Optimization of Peak Power in Vibrating Structures via Semidefinite Programming

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In this paper we consider the problem of designing a truss to minimize peak power input to the structure by a dynamic load. We first consider a model where known sinusoidal forces are applied to the truss, and our goal is to choose rod cross-sectional areas to minimize peak power subject to a constraint on the total mass of the structure. We then consider a second model where unknown forces are applied to the truss, and our goal is to choose rod cross-sectional areas to minimize the worst-case peak power over all unit magnitude forces. In both cases, we show that the optimization problems can be formulated as semidefinite programs. Numerical results are presented for both cases.

I. Background

This paper focuses on optimal design of truss structures for peak power minimization. Optimal design of truss structures has a long history, starting with the papers[5] and.[6] These problems are typically formulated in such a way that a ground structure, rod lengths, and material properties are specified, and the goal is to select rod cross-sectional areas to optimize some objective function. The majority of existing work on this topic involves optimization under static loads, with minimization of compliance subject to weight constraints being the most common formulation.[2]

If the constraint set and objective function of an optimization problem is convex, the globally optimal solution can typically be computed efficiently. Convexity of the objective function and constraint set guarantees that the problem has a single locally optimal solution, which is the global optimum solution. So, gradient-based methods can be applied to compute the global optimum of a convex optimization problem. As is shown in,[4] if the compliance minimization problem is not formulated carefully, the constraint set of the optimization problem will not be convex. In particular, if nodal displacements are not eliminated from the problem formulation, a non-convex bilinear constraint will be present. This lack of convexity creates multiple local optima, and gradient-based methods will not always compute global optima. However, by eliminating nodal displacements, a convex optimization problem results. The paper[7] gives detailed conditions under which optimal truss design problems can be formulated as convex optimization problems.

More recently, it has been shown in[1] that structural optimization problems can be formulated as semidefinite programs. Semidefinite programs are a generalization of linear programs which have many structural properties that can be exploited in their numerical solution. Specifically, semidefinite programs are problems that involve optimization of linear objective functions subject to the constraint that matrix variables are positive (semi)definite. Semidefinite programming formulations of several structural optimization problems are surveyed in.[4] An additional advantage of formulating structural optimization problems as semidefinite programs is that there are several high performance, freely available solvers for semidefinite programs, such as SeDuMi.[8]

The optimization problems that we present in this paper are dynamic extensions of the static formulations surveyed in.[4] We will show that, under certain conditions, minimization of peak power under dynamic loads is a convex optimization problem and can be formulated as a semidefinite program. Any gradient-based search, such as the BIGDOT algorithm used in Nastran's SOL 200,[10] is guaranteed to compute globally optimal solutions to these problems due to convexity. However, we claim that formulating these problems as semidefinite programs and using existing semidefinite programming solvers has certain advantages over using a generic gradient method. The BIGDOT algorithm is an exterior penalty function method, meaning that constraints are replaced with cost components in the objective function that penalize the objective when a constraint is violated. This results in an unconstrained minimization problem that can be solved by standard gradient methods. This algorithm can be applied very generally and is highly scalable to problems with many variables and constraints. However, semidefinite programs have significant structure that can be exploited in their solution. Most existing semidefinite programming solvers are based on customized primal-dual interior-point methods, rather than generic barrier methods.[9] Primal-dual algorithms seek to minimize the gap between the primal objective value and the objective value of the dual problem. Taking this approach has several advantages over primal algorithms, such as simple stopping criteria and simple computation of joint primal and dual search directions. As observed in,[9] these algorithms generally outperform barrier methods when applied to semidefinite programs, typically converging in fewer than 50 iterations even for large problem instances. Solutions to the examples presented in this paper were computed with the SeDuMi solver,[8] a freely-available semidefinite program solver for Matlab that uses primal-dual interior point methods.

