Control Design for Topology-Independent Stability of Interconnected Systems

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Abstract

In this paper we consider the problem of designing scalable controllers for stabilizing collections of identical interconnected subsystems. Such controllers guarantee stability of the overall interconnected system even if we introduce new subsystems into the interconnection or change the system interconnection topology. We present sufficient linear matrix inequality conditions for the synthesis of scalable controllers.

1 Introduction

Many systems of practical interest can be modelled as large collections of interacting subsystems. Examples of such systems include electrical power distribution networks [8], data networks [12], and collections of vehicles travelling in formation [7]. Several practical issues arise when attempting to design controllers for such systems. Implementation of classical control schemes typically requires that each subsystem has access to the states or outputs of each other subsystem. This is often impractical or infeasible. Most practical control schemes for such systems are *decentralized*. That is, each subsystem uses only local information when making control decisions. Another practical issue is uncertainty in the subsystem interconnection topology. With electrical networks, failures of transmission lines or individual generators cause changes in the interconnection topology. With data networks, changes in the number of subsystems and the subsystem interconnection topology are common during normal operation. A collection of subsystems which is stable under one interconnection topology is not necessarily stable under other interconnection topologies. Effective controllers should guarantee stability of a collection of interconnected systems for a wide range of possible interconnection topologies. Also, the number of subsystems comprising the overall system is occasionally large enough to render control synthesis infeasible. Computation of controllers for Internet-sized data networks is clearly a task beyond the limits of most control synthesis procedures.

The issues discussed above will be the focus of this paper. We concentrate on the design of decentralized controllers which are provably scalable. Such controllers guarantee stability of the overall system even if we introduce new subsystems or change the subsystem interconnection topology. This means that we do not have to redesign our control laws as the complexity of the overall system increases.

2 Previous Work

The analysis and control of collections of interconnected systems has been widely studied in the literature. Early work on stability analysis and decentralized control of large-scale interconnected systems is found in [9, 11, 16, 15, 18]. A common theme in many of these works are decompositions which allow a stability analysis for the interconnected system to be performed at a subsystem level. Some of the more widely known stability criteria are the passivity related conditions of [14] and the small-gain related conditions of [2].

The well-known notion of *connective stability* found in [16] is similar in spirit to the concept of topology-independent stability discussed in this paper. An interconnected system possesses connective stability when stability is preserved after removing or weakening links from some given interconnection topology. Rather than considering how system stability changes when links are removed, we would like to consider how stability is affected by the addition of new subsystems into an existing interconnection structure. In this paper we present conditions which determine when stability of an interconnected system is independent of system scale, as well as interconnection topology. This results in a condition which guarantees stability for all topologies with some pre-specified bound on the system connectivity.

The synthesis procedures and stability conditions found in this paper are similar to those found in [1]. In that paper, the authors consider stability of an interconnected system formed by connecting an infinite string of identical subsystems. They prove stability

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using a decentralized Lyapunov function, and exploit shift-invariance in the resulting stability conditions to obtain a collection of uncoupled linear matrix inequalities.

3 Motivating Example

Although the results of this paper apply to general interconnected linear systems, here we will motivate the goals of this paper with an example. Consider the problem of stabilizing an electrical power distribution network [8]. In a later section we will use this as a basis for a numerical example illustrating the use of the methods developed in this paper.

A power distribution network consists of a collection of load-driving generators interconnected by transmission lines. The generator is a dynamic device, with a linearized model given by

$$\begin{aligned} \dot{\delta x}(t) &= A\delta x(t) + L\delta i(t) + Bu(t) \\ \delta v(t) &= C\delta x(t), \end{aligned}$$

where δi and δv are the current and voltage deviations from some operating point, and u is a control torque which can be applied to regulate the generator. For AC power networks δi and δv each typically have two components representing the real and imaginary parts of the current and voltage phasors. If we connect this generator to a load with admittance Y, then the voltage and current are related as $i(t) = Yv(t)$ and we can write the generator dynamics as

$$\dot{\delta x}(t) = (A + LYC)\delta x(t) + Bu(t).$$

In a realistic power distribution network there are several loads driven by multiple interconnected generators. If our system consists of N generators, then the current drawn from generator j in terms of the terminal voltages at each generator is

