound on the average per-period cost. The main tool that is used in this paper to determine upper bounds is the following Lemma. This Lemma is a consequence of Lyapunov's theorem. A proof can be found in [12] and references therein.

Lemma 1. *Suppose $x_0, x_1, \ldots$ is a Markov chain with state space $\mathcal{X}$. Suppose $c\colon \mathcal{X} \to \mathbb{R}$ and $h\colon \mathcal{X} \to \mathbb{R}$. Define*

$$J = \limsup_{t\to\infty} \frac{1}{t} \sum_{k=0}^{t-1} \mathbf{E}[c(x_k)].$$

If $h(x) \geq 0$ for all $x \in \mathcal{X}$, then

$$J \leq \sup_{x\in\mathcal{X}} \left\{ c(x) + \mathbf{E}[h(x_{t+1}) \,|\, x_t = x] - h(x) \right\}.$$

For the event-based control problem discussed in the previous section, we will use this Lemma with quadratic approximate value functions. That is, we will consider functions h of the form $h(x) = x^T Y x$. For such an h to be globally nonnegative we need $Y \succeq 0$. So, we will consider the problem of jointly choosing a policy and a specific quadratic to minimize an upper bound on average per-period cost.

We will first consider the problem where control signals are generated by a given linear state-feedback control law $u_t = Kx_t$. We will then consider the problem of jointly choosing an actuation schedule and the control signals u_t to minimize an upper bound on average per-period cost.

Our first theorem characterizes the optimal achievable upper bound, together with the associated policy, in terms of the solution to a static optimization problem.

Theorem 1. *Consider all upper bounds on average per-period cost that use Lemma 1 with a quadratic function h. The minimum such upper bound is the minimum objective value of the optimization problem*

$$\begin{array}{rl} \textit{min:} & \mathbf{trace}(\Sigma_w Y) + \rho\lambda \\ \textit{s.t.:} & Y \succeq Q + (1-\rho)A_1^T Y A_1 + \rho A_2^T Y A_2 \\ & Y \succeq 0 \\ & \rho \leq 1 \\ & \rho \geq 0 \end{array}$$

Moreover, the policy associated with this upper bound uses

$$a = \begin{cases} 0 & \textit{if } x_t^T \left(A_1^T Y A_1 - A_2^T Y A_2\right) x_t \leq \lambda \\ 1 & \textit{otherwise} \end{cases}$$

Proof. We will start by considering the drift

$$c(x,a) + \mathbf{E}[h(x_{t+1}) \,|\, x_t = x, a_t = a] - h(x) \tag{2}$$

for both values of a. When $a = 0$, (2) is given by

$$x^T Q x + x^T A_1^T Y A_1 x - x^T Y x + \mathbf{trace}(\Sigma_w Y)$$

When $a = 1$, (2) is given by

$$x^T Q x + \lambda + x^T A_2^T Y A_2 x - x^T Y x + \mathbf{trace}(\Sigma_w Y)$$

For a given Y, the policy that minimizes the upper bound in Lemma 1 chooses the value of a that makes (2) smallest. So, this policy chooses

$$a = \begin{cases} 0 & \text{if } x^T A_1^T Y A_1 x \leq x^T A_2^T Y A_2 x + \lambda \\ 1 & \text{otherwise} \end{cases}$$

Suppose there exists x such that

$$x^T(Q + A_1^T Y A_1 - Y)x > 0.$$

For a finite upper bound, it must be the case that

$$x^T(Q + A_1^T Y A_1 - Y)x > 0 \Longrightarrow x^T(Q + A_2^T Y A_2 - Y)x \leq 0.$$

By the *lossless S-procedure* (see, for example, [9]), this condition is equivalent to existence of a scalar $\tau \geq 0$ such that

$$-(Q + A_2^T Y A_2 - Y) \succeq \tau(Q + A_1^T Y A_1 - Y)$$

This is equivalent to existence of a $\tau \geq 0$ such that

$$Y \succeq Q + \frac{\tau}{1+\tau} A_1^T Y A_1 + \frac{1}{1+\tau} A_2^T Y A_2,$$

which is equivalent to existence of $\rho \in (0,1]$ such that

$$Y \succeq Q + (1-\rho)A_1^T Y A_1 + \rho A_2^T Y A_2. \tag{3}$$

Suppose we have ρ satisfying (3) and suppose x is such that

$$x^T(A_1^T Y A_1 - A_2^T Y A_2)x \leq \lambda$$

Then

$$\begin{aligned} &\mathbf{trace}(\Sigma_w Y) + x^T(Q + A_1^T Y A_1 - Y)x \\ &\quad\leq \mathbf{trace}(\Sigma_w Y) + \rho x^T(A_1^T Y A_1 - A_2^T Y A_2)x \\ &\quad\leq \mathbf{trace}(\Sigma_w Y) + \rho\lambda \end{aligned}$$