II. Peak power minimization

In this section we will provide a formulation of the problem of truss design for peak power minimization. We will consider truss composed of linear elastic rods, where the rod cross sectional areas are our design variables. For given material parameters and geometry, we can use standard finite-element methods (as in[3] for example) to obtain the mass and stiffness matrices of the truss. When considering the mass and stiffness matrices as functions of individual rod cross-sectional areas, it turns out that the mass and stiffness matrices are both linear in the cross-sectional areas. That is, the mass and stiffness matrices of the truss can be expressed as

$$M(a) = a_1 M_1 + \cdots + a_n M_n$$

and

$$K(a) = a_1 K_1 + \cdots + a_n K_n,$$

where a_i is the cross-sectional area of the ith rod. Our goal will be to optimize the choice of $a_1, \ldots, a_n$ to minimize peak power delivered to the truss subject to additional constraints on the total structural mass.

For given $M(a)$ and $K(a)$ and a given applied force we will now provide an expression for peak power. This will be the objective that we will optimize over. Let $f(t)$ be the vector of forces applied to nodes of the truss, and let $v(t)$ be the vector of nodal velocities. We define the peak steady-state power delivered to the structure as

$$\limsup_{t\to\infty} |f(t)^T v(t)|.$$

Here we will consider general sinusoidal forces, which can be expressed as

$$\begin{aligned} f(t) &= \cos(\omega t) f_R + \sin(\omega t) f_I \\ &= \frac{1}{2}(e^{i\omega t} + e^{-i\omega t}) f_R - \frac{i}{2}(e^{i\omega t} - e^{-i\omega t}) f_I \\ &= \frac{e^{i\omega t}}{2}(f_R - i f_I) + \frac{e^{-i\omega t}}{2}(f_R + i f_I) \end{aligned}$$

The nodal velocities satisfy the second-order differential equation

$$M(a)v''(t) + K(a)v(t) = f'(t)$$

The solution to this equation is of the form $v(t) = \frac{e^{i\omega t}}{2}(v_R - i v_I) + \frac{e^{-i\omega t}}{2}(v_R + i v_I)$. Substituting this solution and equating real and imaginary parts gives

$$\begin{aligned} (K - \omega^2 M) v_R &= \omega f_I \\ (K - \omega^2 M) v_I &= -\omega f_R. \end{aligned}$$

Power is then computed as

$$
\begin{aligned}
f(t)^T v(t) &= \frac{e^{i2\omega t}}{4}\big((f_R^T v_R - f_I^T v_I) - i(f_R^T v_I + f_I^T v_R)\big) + \frac{e^{-i2\omega t}}{4}\big((f_R^T v_R - f_I^T v_I) + i(f_R^T v_I + f_I^T v_R)\big) \\
&\quad + \frac{1}{2}(f_R^T v_R + f_I^T v_I) \\
&= \frac{1}{2}(f_R^T v_R + f_I^T v_I) + \frac{1}{2}\cos(2\omega t)(f_R^T v_R - f_I^T v_I) - \frac{1}{2}\sin(2\omega t)(f_R^T v_I + f_I^T v_R)
\end{aligned}
$$

Assuming that $(K(a) - \omega^2 M(a))^{-1}$ exists, we can solve for v_R and v_I to obtain

$$
\begin{aligned}
v_R &= \omega(K - \omega^2 M)^{-1} f_I \\
v_I &= -\omega(K - \omega^2 M)^{-1} f_R
\end{aligned}
$$

In our formulation we will place the stronger requirement $(K(a) - \omega^2 M(a)) \succ 0$ to guarantee invertibility. Substituting the expressions for v_R and v_I into the expression for power gives

$$
\begin{aligned}
f(t)^T v(t) &= \omega\big(f_R^T (K - \omega^2 M)^{-1} f_I\big)\cos(2\omega t) + \frac{\omega}{2}\big((f_R + f_I)^T (K - \omega^2 M)^{-1}(f_R - f_I)\big)\sin(2\omega t) \\
&= C\sin(2\omega t + \theta),
\end{aligned}
$$

where

$$
C = \omega\sqrt{\frac{1}{4}\big((f_R + f_I)^T (K - \omega^2 M)^{-1}(f_R - f_I)\big)^2 + \big(f_R^T (K - \omega^2 M)^{-1} f_I\big)^2}
$$

