# On Structured Semidefinite Programs for the Control of Symmetric Systems

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**Abstract**

In this paper we show how the symmetry present in many linear systems can be exploited to significantly reduce the computational effort required for controller synthesis. This approach may be applied when controller design specifications are expressible as a semidefinite program. In particular, when the overall system description is invariant under unitary coordinate transformations of the state space matrices, synthesis semidefinite programs can be decomposed into a collection of smaller, uncoupled, and often repeated semidefinite programs.

## 1 Introduction

This paper focuses on the use of symmetry to reduce computational requirements for the design of controllers for a wide range of control objectives. Many systems with symmetries arise in practice. Examples include symmetrically interconnected loads and generators in an electrical power distribution network, vehicles travelling in a symmetric formation, and temperature control on a symmetric surface. The methods described in this paper can be applied to these problems, as well as many others, to simplify the process of synthesizing controllers.

Specifically, in this paper we focus on controller synthesis problems for which an equivalent formulation as a semidefinite program (SDP) exists. There has been significant research on the properties and decomposition of linear systems with symmetry, and it is known that in many cases the symmetry leads to decomposition of the dynamics into a collection of smaller uncoupled subsystems, for which one may perform controller synthesis directly [6, 7, 12, 21]. This in particular occurs when the performance objective is either an $H_2$ or $H_\infty$ norm. In this paper we show further that, when using semidefinite programming for controller synthesis for symmetric systems, the resulting SDPs are highly structured, which leads to significant computational benefits.

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[1] The first author was partially supported by a Stanford Graduate Fellowship.

[1,2] Partially supported by the Stanford URI *Architectures for Secure and Robust Distributed Infrastructures*, AFOSR DoD award number 49620-01-1-0365.

## 1.1 Previous Work

The branch of group theory known as *representation theory* and the associated notions of symmetry have been well known and applied, most often in chemistry and quantum mechanics, for over half a century [23]. Techniques for exploiting symmetry in semidefinite programs have appeared in the recent paper [9], where the notion of a *symmetric semidefinite program* was introduced and applied to simplify the process of determining sum of squares decompositions of polynomials.

The role of symmetry in dynamical systems has also been previously studied in many different contexts. An important and well-known result is the equivalence of symmetries and conservation laws in Hamiltonian dynamics, and the use of symmetry reduction for such systems [16]. Also, it has been shown how symmetries can be exploited in the study of bifurcations in nonlinear systems [10]. For linear systems, properties resulting from symmetry are well-known, and have been analyzed in [6, 7], where the decomposition of a symmetric system into smaller uncoupled systems is discussed, and it is shown that certain properties of the symmetric system, such as stability, can be determined by examining the individual uncoupled systems. In [13] it is shown that the $H_\infty$ norm of a symmetric system can be determined by from the $H_\infty$ norms of the uncoupled systems. Dynamical systems which are composed of interconnected subsystems have been studied in [15, 21], where the symmetry arises due the an invariance under permutation of the subsystems. Without making use of representation theory, they show that system stability can be analyzed by considering two significantly smaller, uncoupled systems related to the original system. In related work [22], it is shown how to use group-theoretic methods to study the fault-tolerance properties of arrays of symmetric systems.

Recent work [1] describes how spatial invariance can be exploited in controlling distributed systems. Although the systems discussed in their work are infinite dimensional, it is shown how the synthesis problems can be decomposed into an infinite family of finite dimensional problems. This decomposition is achieved by taking a Fourier transform with respect to spatial coordinates. The approach of [1] makes use of a decomposition very similar to that presented in this paper, but focuses on the case where the underlying symmetry group is Abelian. In this paper we consider the non-Abelian case for finite groups, and the resulting decomposition has additional structure, such as repeated diagonal blocks in the synthesis semidefinite programs (SDPs). This repeated block structure is a characteristic of the non-Abelian nature of the groups discussed in this paper, and is not present in systems which are symmetric with respect to Abelian groups.