$$i_j(t) = Y_{jj}v_j(t) + \sum_{k\neq j} Y_{jk}(v_j(t) - v_k(t)),$$

where Y_{jj} is the admittance of the load connected to generator j and Y_{jk} is the admittance of the line connecting generators j and k. Note that we always have $Y_{ij} = Y_{ji}$. The linearized dynamic equations for the N interconnected generators are given by

$$\begin{aligned} \dot{\delta x}_j(t) &= \left(A + \sum_{k=1}^{N} LY_{jk}C\right)\delta x_j(t) \\ &\quad - \sum_{k\neq j}(LY_{jk}C)\delta x_k(t) + Bu_j(t) \end{aligned}$$

for $j = 1, \ldots, N$. Note that many of the interconnection admittances may be equal to zero since there may not be direct connections between many of the generators. The pattern of zero and nonzero admittances determines the interconnection topology of the network. The stabilization problem involves determining a control law for the torques u_j which ensures stability of the overall system. It is not practical to apply a centralized control law to stabilize this system since this would assume that each generator is capable of communicating information about its state to each other generator. The ideal control law is decentralized, so that each generator uses only local information to determine the appropriate control torque. Even if we find a controller which stabilizes the system for a particular interconnection structure, the system can still become unstable if one of the lines connecting two generators fails or if we connect additional generators to the network. We will return to this example in a later section and design a scalable controller which guarantees topology-independent stability.

4 Main Result

In this section we present linear matrix inequality conditions which, when feasible, produce a controller which stabilizes a collection of interconnected subsystems for arbitrary interconnection topologies. We will first present an analysis condition which proves stability of a collection of identical interconnected systems for arbitrary interconnection topologies. This condition is then extended to control synthesis procedures in the following subsections. A method for synthesizing perfectly decentralized controllers is presented at first. This method is then extended to synthesis of distributed controllers.

4.1 Analysis of Identical Interconnected Subsystems

Here we will consider systems formed by interconnecting a collection of identical subsystems by a directed graph. The interconnection structure is specified by a simple directed graph $G = (V, E)$, with N vertices $V = \{1, \ldots, N\}$ and edge set $E \subset V \times V$. Here *simple* means the graph has no self-loops, that is $(i, i) \notin E$ for all i. We say vertices i and j are *adjacent* if $(i, j) \in E$ or $(j, i) \in E$, and define the degree of vertex i as the number of vertices j adjacent to it. In terms of the graph adjacency matrix U, the *degree* of vertex i is

$$d_i(G) = \sum_{j=1}^{N}(1 - (1 - U_{ji})(1 - U_{ij})).$$

We define

$$d_{\max}(G) = \max_i d_i(G)$$

to be the maximum degree of any vertex of G.

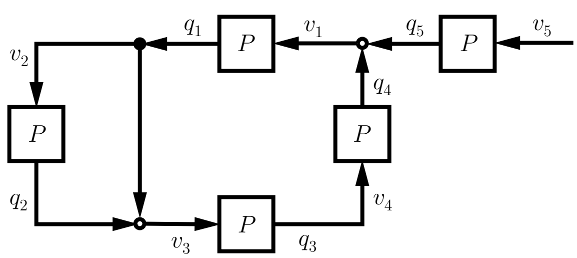
The subsystems are given in terms of state space realizations,

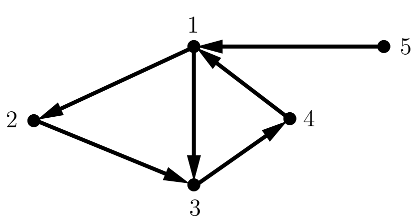
$$\begin{aligned}\dot{x}_i(t) &= Ax_i(t) + Lv_i(t)\\ q_i(t) &= Cx_i(t),\end{aligned}$$

each of which defines a linear map from signals v_i to q_i. These systems are interconnected according to

$$v_i(t) = \sum_{j=1}^{N} U_{ij} q_j(t). \tag{1}$$

Each subsystem corresponds to vertex i in the graph. We interpret edges as signals; all signals entering vertex i are summed to construct the input to system i. Similarly, all signals leaving a vertex are simply copies of the output of system i. This is illustrated in the figure below:





