Coupled stochastic dynamics of water table and soil moisture in bare soil conditions

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[1] The soil water content plays a fundamental role in a number of important environmental processes, including those involved in the water cycle, vegetation dynamics, soil biogeochemical cycles, and land-atmosphere interactions. Despite the recent efforts spent in the analytical modeling of the stochastic soil moisture dynamics in dryland ecosystems, the probabilistic characterization of the soil water balance in groundwater dependent environments is still missing, due to the complexity arising from the existence of shallow aquifers and capillary rise. This paper makes a first step in the direction of studying the stochastic soil water balance in groundwater dependent systems. An analytic probabilistic framework is developed for the modeling of soil moisture dynamics at the daily timescale for the case of bare soil conditions and in the presence of a shallow water table. This framework accounts for random rainfall occurrence, water table fluctuations, capillary rise, and soil evaporation. The stochastic soil water balance equation is analytically solved in the steady state, and the probability density functions of water table depth, soil moisture (as a function of depth), and evaporation are obtained and discussed for a variety of climate conditions and soil properties.


1. Introduction

[2] Capillary rise is the upward flow of water in the soil above the saturated zone [Bear, 1979; Hillel, 1998] and plays a relevant role in the soil water balance, especially in the presence of a shallow water table. Thus, capillary rise is a key feature of humid lands, where shallow phreatic aquifers are common [Mitsch and Gosselink, 2000], though it may be significant also in arid and semiarid regions in the presence of a shallow water table, for example in the riparian zones nearby rivers and lakes [Malanson, 1993; Naiman et al., 2005].

[3] Capillary rise is due to the capillary forces that establish in the pores of the unsaturated soil above the water table [e.g., Jury et al., 1991]. The two-phase (air-water) flow contributing to capillary rise and the related fluctuations of the water table are complex functions of water content and soil properties [Rawls and Brakensiek, 1989; Wosten et al., 2001]. Thus, the modeling of capillary rise and water table dynamics is a difficult task, which remains partly unresolved [Wosten et al., 2001]. For example, the supply of water to the soil in the form of infiltrating precipitation is a random process, which determines the stochastic fluctuations of water table depth and soil moisture. In recent years the stochastic character of the rainfall forcing has been explicitly accounted for in systems where the water table is either absent or has only a negligible influence on soil moisture dynamics [Laio et al., 2001; Rodriguez-Iturbe and Porporato, 2004]. Since in these systems the supply of moisture (from below) by capillary rise can be ignored, the stochastic soil water balance can be studied with existing models of vertically integrated [Laio et al., 2001; Porporato et al., 2002], or vertically explicit soil moisture dynamics [Laio, 2006]. In this context a number of exact results have been obtained to express the dependence of the probabilistic characteristics of the soil moisture dynamics on climate, vegetation and soil properties.

[4] Starting from the seminal work by Eagleson [1978], a number of remarkable studies have investigated systems in which water table and capillary rise play an active role in the stochastic dynamics of soil moisture. For example, Salvucci and Entekhabi [1994] used a numerical approach to study the fluxes and stocks of soil moisture at different timescales and for different water table positions. Salvucci and Entekhabi [1995] investigated the coupled unsaturated and saturated subsurface flows throughout a hillslope. Bierkens [1998] studied the water table dynamics introducing stochasticity in the water balance as a generic noise term. Kim et al. [1999] used a mixed analytical-numerical approach to investigate soil moisture patterns along a hillslope. Despite these important contributions to the study of the interactions between saturated and unsaturated zones under stochastic forcing, an analytical framework to determine the probability density functions of soil moisture and water table depth is still missing. The availability of such a framework could enhance our understanding of important
processes in humid land ecohydrology [Rodriguez-Iturbe et al., 2007], including our ability to explore the relations existing among water table fluctuations, capillary rise, and soil water content. To this end, we provide an analytical description of the stochastic soil-water balance at the daily timescale and for periods of time in which the climate parameters can be considered constant. In this study we will concentrate on the case of a homogeneous bare soil, while we will address in a separate paper the case of vegetated soils.

2. Water Balance of the Saturated Zone

[5] We consider a unit area of bare, homogeneous soil and indicate with $y$ the position of the water table, with $y$ being negatively valued. Here we define as water table the surface marking the transition between saturated and unsaturated soil. The variations in time of the water table depth can be expressed by the stochastic water balance equation

$$\beta(y) \frac{dy}{dt} = -v(y) + I(t) + f(y).$$

(1)

where the “inertia” term, $\beta(y)$, is related to the specific yield of the aquifer (see section 2.4), while on the right-hand side one finds the incoming and outgoing fluxes to the saturated zone: $v(y)$ is the exfiltration rate due to the combined effect of evaporation and capillary rise, modeled as explained in section 2.2; $I(t)$ represents the recharge rate associated with precipitation, as discussed in section 3; $f(y)$ represents the groundwater flow, from a lateral water body. $f(y)$ can be a positive (inflow) or negative (outflow) term, depending on whether $y$ is deeper or shallower than the lateral water body, respectively. Details on the manner how $f(y)$ will be modeled is provided in section 3.