Also, note that the inequality is tight for x such that

$$x^T(A_1^T Y A_1 - A_2^T Y A_2)x = \lambda.$$

Alternatively, suppose that x is such that

$$x^T(A_1^T Y A_1 - A_2^T Y A_2)x > \lambda$$

Then

$$\begin{aligned} &\mathbf{trace}(\Sigma_w Y) + \lambda + x^T(Q + A_2^T Y A_2 - Y)x \\ &\quad\leq \mathbf{trace}(\Sigma_w Y) + \lambda - (1-\rho)x^T(A_1^T Y A_1 - A_2^T Y A_2)x \\ &\quad\leq \mathbf{trace}(\Sigma_w Y) + \rho\lambda \end{aligned}$$

For given $Y \succeq 0$, the optimal upper bound is obtained by finding the ρ that minimizes

$$\begin{array}{rl} \text{min:} & \mathbf{trace}(\Sigma_w Y) + \rho\lambda \\ \text{s.t.:} & Y \succeq Q + (1-\rho)A_1^T Y A_1 + \rho A_2^T Y A_2 \\ & \rho \leq 1 \\ & \rho \geq 0 \end{array}$$

Finally, to determine the Y that produces the optimal upper bound, we can solve this optimization problem jointly over the variables $Y \succeq 0$ and ρ. ■

Note that the optimization problem presented in Theorem 1 contains a constraint that is bilinear in ρ and Y. While problems with bilinear matrix inequality constraints are generally difficult, in this case the variable ρ is a scalar. Hence, this problem can be solved to an arbitrary degree of accuracy by discretizing the interval $[0,1]$, and solving a semidefinite program for each value of ρ in this discretization of $[0,1]$.

In the previous theorem we used a fixed control gain and studied the problem of selecting an optimal control schedule. One reasonable choice of control gain is to use the K that minimizes

$$\limsup_{t\to\infty} \frac{1}{t} \sum_{k=0}^{t-1} \mathbf{E}[x_k^T Q x_k]$$

subject to the dynamics

$$x_{t+1} = (A + BK)x_t + w_t.$$

However, as the example in Section V shows, this does not always lead to the control gain that minimizes the upper bound on (1) obtained by using a quadratic approximate value function. In the following lemma, we show how to jointly compute a policy for selecting a_t and u_t.

Lemma 2. *Consider all upper bounds on average per-period cost that use Lemma 1 with a quadratic function h. When jointly choosing a_t and u_t, the minimum such upper bound is the minimum objective value of the optimization problem*

$$\begin{array}{ll} \textit{min:} & \mathbf{trace}(\Sigma_w Y) + \rho\lambda \\ \textit{s.t.:} & Y \succeq Q + A^T Y A - \rho A^T Y B (B^T Y B)^{-1} B^T Y A \\ & Y \succeq 0 \\ & \rho \leq 1 \\ & \rho \geq 0 \end{array}$$

Moreover, the policy for jointly choosing a and u that minimizes this upper bound uses

$$a = \begin{cases} 0 & \textit{if } x^T\big(A^T Y B (B^T Y B)^{-1} B^T Y A\big) x \leq \lambda \\ 1 & \textit{otherwise} \end{cases}$$

and

$$u = -(B^T Y B)^{-1} B^T Y A x.$$

Proof. We will start by considering the drift

$$c(x, a) + \mathbf{E}[h(x_{t+1}) \,|\, x_t = x, a_t = a, u_t = u] - h(x) \qquad (4)$$

for all values of a and u. When $a = 0$, (4) is given by

$$x^T Q x + x^T A^T Y A x - x^T Y x + \mathbf{trace}(\Sigma_w Y)$$

In this case, (4) is independent of u. When $a = 1$, (4) is given by

$$x^T Q x + \lambda + (Ax + Bu)^T Y (Ax + Bu) - x^T Y x + \mathbf{trace}(\Sigma_w Y)$$

The u that minimizes the expression above is

$$u = -(B^T Y B)^{-1} B^T Y A x$$

Let

$$\begin{array}{rcl} A_1 & = & A \\ A_2 & = & A - (B^T Y B)^{-1} B^T Y A \end{array}$$

Using these A_1 and A_2, we immediately obtain the optimization problem for finding Y and the policy for choosing a_t from Theorem 1. ■

In the next section we show that these computational approaches can be applied to more general architectures than the one considered in this section.