A consequence of interconnecting the systems via the graph G is that the dynamics of the subsystems becomes coupled. For a specific interconnection topology, the dynamics of the interconnected system can be expressed as

$$\dot{x}_i(t) = Ax_i(t) + \sum_{j=1}^{N} U_{ij} LC x_j(t)$$

for all $i = 1, \dots, N$. We can write the equations above as $\dot{x}(t) = \mathcal{A}_G x(t)$, where $\mathcal{A}_G = (I_N \otimes A) + (U \otimes LC)$ and the state vector x is formed by concatenating each of the subsystem state vectors. Here, I_N denotes the $N \times N$ identity matrix. The dynamics of the resulting interconnected system depend on the graph G. The following result determines when $\mathcal{A}_G$ is stable for any graph G such that $d_{\max}(G) \le d$:

Theorem 1. *Suppose there exists a solution $X \succ 0$ to the matrix inequalities*

$$\begin{bmatrix} A & dLC \\ dLC & A \end{bmatrix} \begin{bmatrix} X & 0 \\ 0 & X \end{bmatrix} + \begin{bmatrix} X & 0 \\ 0 & X \end{bmatrix} \begin{bmatrix} A & dLC \\ dLC & A \end{bmatrix}^T \prec 0,$$

$$\begin{bmatrix} A & dLC \\ 0 & A \end{bmatrix} \begin{bmatrix} X & 0 \\ 0 & X \end{bmatrix} + \begin{bmatrix} X & 0 \\ 0 & X \end{bmatrix} \begin{bmatrix} A & dLC \\ 0 & A \end{bmatrix}^T \prec 0.$$

Then $\mathcal{A}_G$ is stable for all G such that $d_{max}(G) \le d$.

Proof of this theorem is presented at the end of this subsection.

It is well known that for fixed G, $\mathcal{A}_G$ is stable if and only if there exists a matrix $\mathcal{X} \succ 0$ such that $\mathcal{A}_G \mathcal{X} + \mathcal{X} \mathcal{A}_G^T \prec 0$. A sufficient LMI condition for stability of $\mathcal{A}_G$ can be obtained if we restrict ourselves to an $\mathcal{X}$ of the form $\mathcal{X} = I_N \otimes X$. This restriction will allow us to prove stability of the interconnected system for multiple topologies. This restriction will also render the decentralized control problem computationally tractable, as shown in the next subsection. With the variable $\mathcal{X}$ restricted as such, the i, j block of the matrix $\mathcal{A}_G \mathcal{X} + \mathcal{X} \mathcal{A}_G^T$ is

$$\begin{aligned} AX + XA^T &\quad \text{for } i = j \\ U_{ij}(LC)X + U_{ji}X(LC)^T &\quad \text{for } i \ne j. \end{aligned}$$

The proof of Theorem 1 will involve relating properties of these blocks to negative definiteness of the matrix $\mathcal{A}_G \mathcal{X} + \mathcal{X} \mathcal{A}_G^T$. The following theorem is used in the proof of Theorem 1:

Theorem 2. *Let H be a Hermitian matrix partitioned into blocks H_{ij}, where $i, j = 1, \dots, N$. Let m_i be the number of nonzero off-diagonal blocks in row i of H. Suppose, without loss of generality, that each row has at least one nonzero off-diagonal block. If*

$$\begin{bmatrix} \frac{1}{m_i} H_{ii} & H_{ij} \\ H_{ji} & \frac{1}{m_j} H_{jj} \end{bmatrix} \succ 0$$

for all $i, j = 1, \dots, N$, $i \ne j$, then $H \succ 0$.