It is known that, when decomposing a symmetric system into decoupled subsystems, the $H_2$ norm of the original system is given by the sum of the $H_2$ norms of the decoupled subsystems. Similarly, the $H_\infty$ norm of the original system is the maximum of the $H_\infty$ norm of the decoupled subsystems. This leads to immediate computational benefits when using any numerical approach to minimize either of those performance indices. However, for general control design objectives, such as multiobjective problems, such a norm decomposition does not apply. In these cases, the benefits of symmetry can still be obtained from the structure in the semidefinite program. In particular, it is in this situation that the methods in this paper offer a new benefit over previously analyzed approaches. Application of the methods in this paper to either $H_2$ or $H_\infty$ synthesis will also offer computational benefits similar to those obtained by previous decoupling techniques, and in this sense this paper unifies these different approaches.

# 2 Group Theoretic Preliminaries

Here we discuss the general theory of matrices which have the symmetry of some group, and review the use of this for system decomposition. The following notation and results are standard; proofs of the theorems presented in this section can be found, for example, in [8] and [19]. A map $\Theta : G \to \mathbb{C}^{n\times n}$ is called a *representation* if it is a group homomorphism; *i.e.*, $\Theta(gh) = \Theta(g)\Theta(h)$ for all $g, h \in G$. Two representations $\Theta_1$ and $\Theta_2$ are called *equivalent* if there exists a nonsingular matrix $T$ such that $\Theta_1(g) = T\Theta_2(g)T^{-1}$ for all $g \in G$. A subspace $V \subseteq \mathbb{C}^n$ such that $\Theta(g)x \in V$ for all $x \in V$ and all $g \in G$ is said to be an *invariant subspace* of the representation $\Theta$. It is called *proper* if it is a strict subset of $\mathbb{C}^n$. An *irreducible* representation is a representation which has no proper invariant subspace. If $V \subset \mathbb{C}^n$ is a subspace, a subspace $W \subset \mathbb{C}^n$ is called a *complementary subspace of* $V$ if $V \cap W = 0$ and $V \oplus W = \mathbb{C}^n$. The following theorem is used to show that representations have special structure.

**Theorem 1.** *Suppose $V$ is an invariant subspace of the representation $\Theta : G \to \mathbb{C}^{n\times n}$. Then there is a subspace $W$ which is a complement of $V$ and is also an invariant subspace of $\Theta$.*

The fact that we can block diagonalize representations directly follows.

**Theorem 2.** *Every reducible representation $\Theta$ of a finite group $G$ is equivalent to a representation $\hat{\Theta}$ such that for each $g \in G$, $\hat{\Theta}(g)$ is block diagonal and each diagonal block corresponds to an irreducible representation of $G$.*

If we let $\Omega_1, \dots, \Omega_J$ denote the $J$ inequivalent irreducible representations of a group $G$ (each relative to specific coordinates), then we can write the representation in Theorem 2 as

$$\hat{\Theta}(g) = \bigoplus_{i=1}^{J} (\Omega_i(g) \otimes I_{p_i}). \tag{1}$$

Here we have denoted the block diagonal matrix consisting of $p_i$ copies of $\Omega_i(g)$ by the Kronecker product $\Omega_i(g) \otimes I_{p_i}$ (where $p_i$ may be zero). Also, we have written the block diagonal matrix with $\Omega_i(g) \otimes I_{p_i}$ as its blocks as a *matrix direct sum.*

## 2.1 Equivariant Matrices

A matrix $X \in \mathbb{C}^{m\times n}$ is said to be *equivariant* with respect to the representations $\Theta_1$ and $\Theta_2$ if $X = \Theta_1(g)^{-1}X\Theta_2(g)$ for all $g \in G$. Theorem 2 shows that a matrix $T$ exists such that $\hat{\Theta}(g) = T^{-1}\Theta(g)T$ has the structure of equation (1). It is clear that $\hat{X} = T_1^{-1}XT_2$ is equivariant with respect to $\hat{\Theta}_1(g) = T_1^{-1}\Theta_1(g)T_1$ and $\hat{\Theta}_2(g) = T_2^{-1}\Theta_2(g)T_2$. The following results will be used to establish the structure of $\hat{X}$.