So,

$$
\limsup_{t\to\infty} |f(t)^T v(t)| = C.
$$

We will consider the problem of minimizing C subject to the constraint that

$$
K(a) - \omega^2 M(a) \succ 0,
$$

together with a constraint on the total mass of the structure. Since C is positive and C^2 is monotonically increasing in C, we will work with the equivalent problem of minimizing C^2. If we let q be a vector of densities times lengths of each rod, $q^T a$ gives the mass of the structure. We would like to design a structure with minimum peak power subject to a constraint on the total mass of the structure. If m is the maximum allowable mass, this constraint is expressed as $q^T a \le m$. This leads to the peak power minimization problem:

$$
\begin{aligned}
\text{minimize:} \quad & \tfrac{1}{4}\big((f_R + f_I)^T (K(a) - \omega^2 M(a))^{-1}(f_R - f_I)\big)^2 + \big(f_R^T (K(a) - \omega^2 M(a))^{-1} f_I\big)^2 \\
\text{subject to:} \quad & q^T a \le m \\
& K(a) - \omega^2 M(a) \succ 0
\end{aligned}
$$

In the next section we will discuss computational algorithms for solving this problem.

III. Semidefinite programming formulation

In this section we will consider semidefinite programming formulations of the peak power minimization problem. At this point it is not clear whether the the general peak power minimization problem for any specific applied force can be equivalently posed as a semidefinite program. What we will show are the following:

- When all forces applied to the truss are in-phase, the problem of peak power minimization can be cast as a semidefinite program.
- We also consider the problem of minimizing worst-case peak power over all unit-magnitude forces. We show that this problem has an equivalent semidefinite programming formulation. This is true even when considering nodal forces that may not be in-phase.

III.A. Peak power minimization: forces in-phase

If all applied nodal forces are in-phase, we will show that the peak power minimization problem can be reduced to a semidefinite program and solved efficiently. All applied forces are in phase if and only if there exist some constants c_1 and c_2 such that $c_1 f_R = c_2 f_I$. Suppose, without loss of generality, that $c_2 \neq 0$. Then the objective function of the peak power minimization problem reduces to

$$\left(1 + 2\left(\frac{c_1}{c_2}\right)^2 + \left(\frac{c_1}{c_2}\right)^4\right)\left(f_R^T (K(a) - \omega^2 M(a))^{-1} f_R\right)^2$$

Under the constraint that $K(a) - \omega^2 M(a) \succ 0$, minimizing this objective is equivalent to minimizing

$$f_R^T (K(a) - \omega^2 M(a))^{-1} f_R.$$

By the Schur complement formula, the conditions $K(a) - \omega^2 M(a) \succ 0$ and $t > f_R^T (K(a) - \omega^2 M(a))^{-1} f_R$ are equivalent to

$$\begin{bmatrix} t & f_R^T \\ f_R & K(a) - \omega^2 M(a) \end{bmatrix} \succ 0$$

So, in the case $c_1 f_R = c_2 f_I$, the problem of minimizing peak power subject to a constraint on total structural mass is expressed as

$$\begin{array}{rl} \text{minimize:} & t \\ \text{subject to:} & q^T a \leq m \\ & \begin{bmatrix} t & f_R^T \\ f_R & K(a) - \omega^2 M(a) \end{bmatrix} \succ 0 \end{array}$$

If t^* is the optimal value of this semidefinite program, then $\frac{\omega}{2} t^*$ is the peak power for the force $f(t)$. In the next section, we will show a numerical example of this approach.