IV. Extensions

It turns out that the approach described in the previous section can be applied to control and estimation architectures more general than the one shown in Figure 1. As another possibility, consider the architecture pictured in Figure 2. This architecture represents a system where actuation is inexpensive, but transmission of state measurements to the controller is expensive. This might be the case if measurements are transmitted to the controller by sensors in a sensor network, and each sensor has a limited power budget. In this case, each time the state is sampled, a control signal is computed and applied until the next state measurement. Equivalently, the signal $\widehat{x}_t$ gives the value of the most recent state measurement, and the constant control signal $u_t = K\widehat{x}_t$ is applied until a new measurement is received. We note that a similar architecture is considered in [26].

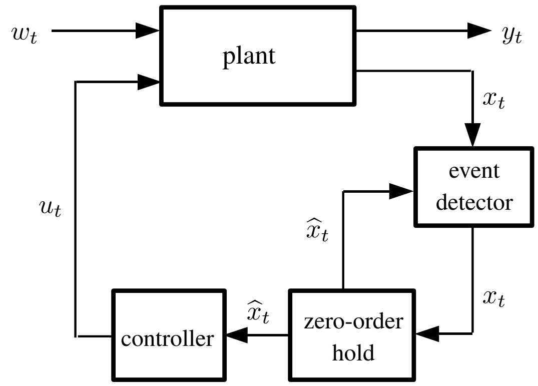

Fig. 2. Event-based sampling for control

In this system, the state of the plant evolves as

$$x_{t+1} = Ax_t + BK\widehat{x}_t + w_t$$

If the control variable a_t denotes the times that the state is sampled, then $\widehat{x}_t$ evolves as

$$\widehat{x}_{t+1} = (1 - a_t)\widehat{x}_t + a_t x_t$$

Equivalently, we can define the error as $e_t = x_t - \widehat{x}_t$, and show that the error evolves as

$$e_{t+1} = (1 - a_t)\big((A + BK - I)x_t + (I - BK)e_t\big) + w_t$$

So, letting $z_t^T = \begin{bmatrix} x_t^T & e_t^T \end{bmatrix}$, the dynamics of the state and error evolve as

$$z_{t+1} = \big((1 - a_t)A_1 + a_t A_2\big) z_t + v_t,$$

where

$$A_1 = \begin{bmatrix} A + BK & -BK \\ A + BK - I & I - BK \end{bmatrix} \qquad A_2 = \begin{bmatrix} A + BK & 0 \\ 0 & 0 \end{bmatrix}$$

and

$$\Sigma_v = \begin{bmatrix} \Sigma_w & \Sigma_w \\ \Sigma_w & \Sigma_w \end{bmatrix}$$

This problem is in the same general form as the event-based control system depicted in Figure 1, and hence the same procedure can be applied to compute an event-based sampling strategy.

Finally, we will show that state estimation problems can also be cast in the framework of the problems discussed above. Recall that the event-based sampling scheme in [13], [14] transmits the entire system state to the estimator at sampling times. This considerably simplifies analysis since the estimation error resets to zero at each sampling time. The analysis becomes more complicated when only output measurements are sent in each time period. In this case, state estimates must be maintained and updated when new measurements are received. Fortunately, we can extend the control approach discussed previously to the problem of estimation with output measurements.