**Theorem 3 (Schur).** *Let $\Omega_1 : G \to \mathbb{C}^{m\times m}$ and $\Omega_2 : G \to \mathbb{C}^{n\times n}$ be irreducible representations of some group $G$. Suppose the matrix $X \in \mathbb{C}^{m\times n}$ satisfies $X = \Omega_1(g)^{-1}X\Omega_2(g)$ for all $g \in G$. Then one of the following alternatives holds:*

*(i)* $X = 0$.

*(ii) $X$ is square and nonsingular.*

**Corollary 4.** *Let $\Omega : G \to \mathbb{C}^{n\times n}$ be an irreducible representation of some group $G$. If the matrix $X \in \mathbb{C}^{n\times n}$ satisfies $X = \Omega(g)^{-1}X\Omega(g)$ for all $g \in G$, then $X = \lambda I$ where $\lambda$ is some scalar and $I$ is the $n \times n$ identity matrix.*

Now we can describe the structure of $\hat{X}$. As we already know, $T_1$ and $T_2$ are chosen so that $\hat{\Theta}_1(g) = T_1^{-1}\Theta_1(g)T_1 = \bigoplus_{i=1}^{J}(\Omega_i(g)\otimes I_{p_i})$ and $\hat{\Theta}_2(g) = T_2^{-1}\Theta_2(g)T_2 = \bigoplus_{i=1}^{J}(\Omega_i(g)\otimes I_{q_i})$, where the matrices $\Omega_i(g)$ are $n_i \times n_i$. We can partition $\hat{X}$ into blocks:

$$\hat{X} = \begin{bmatrix} \hat{X}_{11} & \hat{X}_{12} & \cdots & \hat{X}_{1J} \\ \hat{X}_{21} & \hat{X}_{22} & \cdots & \hat{X}_{2J} \\ \vdots & & \ddots & \\ \hat{X}_{J1} & \hat{X}_{J2} & \cdots & \hat{X}_{JJ} \end{bmatrix}$$

where $\hat{X}_{ij} \in \mathbb{C}^{n_ip_i\times n_jq_j}$. Therefore $\hat{\Theta}_1(g)\hat{X} = \hat{X}\hat{\Theta}_2(g)$ if and only if $(\Omega_i(g)\otimes I_{p_i})\hat{X}_{ij} = \hat{X}_{ij}(\Omega_j(g)\otimes I_{q_j})$ for $i, j = 1, \ldots, J$. Using the results above,

$$\hat{X}_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ I_{n_i}\otimes\Lambda_i & \text{if } i = j \end{cases}$$

where $\Lambda_i$ is some $p_i \times q_i$ matrix (and either dimension may be zero). In other words, $\hat{X}$ is a block diagonal matrix where each of the diagonal blocks has special structure. We can express $\hat{X}$ in matrix direct sum notation as $\hat{X} = \bigoplus_{i=1}^{J}(I_{n_i}\otimes\Lambda_i)$.

We can refine the block diagonal structure of $\hat{X}$ by noting that there are permutation matrices $L_i$ and $R_i$ such that $L_i^T(I_{n_i}\otimes\Lambda_i)R_i = \Lambda_i\otimes I_{n_i}$ [11]. We can apply the permutations $\bigoplus_{i=1}^{J} L_i$ and $\bigoplus_{i=1}^{J} R_i$ to the matrix $\hat{X}$ to obtain a block diagonal matrix $\tilde{X}$ with finer block structure:

$$\begin{aligned} \tilde{X} &= \left(\bigoplus_{i=1}^{J} L_i\right)^T \left(\bigoplus_{i=1}^{J}(I_{n_i}\otimes\Lambda_i)\right)\left(\bigoplus_{i=1}^{J} R_i\right) \\ &= \bigoplus_{i=1}^{J} L_i^T(I_{n_i}\otimes\Lambda_i)R_i \\ &= \bigoplus_{i=1}^{J}(\Lambda_i\otimes I_{n_i}). \end{aligned}$$

The matrix $\tilde{X}$ is a block diagonal matrix consisting of $J$ distinct $p_i \times q_i$ blocks, each with multiplicity $n_i$. To avoid confusion with the block structure of $\hat{\Theta}(g)$, note that $\hat{\Theta}(g)$ is a block diagonal matrix consisting of $J$ distinct $n_i \times n_i$ blocks, each with multiplicity $p_i$. The key fact here is that there exists a fixed coordinate transformation under which all equivariant matrices (with respect to some representations) are block diagonal with specific block structure. In the next section we will discuss systems in which each of the state space matrices are equivariant.