Proof. Let U be the $N \times N$ matrix such that for $i \ne j$, $U_{ij} = 1$ if $H_{ij} \ne 0$. Otherwise, $U_{ij} = 0$. Note that $U_{ii} = 0$ for all i. For any vector x,

$$\begin{aligned} x^* H x &= \sum_i^n x_i^* H_{ii} x_i + \sum_{i=1}^n \sum_{j>i} (x_i^* H_{ij} x_j + x_j^* H_{ji} x_i) \\ &= \sum_{i=1}^n \sum_{j>i} U_{ij} \left(\frac{1}{m_i} x_i^* H_{ii} x_i + \frac{1}{m_j} x_j^* H_{jj} x_j \right) \\ &\quad + \sum_{i=1}^n \sum_{j>i} U_{ij} (x_i^* H_{ij} x_j + x_j^* H_{ji} x_i) \\ &= \sum_{i=1}^n \sum_{j>i} U_{ij} \begin{bmatrix} x_i \\ x_j \end{bmatrix}^* \begin{bmatrix} \frac{1}{m_i} H_{ii} & H_{ij} \\ H_{ji} & \frac{1}{m_j} H_{jj} \end{bmatrix} \begin{bmatrix} x_i \\ x_j \end{bmatrix} \end{aligned}$$

Clearly, if

$$\begin{bmatrix} \frac{1}{m_i} H_{ii} & H_{ij} \\ H_{ji} & \frac{1}{m_j} H_{jj} \end{bmatrix} \succ 0$$

for all $i, j = 1, \ldots, N$, $i \neq j$, then $x^* H x > 0$ for all nonzero x, or $H \succ 0$. ■

We can now apply this result to prove Theorem 1:

Proof of Theorem 1. If the two matrix inequalities are satisfied, then

$$\begin{bmatrix} \frac{1}{d_1}(AX + XA^T) & (LC)X + X(LC)^T \\ (LC)X + X(LC)^T & \frac{1}{d_2}(AX + XA^T) \end{bmatrix} \prec 0 \quad (2)$$

and

$$\begin{bmatrix} \frac{1}{d_1}(AX + XA^T) & (LC)X \\ X(LC)^T & \frac{1}{d_2}(AX + XA^T) \end{bmatrix} \prec 0 \quad (3)$$

for all $1 \leq d_1, d_2 \leq d$. Let U be any adjacency matrix defining an interconnection topology such that each subsystem has degree less than or equal to d. Let d_i be the degree of subsystem i. The above matrix inequalities imply

$$\begin{bmatrix} \frac{1}{d_i}(AX + XA^T) & U_{ij}(LC)X + U_{ji}X(LC)^T \\ U_{ji}(LC)X + U_{ij}X(LC)^T & \frac{1}{d_j}(AX + XA^T) \end{bmatrix} \prec 0$$

for all $i \neq j$. To show this, consider the four possible cases: $(U_{ij} = U_{ji} = 0)$, $(U_{ij} = U_{ji} = 1)$, $(U_{ij} = 1, U_{ji} = 0)$, and $(U_{ij} = 0, U_{ji} = 1)$. The matrix inequality (2) clearly implies that the above inequality holds in the first two cases. The matrix inequality (3) clearly implies that the above inequality holds in the third case. The above inequality holds in the fourth case since we get

$$\begin{bmatrix} \frac{1}{d_j}(AX + XA^T) & X(LC)^T \\ (LC)X & \frac{1}{d_i}(AX + XA^T) \end{bmatrix} \prec 0$$

by permuting the blocks in (3). Since the above inequality holds for all $i \neq j$, this implies $\mathcal{A}\mathcal{X} + \mathcal{X}\mathcal{A}^T \prec 0$ by Theorem 2. Hence, the system with interconnection topology specified by U is stable. ■

Note that we only need the first inequality in Theorem 1 when considering interconnected systems where each of the links are bi-directional, *i.e.* the adjacency matrix U is symmetric. We only need the second inequality when considering interconnected systems where none of the links are bi-directional, *i.e.* $U_{ij} = 1$ if and only if $U_{ji} = 0$.

4.2 Decentralized Control Synthesis

At this point we extend the stability condition presented in the previous subsection to a procedure for designing decentralized controllers which guarantee topology-independent stability. Each subsystem now has a control input u_i:

$$\begin{aligned} \dot{x}_i(t) &= Ax_i(t) + Lv_i(t) + Bu_i(t) \\ q_i(t) &= Cx_i(t). \end{aligned}$$

The desired control law determines control inputs for each subsystem using only measurements of the local subsystem state. In general, there is no known computationally tractable procedure guaranteed to generate a decentralized controller for a linear system, given that one exists. Conservative approaches do exist, however. Here we can extend our stability analysis condition to a condition guaranteeing the existence of a stabilizing decentralized controller without adding any additional conservatism. This is because of the restricted form of the matrix $\mathcal{X}$ used to prove stability.