III.B. Worst-case peak power minimization

The formulation discussed previously optimizes a truss for a specific applied force. More generally, we could consider the problem of minimizing the worst-case peak power over a collection of applied forces. For example, we could minimize the worst-case peak power over all unit magnitude forces. That is, we could minimize peak power over all forces satisfying $f_R^T f_R + f_I^T f_I = 1$. It turns out, quite surprisingly, that this problem has a simple semidefinite programming formulation.

We will start out by showing that the the worst-case force applied to any truss has all nodal forces in-phase. To show this, suppose f_R^* and f_I^* are the worst-case force vectors applied to a truss with given cross-sectional areas a. We will show that f_R^* and f_I^* can always be chosen so that either $f_R^* = 0$ or $f_I^* = 0$.

We will suppose that we must have ${f^*}_R^T f_R^* > 0$ and ${f^*}_I^T f_I^* > 0$, and show that this leads to a contradiction. In this case, we can define

$$f_R = \frac{1}{\sqrt{{f^*}_R^T f_R^*}} f_R^*$$

and

$$f_I = \frac{1}{\sqrt{{f^*}_I^T f_I^*}} f_I^*.$$

Consider the applied force

$$f(t) = \frac{c_1}{2} f_R (e^{i\omega t} + e^{-i\omega t}) - \frac{i c_2}{2} f_I (e^{i\omega t} - e^{-i\omega t})$$

where $c_1^2 + c_2^2 = 1$. We will show that peak power is maximized with either $c_1 = 0$ or $c_2 = 0$. The square of peak power is

$$\frac{1}{4}\left(c_1^2 f_R^T (K(a)-\omega^2 M(a))^{-1} f_R - c_2^2 f_I^T (K(a)-\omega^2 M(a))^{-1} f_I\right)^2 + \left(c_1 c_2 f_R^T (K(a)-\omega^2 M(a))^{-1} f_I\right)^2$$

$$\begin{aligned}
&\leq \frac{1}{4}\left(c_1^2 f_R^T (K(a)-\omega^2 M(a))^{-1} f_R - c_2^2 f_I^T (K(a)-\omega^2 M(a))^{-1} f_I\right)^2 \\
&\qquad + (c_1^2 f_R^T (K(a)-\omega^2 M(a))^{-1} f_R)(c_2^2 f_I^T (K(a)-\omega^2 M(a))^{-1} f_I) \\
&= \frac{c_1^4}{4}\left(f_R^T (K(a)-\omega^2 M(a))^{-1} f_R\right)^2 + \frac{c_2^4}{4}\left(f_I^T (K(a)-\omega^2 M(a))^{-1} f_I\right)^2 \\
&\qquad + \frac{c_1^2 c_2^2}{2}(f_R^T (K(a)-\omega^2 M(a))^{-1} f_R)(f_I^T (K(a)-\omega^2 M(a))^{-1} f_I) \\
&= \frac{1}{4}\left(c_1^2 f_R^T (K(a)-\omega^2 M(a))^{-1} f_R + c_2^2 f_I^T (K(a)-\omega^2 M(a))^{-1} f_I\right)^2
\end{aligned}$$

The inequality above follows from the Cauchy-Schwarz inequality. The choice of c_1 and c_2 that maximizes this upper bound is $c_1 = 1$ if

$$f_R^T (K(a)-\omega^2 M(a))^{-1} f_R \geq f_I^T (K(a)-\omega^2 M(a))^{-1} f_I$$

and $c_2 = 1$ if

$$f_I^T (K(a)-\omega^2 M(a))^{-1} f_I > f_R^T (K(a)-\omega^2 M(a))^{-1} f_R$$

However, the upper bound is tight for this choice of c_1 and c_2. So, it is necessary that either $f_R^* = 0$ or $f_I^* = 0$.