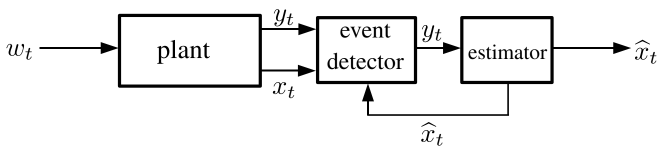

Fig. 3. Event-based estimation

In the estimation problem, we will consider a system with dynamics

$$
\begin{aligned}
x_{t+1} &= Ax_t + w_t \\
y_t &= Cx_t + v_t
\end{aligned}
$$

When all output measurements are available, the steady state Kalman filter produces optimal state estimates $\widehat{x}_t$ according to the recursion

$$\widehat{x}_{t+1} = A\widehat{x}_t + L(y_t - C\widehat{x}_t)$$

Here, L is the steady state Kalman filter observer gain. This estimator is known to minimize

$$\limsup_{t\to\infty} \frac{1}{t}\sum_{k=0}^{t-1} \mathbf{E}[(x_k - \widehat{x}_k)^T Q (x_k - \widehat{x}_k)]$$

We will consider a system composed of a plant and an estimator, where measurements of the outputs from the plant are intermittently transmitted to the estimator. This is depicted in Figure 3. The estimator operates as follows. If $\widehat{x}_t$ is the current state estimate and no measurement is available at time t, the estimate of state $\widehat{x}_{t+1}$ is $\widehat{x}_{t+1} = A\widehat{x}_t$. If a measurement is available at time t, the estimate of state $\widehat{x}_{t+1}$ is

$$\widehat{x}_{t+1} = A\widehat{x}_t + L(y_t - C\widehat{x}_t).$$

Using the variable $a_t \in \{0, 1\}$ to indicate that a measurement has been taken, state estimates evolve as

$$\widehat{x}_{t+1} = A\widehat{x}_t + a_t L(y_t - C\widehat{x}_t).$$

Similarly, we can write the dynamic equations for the state estimation error $e_t = \widehat{x}_t - x_t$ as

$$
\begin{aligned}
e_{t+1} &= \widehat{x}_{t+1} - x_{t+1} \\
&= \big(A\widehat{x}_t + a_t L(y_t - C\widehat{x}_t)\big) - \big(Ax_t + w_t\big) \\
&= \big(A\widehat{x}_t + a_t L(Cx_t + v_t - C\widehat{x}_t)\big) - \big(Ax_t + w_t\big) \\
&= (A + a_t LC)e_t - w_t + a_t L v_t
\end{aligned}
$$

As in the control case, we will simplify notation by using $A_1 = A$ to denote the open-loop estimator dynamics and $A_2 = A + LC$ to denote the closed loop estimator dynamics.

When considering event-based estimation strategies, the goal is to schedule measurements to minimize a trade-off between the transmission rate and the estimation error. That is, our goal is to determine a policy for choosing a_t that minimizes

$$\limsup_{t\to\infty} \frac{1}{t}\sum_{k=0}^{t-1} \mathbf{E}[e_k^T Q e_k + \lambda a_k].$$

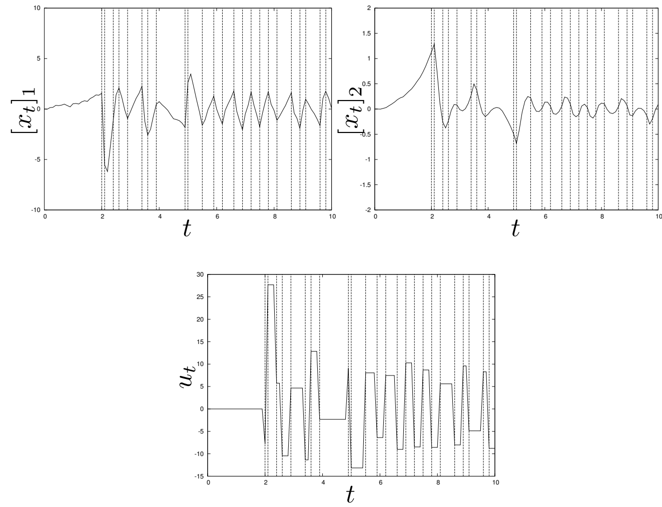

Fig. 4. State trajectories and control signals for the double integrator. Vertical dashed lines indicate sampling times.

In the architecture depicted in Figure 3, an event detector can observe the current state of the plant as well as the current state estimate used by the estimator (possibly by running an identical estimator locally). When some event dependent on the estimation error occurs, the current output measurement y_t is sent to the estimator, which then updates its state estimate accordingly.

The model of this estimation architecture can be cast in a form identical to the models for the control architectures described above. As with the control problems, we propose the following event-based transmission strategy for selecting a_t.