The desired control law is a decentralized state feedback control law where each local controller is identical. In other words, we would like to stabilize $\mathcal{A}_G + \mathcal{B}\mathcal{K}$ with a controller of the form $\mathcal{K} = I_N \otimes K$, where $\mathcal{B} = I_N \otimes B$. When there are no constraints on the structure of $\mathcal{K}$, the LMI approach to state feedback synthesis involves introducing a variable $\mathcal{Z} = \mathcal{K}\mathcal{X}$ and finding $\mathcal{Z}$ and $\mathcal{X} \succ 0$ such that

$$\mathcal{A}_G \mathcal{X} + \mathcal{X}\mathcal{A}_G^T + \mathcal{B}\mathcal{Z} + \mathcal{Z}^T \mathcal{B}^T \prec 0.$$

Upon finding such an $\mathcal{X}$ and $\mathcal{Z}$, we can construct a control law as $\mathcal{K} = \mathcal{Z}\mathcal{X}^{-1}$. Existence of a solution to these LMIs is equivalent to existence of a stabilizing controller. However, when the desired controller has special structure, there is no known equivalent LMI condition. This is because the resulting constraints on $\mathcal{X}$ and $\mathcal{Z}$ are typically non-convex. However, recall that for our stability condition we are restricting ourselves to an $\mathcal{X}$ of the form $\mathcal{X} = I_N \otimes X$. When restricting $\mathcal{X}$ to this form, we can make a change of variables $\mathcal{Z} = \mathcal{K}\mathcal{X}$, where $\mathcal{K}$ is of the desired form if and only if $\mathcal{Z}$ is of the form $\mathcal{Z} = I_N \otimes Z$. This provides a computationally tractable sufficient condition for synthesis of a stabilizing decentralized controller. With the variables $\mathcal{X}$ and $\mathcal{Z}$ restricted as such, the i, j block of the matrix

$\mathcal{A}_G\mathcal{X} + \mathcal{X}\mathcal{A}_G^T + \mathcal{B}\mathcal{Z} + \mathcal{Z}^T\mathcal{B}^T$ is

$$\begin{array}{ll} AX + XA^T + BZ + Z^TB^T & \text{for } i = j \\ U_{ij}(LC)X + U_{ji}X(LC)^T & \text{for } i \neq j. \end{array} \tag{4}$$

We can use this fact to obtain the following synthesis condition:

Theorem 3. *Suppose there exist solutions Z and $X \succ 0$ to the linear matrix inequalities*

$$\begin{array}{lll} \hat{A}_1\hat{X} + \hat{X}\hat{A}_1^T + \hat{B}\hat{Z} + \hat{Z}^T\hat{B}^T & \prec & 0 \\ \hat{A}_2\hat{X} + \hat{X}\hat{A}_2^T + \hat{B}\hat{Z} + \hat{Z}^T\hat{B}^T & \prec & 0 \end{array}$$

where

$$\hat{A}_1 = \begin{bmatrix} A & dLC \\ dLC & A \end{bmatrix}, \qquad \hat{A}_2 = \begin{bmatrix} A & dLC \\ 0 & A \end{bmatrix},$$

$$\hat{B} = \begin{bmatrix} B & 0 \\ 0 & B \end{bmatrix}, \; \hat{X} = \begin{bmatrix} X & 0 \\ 0 & X \end{bmatrix}, \; \hat{Z} = \begin{bmatrix} Z & 0 \\ 0 & Z \end{bmatrix}.$$

When such solutions exist, the decentralized control law $u_i(t) = Kx_i(t) = ZX^{-1}x_i(t)$ will stabilize $\mathcal{A}_G + \mathcal{B}\mathcal{K}$ for all G such that $d_{max}(G) \leq d$.

Proof of this theorem is not given since it is nearly identical to the proof of Theorem 1.