# 3 LMI-based Synthesis for Symmetric Systems

## 3.1 Symmetric Systems

In this section we will discuss the notion of a symmetric state space system and show that under the appropriate transformation, the transformed system decouples into a collection of independent subsystems.

**Definition 5.** *Suppose $G$ is a finite group and $\Theta_x : G \to \mathbb{C}^{n\times n}$, $\Theta_y : G \to \mathbb{C}^{m\times m}$, $\Theta_u : G \to \mathbb{C}^{p\times p}$ are unitary representations. Suppose we have a system with a state space realization $(A, B, C, D)$. We say that this system is a symmetric system if the following equations hold:*

$$\begin{aligned} A = \Theta_x(g)^* A \Theta_x(g), &\quad B = \Theta_x(g)^* B \Theta_u(g), \\ C = \Theta_y(g)^* C \Theta_x(g), &\quad D = \Theta_y(g)^* D \Theta_u(g) \end{aligned} \tag{2}$$

*for all $g \in G$.*

Note that our definition of symmetric system is similar, but slightly stricter than other definitions appearing in the literature [6, 7, 13], where invariance of the input-output behavior under static symmetry transformations of the input and output are the only requirements. The reason for this is that complete decoupling of the state space matrices under unitary transformations is required in order to decouple the control synthesis SDPs, as we will see in the next section.

A key fact used here is that unitary representations can be brought to their block diagonal form by unitary coordinate transformations [8]. Therefore, there exist unitary matrices $T_x$, $T_y$, and $T_u$ such that

$$\begin{aligned} T_x^* \Theta_x(g) T_x &= \bigoplus_{i=1}^{J} (\Omega_i(g) \otimes I_{p_i}) \\ T_y^* \Theta_y(g) T_y &= \bigoplus_{i=1}^{J} (\Omega_i(g) \otimes I_{q_i}) \\ T_u^* \Theta_u(g) T_u &= \bigoplus_{i=1}^{J} (\Omega_i(g) \otimes I_{r_i}). \end{aligned}$$

Moreover, these matrices can be computed by standard algorithms from computational group theory. More details on these algorithms can be found in [8], chapter 5. From the results of the last section, if we apply these unitary transformations followed by the appropriate permutation to the matrices $A, B, C, D$ we obtain the matrices

$$\begin{aligned} \tilde{A} = \bigoplus_{i=1}^{J} (\tilde{A}_i \otimes I_{n_i}) &\qquad \tilde{B} = \bigoplus_{i=1}^{J} (\tilde{B}_i \otimes I_{n_i}) \\ \tilde{C} = \bigoplus_{i=1}^{J} (\tilde{C}_i \otimes I_{n_i}) &\qquad \tilde{D} = \bigoplus_{i=1}^{J} (\tilde{D}_i \otimes I_{n_i}) \end{aligned} \tag{3}$$

where $\tilde{A}_i \in \mathbb{C}^{p_i\times p_i}$, $\tilde{B}_i \in \mathbb{C}^{p_i\times r_i}$, $\tilde{C}_i \in \mathbb{C}^{q_i\times p_i}$, and $\tilde{D}_i \in \mathbb{C}^{q_i\times r_i}$. Note that the block structure of these matrices is such that the state space system $\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}$ can be completely decoupled into subsystems $\tilde{A}_i, \tilde{B}_i, \tilde{C}_i, \tilde{D}_i$. This will allow us to completely decouple the control synthesis SDPs in the next section.