When considering the worst-case peak power delivered to a truss with given cross-sectional areas a, it is sufficient to search for f_R that satisfies $f_R^T f_R = 1$ and maximizes

$$\frac{1}{4}\left(f_R^T (K(a)-\omega^2 M(a))^{-1} f_R\right)^2.$$

We would like to find the cross sectional areas that minimize this maximum. Since $f_R^T (K(a)-\omega^2 M(a))^{-1} f_R > 0$, we can equivalently search for cross sectional areas that minimize the maximum value of

$$f_R^T (K(a)-\omega^2 M(a))^{-1} f_R$$

over all f_R satisfying $f_R^T f_R = 1$. This is equivalent to minimizing t subject to

$$t > f_R^T (K(a)-\omega^2 M(a))^{-1} f_R$$

for all f_R satisfying $f_R^T f_R = 1$. This again is equivalent to minimizing t subject to

$$f_R^T \left(tI - (K(a)-\omega^2 M(a))^{-1}\right) f_R > 0$$

for all f_R, or equivalently

$$tI - (K(a)-\omega^2 M(a))^{-1} \succ 0.$$

Using the Schur complement formula we arrive at the equivalent semidefinite program

$$\begin{aligned}
\text{minimize:} \quad & t \\
\text{subject to:} \quad & q^T a \leq m \\
& \begin{bmatrix} tI & I \\ I & K(a)-\omega^2 M(a) \end{bmatrix} \succ 0
\end{aligned}$$

If t^* is the optimal value of this semidefinite program, then $\frac{\omega}{2}t^*$ is the worst-case peak power.

IV. Examples

In the following examples, we will optimize the cross sectional areas of the truss shown in Figure 1 for various optimization criteria. Each rod i is made from material with density $\rho_i = 1$ and Young's modulus $E_i = 2.5 \times 10^4$. Vertical and horizontal rods have a length of 1 meter, and diagonal rods have length $\sqrt{2}$ meters. In all cases we will look at applied forces with a frequency of $\omega = 15$ rad/sec.

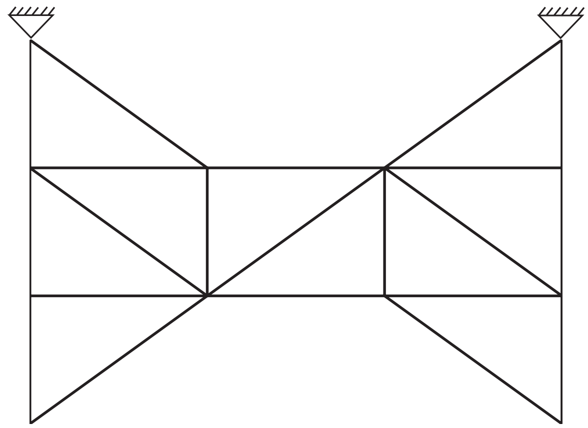

Figure 1. Uniform truss. All rods have equal cross-sectional areas.

IV.A. Peak power minimization: forces in phase

We will first consider the case when forces of $\frac{1}{\sqrt{2}} \cos(\omega t)$ are applied to two nodes of the truss, as shown in Figure 2. The optimal cross sectional areas for this force are depicted by line thicknesses in Figure 2. For this set of cross-sectional areas, the peak power is 0.007 Watts.

It is interesting to compare the peak power of the optimal truss with that of the uniform truss, where all rod cross-sectional areas identical. Suppose the forces $\frac{1}{\sqrt{2}} \cos(\omega t)$ is applied to the optimal and uniform trusses, as in Figure 2. For the uniform truss, the peak power is approximately 0.022 Watts, more than three times the value for the optimal truss.

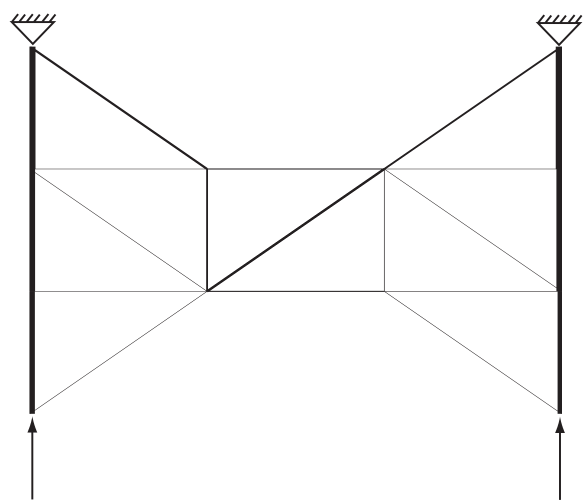

Figure 2. Optimal truss for the shown applied force.