4.3 Distributed Control Synthesis

In the previous subsection we considered control schemes in which each subsystem determines control inputs based only on its own state. We can extend the previous control synthesis method to accommodate a wider class of control policies. In particular, we can synthesize distributed controllers which determine the control input for a subsystem based on the state of this subsystem, as well as the states of neighboring subsystems. The control input u_i is determined under such a control law as

$$u_i(t) = K_Sx_i(t) + \sum_{j=1}^{N} U_{ij}K_Ix_j(t).$$

In this setting, state information is shared according to an interconnection topology matching that of the subsystem interconnections. As with the perfectly decentralized case, each subsystem uses an identical control law. In the perfectly decentralized case we used a control law of the form $\mathcal{K} = I_N \otimes K$. In this case we use a control law of the form $\mathcal{K}_G = (I_N \otimes K_S) + (U \otimes K_I)$. Since $\mathcal{X}$ is restricted as $\mathcal{X} = I_N \otimes X$, we can make a change of variables $\mathcal{Z}_G = \mathcal{K}_G\mathcal{X}$, where $\mathcal{K}_G$ is of the desired form if and only if $\mathcal{Z}_G$ is of the form $\mathcal{Z}_G = (I_N \otimes Z_S) + (U \otimes Z_I)$. This leads to the following synthesis condition:

Theorem 4. *Suppose there exist solutions Z_S, Z_I, and $X \succ 0$ to the linear matrix inequalities*

$$\begin{array}{lll} \hat{A}_1\hat{X} + \hat{X}\hat{A}_1^T + \hat{B}\hat{Z}_1 + \hat{Z}_1^T\hat{B}^T & \prec & 0 \\ \hat{A}_2\hat{X} + \hat{X}\hat{A}_2^T + \hat{B}\hat{Z}_2 + \hat{Z}_2^T\hat{B}^T & \prec & 0 \end{array}$$

where

$$\hat{A}_1 = \begin{bmatrix} A & dLC \\ dLC & A \end{bmatrix}, \qquad \hat{A}_2 = \begin{bmatrix} A & dLC \\ 0 & A \end{bmatrix},$$

$$\hat{B} = \begin{bmatrix} B & 0 \\ 0 & B \end{bmatrix}, \qquad \hat{X} = \begin{bmatrix} X & 0 \\ 0 & X \end{bmatrix}$$

$$\hat{Z}_1 = \begin{bmatrix} Z_S & dZ_I \\ dZ_I & Z_S \end{bmatrix}, \qquad \hat{Z}_2 = \begin{bmatrix} Z_S & dZ_I \\ 0 & Z_S \end{bmatrix}$$

When such solutions exist, the distributed control law $u_i(t) = K_Sx_i(t) + \sum_{j=1}^{N} U_{ij}K_Ix_j(t)$ with $K_S = Z_SX^{-1}$ and $K_I = Z_IX^{-1}$ will stabilize $\mathcal{A}_G + \mathcal{B}\mathcal{K}_G$ for all G such that $d_{max}(G) \leq d$.

Again, proof of this theorem is not given since it is nearly identical to the proof of Theorem 1.

5 Numerical Example

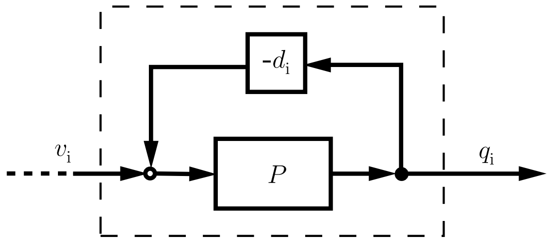
Here we will illustrate the ideas presented in the previous section with an example. Consider the power distribution network discussed in Section 3. Suppose that our network consists of a collection of N identical generators and loads connected by identical lines. As before, let U be the adjacency matrix describing the interconnection topology of the system. Since the admittance matrix is block symmetric, the adjacency matrix U will always be symmetric. We can write the dynamic equations as

$$\begin{aligned} \dot{\delta x}_i(t) &= \left(A + LY_1C + \left(\sum_{j=1}^{N} U_{ij}\right) LY_2C\right)\delta x_i(t) \\ &\quad - \sum_{j=1}^{N} U_{ij}(LY_2C)\delta x_j(t) + Bu_i(t) \end{aligned}$$

for $j = 1, \ldots, N$, where Y_1 is the load admittance and Y_2 is the line admittance. In order to put this problem in the framework of the previous section, we can write each subsystem as

$$\begin{aligned} &\dot{\delta x}_i(t) = (A + LY_1C + d_iLY_2C)\,\delta x_i(t) + Lv_i(t) + Bu_i(t) \\ &q_i(t) = -Y_2C\delta x_i(t). \end{aligned}$$