## 3.2 LMI-based synthesis

Here we show how symmetry can be exploited to obtain computational simplification when solving control synthesis semidefinite programs. Due to space constraints, we only discuss the $H_2$ state feedback case in detail. However, the point of this section is to demonstrate a procedure which can be applied to many different LMI-based synthesis procedures. Decompositions can be obtained in a nearly identical manner for most other cases.

Suppose we have a system with a state space realization

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_u u(t) + B_w w(t) \\ y(t) &= Cx(t) + D_u u(t) \end{aligned} \tag{4}$$

where $u$ is the control input and $w$ is the disturbance input. To determine the static state feedback control law $u(t) = Kx(t)$ which minimizes the closed loop $H_2$ norm of the system, we solve the standard optimization problem

$$\begin{aligned} \text{minimize} \quad & \operatorname{trace}(W) \\ \text{subject to} \quad & AX + XA^T + B_u Z + Z^T B_u^T + B_w B_w^T < 0 \\ & \begin{bmatrix} X & XC^T + Z^T D_u^T \\ CX + D_u Z & W \end{bmatrix} > 0 \end{aligned} \tag{5}$$

where $W$, $X$, and $Z$ are the decision variables. Given a solution, a state feedback controller can be obtained as $K = ZX^{-1}$.

Suppose each of the state space matrices are equivariant with respect to some group representation. As in Theorem 3.3 of [9], we can show that there exist $\hat{W}$, $\hat{X}$, and $\hat{Z}$ which solve (5) and are also equivariant with respect to $G$. The inequality

$$AX + XA^T + B_u Z + Z^T B_u^T + B_w B_w^T < 0$$

is equivalent to

$$\Theta_x(g)^*(AX + XA^T + B_u Z + Z^T B_u^T + B_w B_w^T)\Theta_x(g) < 0.$$

This inequality holds if and only if

$$\begin{aligned} &A(\Theta_x(g)^* X \Theta_x(g)) + (\Theta_x(g)^* X \Theta_x(g))A^T \\ &\qquad + B_u(\Theta_u(g)^* Z \Theta_x(g)) + (\Theta_u(g)^* Z \Theta_x(g))^T B_u^T + B_w B_w^T < 0 \qquad \text{for all } g \in G. \end{aligned}$$

Similarly

$$\begin{bmatrix} X & XC^T + Z^T D_u^T \\ CX + D_u Z & W \end{bmatrix} > 0$$

holds if and only if

$$\begin{bmatrix} (\Theta_x(g)^* X \Theta_x(g)) & (\Theta_x(g)^* X \Theta_x(g))C^T + (\Theta_u(g)^* Z \Theta_x(g))^T D_u^T \\ C(\Theta_x(g)^* X \Theta_x(g)) + D_u(\Theta_u(g)^* Z \Theta_x(g)) & (\Theta_y(g)^* W \Theta_y(g)) \end{bmatrix} > 0$$

for all $g \in G$.

Suppose we define

$$\begin{aligned} \hat{W} &= \frac{1}{|G|} \sum_{g \in G} \Theta_y(g)^* W \Theta_y(g) \\ \hat{X} &= \frac{1}{|G|} \sum_{g \in G} \Theta_x(g)^* X \Theta_x(g) \\ \hat{Z} &= \frac{1}{|G|} \sum_{g \in G} \Theta_u(g)^* Z \Theta_x(g) \end{aligned}$$

where $|G|$ denotes the number of elements in $G$. Then by linearity of the objective function, $\operatorname{trace}(\hat{W}) = \operatorname{trace}(W)$. By convexity of the feasible set, $\hat{W}$, $\hat{X}$, and $\hat{Z}$ satisfy the inequality constraints (5). The matrices $\hat{W}$, $\hat{X}$, and $\hat{Z}$ are equivariant since they satisfy

$$\hat{W} = \Theta_y(g)^* \hat{W} \Theta_y(g), \quad \hat{X} = \Theta_x(g)^* \hat{X} \Theta_x(g), \quad \hat{Z} = \Theta_u(g)^* \hat{Z} \Theta_x(g)$$

for all $g \in G$. Therefore, we can restrict our search to equivariant decision variables when solving the problem (5).