Let $\widetilde{\lambda} = \lambda + \mathbf{trace}(L\Sigma_v L^T Y)$ and let ρ and Y be solutions to the optimization problem

$$
\begin{aligned}
\text{min:}\quad & \mathbf{trace}(\Sigma_w Y) + \rho\widetilde{\lambda} \\
\text{s.t.:}\quad & Y \succeq Q + (1-\rho)A_1^T Y A_1 + \rho A_2^T Y A_2 \\
& Y \succeq 0 \\
& \rho \le 1 \\
& \rho \ge 0
\end{aligned}
$$

A measurement is sent to the estimator by setting

$$a_t = \begin{cases} 0 & \text{if } e_t^T\big(A_1^T Y A_1 - A_2^T Y A_2\big)e_t \le \widetilde{\lambda} \\ 1 & \text{otherwise} \end{cases}$$

As with the control architectures discussed above, it can be shown that an upper bound on the cost incurred by this policy is given by $\mathbf{trace}(\Sigma_w Y) + \rho\widetilde{\lambda}$.

V. Example

To conclude this paper, we will show a simple example that demonstrates the significant improvement in control cost that can be obtained by using an event-based policy in place of a periodic sampling strategy. Consider a discrete-time approximation of a double-integrator. This system has dynamics

$$x_{t+1} = Ax_t + Bu_t + w_t$$

with

$$A = \begin{bmatrix} 1 & 0 \\ 0.1 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 0.1 \\ 0 \end{bmatrix} \qquad \Sigma_w = \begin{bmatrix} 0.01 & 0 \\ 0 & 0 \end{bmatrix}$$

In this system, measuring the system state is expensive, so we would like to control the system with as few measurements of the state as possible. The desire to limit measurements is modeled as a measurement cost $\lambda = 100$ charged each time the state is measured. So, the overall cost we would like to minimize is of the form

$$\limsup_{t\to\infty} \frac{1}{t} \sum_{k=0}^{t-1} \mathbf{E}[x_k^T Q x_k + \lambda a_k],$$

where

$$Q = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

We will control this system using the architecture shown in Figure 2. Control signals are generated using the state-feedback controller

$$K = \begin{bmatrix} -10 & -50 \end{bmatrix}.$$

Each time the state is measured, a constant control signal is applied until the next state measurement. For this system, we can construct the matrices A_1 and A_2 as

$$A_1 = \begin{bmatrix} 0 & -5 & 1 & 5 \\ 0.1 & 1 & 0 & 0 \\ -1 & -5 & 2 & 5 \\ 0.1 & 0 & 0 & 1 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 0 & -5 & 0 & 0 \\ 0.1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

By constructing the threshold region using

$$A_1^T Y A_1 - A_2^T Y A_2,$$

we sample the state whenever the state x and error e satisfy

$$\begin{bmatrix} x_1 \\ x_2 \\ e_1 \\ e_2 \end{bmatrix}^T \begin{bmatrix} 0.186 & 1.483 & -0.352 & -1.762 \\ 1.483 & 11.024 & -2.705 & -13.529 \\ -0.352 & -2.705 & 0.642 & 3.210 \\ -1.762 & -13.529 & 3.210 & 16.052 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ e_1 \\ e_2 \end{bmatrix} > 1$$

Figure 4 shows a sample path of the state and control signals generated when using this policy. Under this policy the state is sampled at a rate of approximately 2.8 samples per second. The resulting control cost is

$$\limsup_{t\to\infty} \frac{1}{t} \sum_{k=0}^{t-1} \mathbf{E}[x_k^T Q x_k] \approx 0.021$$

We can compare this to the control cost achieved when applying control in every time step (*i.e.*, 10 samples per second), which is approximately 0.00024.

We can also compare this policy to a policy that randomly samples the state in each time step with probability 0.28, also achieving an average sampling rate of 2.8 samples per second. Under this policy, *the resulting system is unstable*. So, for a sampling rate that does not stabilize the system under open-loop sampling, we can achieve acceptable performance using event-based sampling.

VI. Conclusions

In this paper we developed an approach to event-based control that minimizes an upper bound on system performance. This approach minimizes an upper bound on system performance associated with using a quadratic approximate value function for the underlying Markov decision process. This approach reduces to the problem of solving a sequence of semidefinite programs indexed by a scalar parameter.

This approach is first developed for a simple control architecture, where the goal is to maintain satisfactory regulation of a plant when actuating the system infrequently. We then showed that our approach can be applied to more general control and estimation architectures.

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