One way to think of such subsystems is as a generator/load subsystem with a gain of $-d_i$ placed in feedback:



Although we can now cast this problem in our framework, we now face the problem that each subsystem contains parameters which depend on the interconnection topology. We can easily get around this problem with a simple modification to our control synthesis procedure. Since d is the maximum degree for any subsystem, we can perform the synthesis procedure using subsystems of the form

$$\begin{aligned}\dot{\delta x}_i(t) &= (A + LY_1C + dLY_2C)\,\delta x_i(t) + Lv_i(t) + Bu_i(t)\\ q_i(t) &= -Y_2C\delta x_i(t).\end{aligned}$$

To synthesize a controller, we find $X \succ 0$ and Z such that

$$\begin{aligned}(A + LY_1C)X + X(A + LY_1C)^T + BZ + Z^TB^T &\prec 0\\ \hat{A}\hat{X} + \hat{X}\hat{A} + \hat{B}\hat{Z} + \hat{Z}^T\hat{B}^T &\prec 0,\end{aligned}$$

where

$$\hat{A} = \begin{bmatrix} (A{+}LY_1C){+}dLY_2C & -dLY_2C \\ -dLY_2C & (A{+}LY_1C){+}dLY_2C \end{bmatrix}$$

$$\hat{B} = \begin{bmatrix} B & 0 \\ 0 & B \end{bmatrix} \quad \hat{X} = \begin{bmatrix} X & 0 \\ 0 & X \end{bmatrix} \quad \hat{Z} = \begin{bmatrix} Z & 0 \\ 0 & Z \end{bmatrix}.$$

We can then show that the inequalities still hold for subsystems with $d_i < d$ by taking the appropriate conic combinations of the first inequality with the diagonal blocks of the second inequality.

Now we will compute a controller for a specific model. We will use the simple model given by

$$\begin{aligned}\frac{d}{dt}\begin{bmatrix} \delta\omega(t) \\ \delta\theta(t) \end{bmatrix} &= \begin{bmatrix} -0.5 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \delta\omega(t) \\ \delta\theta(t) \end{bmatrix}\\ &+ \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \delta i_R(t) \\ \delta i_I(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t)\\ \begin{bmatrix} \delta v_R(t) \\ \delta v_I(t) \end{bmatrix} &= \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} \delta\omega(t) \\ \delta\theta(t) \end{bmatrix},\end{aligned}$$

where $\delta\omega$ gives the deviation of the rotor angular velocity from some fixed operating condition and $\delta\theta$ gives the deviation of the rotor angle from some uniformly increasing reference angle. This system is marginally stable. That is, if the rotor angle drifts to some offset then it will remain at that offset. Suppose each generator drives a load with admittance $y_1 = 1 + 0.1i$ and can be connected by a transmission line with admittance $y_2 = 0.3 - i$. These admittances are represented in the model in matrix form as

$$Y_1 = \begin{bmatrix} 1 & -0.1 \\ 0.1 & 1 \end{bmatrix} \quad \text{and} \quad Y_2 = \begin{bmatrix} 0.3 & 1 \\ -1 & 0.3 \end{bmatrix}.$$

When we connect a single generator to a load, the resulting system becomes stable. However, the system actually becomes unstable when we connect a pair of generators driving loads by a single transmission line.

We can solve the synthesis LMIs for this system (with $d = 3$ chosen arbitrarily) to obtain the decentralized state feedback controller

$$K = \begin{bmatrix} -6.57 & -24.75 \end{bmatrix}.$$

This controller is guaranteed to stabilize any interconnection between any number of generators, as long as each generator is connected to no more than three other generators.

6 Conclusions

In this paper we addressed the problem of designing scalable controllers for collections of interconnected subsystems. We derived sufficient linear matrix inequality conditions for the existence of such controllers. This paper exclusively covered the case where all subsystems are identical. Although it is not discussed here, the methods of this paper may easily be extended to the case where subsystems of various types are interconnected.

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