We can also show that any achievable closed loop norm can be achieved with a controller $\hat{K}$ that is equivariant. This is based on the fact that any achievable value of the objective in an invariant SDP can be achieved with an equivariant solution. Note that $\hat{X} = \Theta_x(g)^* \hat{X} \Theta_x(g) \implies \hat{X}^{-1} = (\Theta_x(g)^* \hat{X} \Theta_x(g))^{-1} = \Theta_x(g)^* \hat{X}^{-1} \Theta_x(g)$ for all $g \in G$. Therefore,

$$\begin{aligned}
\hat{K} &= \hat{Z}\hat{X}^{-1} \\
&= (\Theta_u(g)^* \hat{Z} \Theta_x(g))(\Theta_x(g)^* \hat{X}^{-1} \Theta_x(g)) \\
&= \Theta_u(g)^* \hat{Z}\hat{X}^{-1} \Theta_x(g) \\
&= \Theta_u(g)^* \hat{K} \Theta_x(g)
\end{aligned}$$

for all $g \in G$.

From the previous section we know that there exist unitary matrices $T_y$, $T_x$, $T_u$, and $T_w$ such that $\tilde{A} = T_x^* A T_x$, $\tilde{B}_u = T_x^* B_u T_u$, $\tilde{B}_w = T_x^* B_w T_w$ $\tilde{C} = T_y^* C T_x$, and $\tilde{D}_u = T_y^* D_u T_u$ are block diagonal. We can apply these same transformations to obtain $\tilde{W} = T_y^* \hat{W} T_y$, $\tilde{X} = T_x^* \hat{X} T_x$, and $\tilde{Z} = T_u^* \hat{Z} T_x$ which are also block diagonal with block structures matching that of $\tilde{D}_u$, $\tilde{A}$, and $\tilde{B}_u$, respectively. As in the previous section, each of the transformed matrices consists of $J$ distinct diagonal blocks, each with multiplicity $n_i$. We will denote the distinct diagonal blocks of $\tilde{A}$, for example, as $\tilde{A}_i$ for $i = 1, \ldots, J$. Then it is easy to show that

$$\begin{aligned}
\operatorname{trace}(W) &= \operatorname{trace}(\tilde{W}) \\
&= \sum_{i=1}^{J} \operatorname{trace}(\tilde{W}_i \otimes I_{n_i}) \\
&= \sum_{i=1}^{J} n_i \operatorname{trace}(\tilde{W}_i)
\end{aligned}$$

Since $A\hat{X} + \hat{X}A^T + B_u\hat{Z} + \hat{Z}^T B_u^T + B_w B_w^T$ is equivariant with respect to the representation $\Theta_x$,

$$A\hat{X} + \hat{X}A^T + B_u\hat{Z} + \hat{Z}^T B_u^T + B_w B_w^T < 0,$$

holds if and only if

$$\tilde{A}_i\tilde{X}_i + \tilde{X}_i\tilde{A}_i^* + \tilde{B}_{ui}\tilde{Z}_i + \tilde{Z}_i^*\tilde{B}_{ui}^* + \tilde{B}_{wi}\tilde{B}_{wi}^* < 0 \quad \text{for } i = 1, \ldots, J.$$

Similarly, the matrix

$$\begin{bmatrix} \hat{X} & \hat{X}C^T + \hat{Z}^T D_u^T \\ C\hat{X} + D_u\hat{Z} & \hat{W} \end{bmatrix}$$

is equivariant with respect to the representation $\Theta_x \oplus \Theta_y$, so

$$\begin{bmatrix} \hat{X} & \hat{X}C^T + \hat{Z}^T D_u^T \\ C\hat{X} + D_u\hat{Z} & \hat{W} \end{bmatrix} > 0$$

holds if and only if

$$\begin{bmatrix} \tilde{X}_i & \tilde{X}_i\tilde{C}_i^* + \tilde{Z}_i^*\tilde{D}_{ui}^* \\ \tilde{C}_i\tilde{X}_i + \tilde{D}_{ui}\tilde{Z}_i & \tilde{W}_i \end{bmatrix} > 0 \quad \text{for } i = 1, \ldots, J.$$