IV.B. Worst-case peak power minimization

For worst-case peak power minimization, we consider the set of all applied forces with $f_R^T f_R + f_I^T f_I = 1$. Cross-sectional areas are chosen to minimize the worst-case peak power over all forces satisfying $f_R^T f_R + f_I^T f_I = 1$. The optimal cross sectional areas are depicted by line thicknesses in Figure 3. For this set of cross-sectional areas, the worst-case peak power is 0.175 Watts.

We will again compare the peak power of the optimal truss with that of the uniform truss, where all rod cross-sectional areas identical. For the uniform truss, depicted in Figure 1, the worst-case peak power is 0.525 Watts, approximately three times that of the optimal truss. We can also consider the peak power for a typical applied force. Now we will consider an applied force with components that are not in phase. We will apply the forces $\frac{1}{2}\cos(\omega t)$ as shown by the arrows marked "A" in Figure 4 and will apply the forces $\frac{1}{2}\sin(\omega t)$ as shown by the arrows marked "B". This force is chosen to model an unbalanced rotating load. For the optimal truss, the peak power associated with this force is approximately 0.041 Watts. For the uniform truss, the peak power is approximately 0.099 Watts, more than twice the value for the optimal truss.

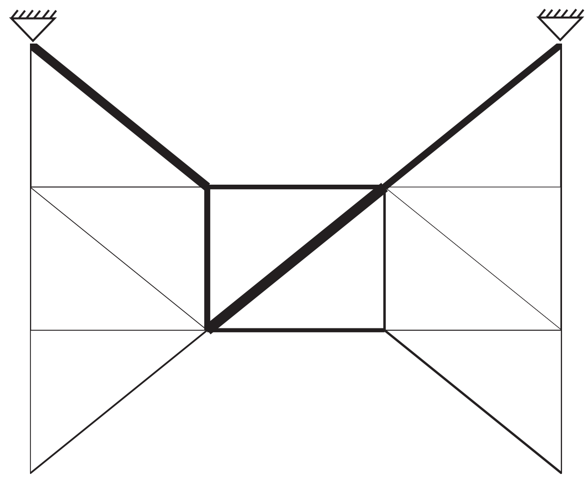

Figure 3. Optimal truss for worst-case peak power minimization.

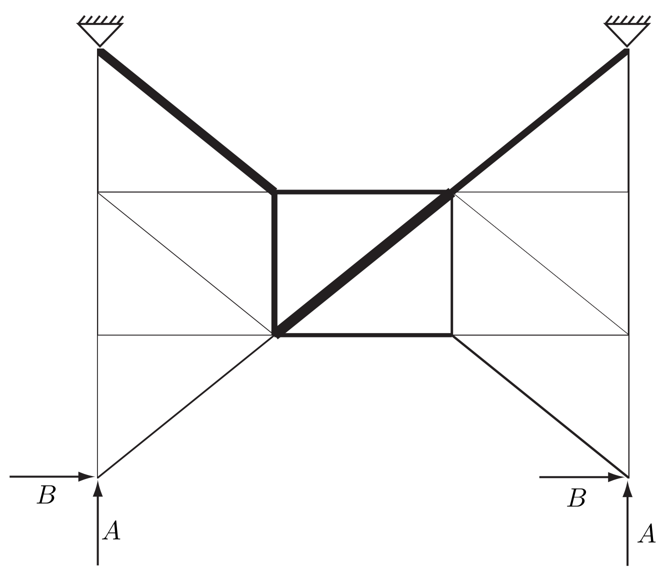

Figure 4. Forces applied to the optimal worst-case design.

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