Hence, the problem (5) can be solved by solving the $J$ uncoupled problems:

$$\begin{array}{ll} \text{minimize} & \operatorname{trace}(\tilde{W}_i) \\ \text{subject to} & \tilde{A}_i\tilde{X}_i + \tilde{X}_i\tilde{A}_i^* + \tilde{B}_{ui}\tilde{Z}_i + \tilde{Z}_i^*\tilde{B}_{ui}^* + \tilde{B}_{wi}\tilde{B}_{wi}^* < 0 \\ & \begin{bmatrix} \tilde{X}_i & \tilde{X}_i\tilde{C}_i^* + \tilde{Z}_i^*\tilde{D}_{ui}^* \\ \tilde{C}_i\tilde{X}_i + \tilde{D}_{ui}\tilde{Z}_i & \tilde{W}_i \end{bmatrix} > 0 \end{array}$$

for $i = 1, \ldots, J$.

Note that each of the resulting SDPs are complex and have complex decision variables. When we convert our solutions back to the original coordinates, we find solutions $\hat{W}$, $\hat{X}$, and $\hat{Z}$ which may be complex. Complex solutions to the original problem are undesirable since we typically wish to construct a real state feedback controller. This is not a problem however, since a real controller can be easily constructed from complex solutions. For any complex positive definite matrix $X$, $\operatorname{Re}(X)$ is also positive definite. Since each of the data matrices in (5) are real, $\operatorname{Re}(\hat{W})$, $\operatorname{Re}(\hat{X})$, and $\operatorname{Re}(\hat{Z})$ are also feasible solutions for (5). Since $\hat{W}$ is Hermitian, $\operatorname{trace}(\hat{W}) = \operatorname{trace}(\operatorname{Re}(\hat{W}))$, so $\operatorname{Re}(\hat{W})$, $\operatorname{Re}(\hat{X})$, and $\operatorname{Re}(\hat{Z})$ are optimal solutions. We can then construct a real controller as $\hat{K} = \operatorname{Re}(\hat{Z})(\operatorname{Re}(\hat{X}))^{-1}$. Since each of the group representations that we are concerned with are real, we can show that the resulting $\hat{K}$ is still equivariant.

One could avoid dealing with complex SDPs altogether by decomposing the representations into *real irreducible representations*, as discussed in [19]. However, we have avoided this approach here since a discussion of the theory of real representations is more involved than the complex case and the complex SDPs that we encounter do not pose any real problems.

Here we have shown that symmetry present in a linear system allows us to replace the $H_2$ control synthesis SDP with a set of SDPs, with one for each irreducible representation of our symmetry group. We obtained these uncoupled SDPs by a simple procedure which can be repeated for many other control synthesis SDPs. This procedure will be summarized briefly. Since each of the state space matrices are equivariant, we can use convexity of the constraints and linearity of the objective function to show that we can restrict our search to equivariant decision variables. Since the decision variables are chosen to be equivariant, each of the inequality constrained matrices is equivariant with respect to some representations. For this reason we can then decompose each of the inequality constraints into a collection of uncoupled inequality constraints. Since the objective function is a symmetric linear function of an equivariant matrix, it can be decomposed into a sum of individual components as well.

The exact procedure described here can be applied to completely decouple the semidefinite programs associated with other many other control objectives, such as stabilization, $H_\infty$, pole placement, bounds on peak-to-peak gain, and multiobjective problems. It is important to note that we can also apply this procedure to problems where the variables are required to have special structure, such as decentralized control problems. In this case, the problems do not necessarily decouple, but computational savings can still be obtained due to reduced dimension of the variables and the presence of repeated blocks.

# 4 Conclusions

In this paper it was shown that certain types of symmetries which may be present in the state space realizations of some linear systems may be exploited to significantly reduce the amount of computation required for controller synthesis. Specifically, when symmetries are present, we can decompose the synthesis SDPs into a collection of uncoupled SDPs. Depending on the particular symmetry of the system, we may also have repeated SDPs in the decomposition, further simplifying the synthesis of optimal controllers.

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