[6] The aim of this paper is to solve equation (1) in the presence of a stochastic rainfall forcing $I(t)$. The main difficulty while dealing with equation (1) arises from the necessity to correctly specify the terms $v(y)$ and $\beta(y)$, which depend on $y$ and on the soil water content in the whole soil column above the water table. Thus, the solution of equation (1) requires a preliminary study of the processes regulating the water movement in the unsaturated portion of the soil profile. To this end, in section 2.1 we determine the steady state profile of soil moisture as a function of $y$. This result is then used in the modeling of exfiltration rate, $v(y)$ (section 2.2) and “inertia” coefficient $\beta(y)$ (in section 2.4).

2.1. Steady State Soil Moisture Profile

[7] The water motion in an unsaturated media is described by the well-known Richards’ equation [Richards, 1931; Jury et al., 1991; Dingman, 1994]

$$\frac{\partial \theta}{\partial t} - \frac{\partial}{\partial z} \left[ K(\theta) \left( \frac{\partial \psi}{\partial z} + 1 \right) \right].$$

(2)

where $z$ is the vertical, upward oriented coordinate $z$ with origin at the ground surface, $t$ is time, $\theta$ is the volumetric soil moisture (equal to the ratio between the water volume and the soil volume), $\psi(\theta)$ is the soil matrix potential, and $K(\theta)$ is the unsaturated hydraulic conductivity.

[8] The solution of Richards’ equation requires that two empirical relations, $K = K(\theta)$ and $\psi = \psi(\theta)$, are specified to relate the soil matrix potential and the hydraulic conductivity to the soil water content. The parametrization of these functions represents a difficult task [Wosten et al., 2001]. In fact, they do not depend only on the textural classes (i.e., the percentages of sand, silt, and clay) but on several other soil properties, including soil organic matter content, macroporosity, soil crust, and water repellency [Rawls and Brakensiek, 1989]. The experimental determination of these functions is also prone to uncertainty, in particular at low soil water contents. These factors explain both the variety of mathematical expressions proposed for the parametric representation of these functions, and the remarkable dispersion of the experimental data [Wosten et al., 2001; Cornelis et al., 2005].

[9] We adopt the model by Brooks and Corey [1964], which is simple, requires only two parameters, and has been commonly used and parameterized for a broad variety of soils. The simple mathematical structure of these functions will allow us to resolve analytically the stochastic soil moisture dynamics. By assuming the residual water content equal to zero, the Brooks and Corey [1964] model reads

$$\psi(s) = \psi_1 \left( \frac{1}{s} \right)^{1/m} \quad (0 \leq s \leq 1).$$

(3)

where $s = \theta/n$ is the relative soil moisture content, $n$ is the soil porosity, $\psi_1(<0)$ is the air-entry value of $\psi$ (or “bubbling pressure head”), and $m$ is the pore size index. We set to zero the residual water content because it does not contribute to water motion, it is not involved in the water balance, it is generally very small, and it is often not even defined as a measurable value [Wosten et al., 2001, p. 126]. The bubbling pressure, $\psi_1$, represents the minimal suction necessary to allow the entrance of air in a saturated porous media. As the soil remains saturated as long as $\psi > \psi_1$, we take the horizontal surface at the bubbling pressure, $\psi = \psi_1$, as the separation between the saturated and unsaturated zone, and we define this surface as the water table. In our modeling framework the water table is therefore positioned at a depth $|y|$ which is located at a height $|\psi_1|$ above the phreatic surface (where $\psi = 0$).

[10] For the hydraulic conductivity we adopt the power law representation [e.g., Clapp and Hornberger, 1978; Hillel, 1998]

$$K(s) = K_1 s^c = K_1 \left( \frac{\theta}{n} \right)^c = \frac{2 + 3m}{m},$$

(4)

where $K_1$ is the conductivity in saturated conditions.

[11] Table 1 reports the outcomes of two detailed experimental investigations by Cosby et al. [1984] and Rawls and Brakensiek [1989] on the value of the parameters of equations (3) and (4). We notice that, despite the detailed classification of the soil texture provided by these authors, their results exhibit a broad variability, and the parameter estimates reported in these studies differ by 50–100%, further indicating that the parameter estimation for soil hydraulic properties is a difficult task. Unless it is otherwise specified, in what follows all the examples will be based on the mean values of the parameters provided by Rawls and Brakensiek [1989].
where $n$ is the steady, $z$-independent, exfiltration rate, which is positive (i.e., upward oriented), when the pressure gradient, $\frac{\partial \psi}{\partial z}$, exceeds the gravity term.

[13] Considering the boundary condition $s_{\text{surf}} = 1$ (hence, using equation (3), with $\psi_{s=1} = \psi_1$), the differential equation (5) can be further integrated leading to the analytical solution

$$y = z - \psi_1 F_1\left[1 + \frac{1}{2 + 3m}, 1 + \frac{1}{2 + 3m} - \frac{\psi_1(y)}{K_1}, \psi_{s=0} \right],$$

where $F_1[\cdot; \cdot; \cdot]$ is the hypergeometric function [Abramowitz and Stegun, 1965]. Our exact solution presented in equation (5) has been obtained using the functions (3) and (4). Other analytical steady state solutions can be found in the literature for different functions [e.g., Gardner, 1958; Salvucci, 1993]. The relation between $y$ and soil moisture can be made explicit inserting equation (3) in (5).

In the particular case when $\psi(y) = 0$, i.e., when there is an equilibrium between gravity and capillary tensions due to the soil moisture gradient, equation (5) has the well-known solution

$$y = z + (\psi - \psi_1) \rightarrow \psi = (y - z) + \psi_1,$$

i.e., $\psi$ is linearly dependent on $z$. Notice that the variables $y$, $z$ and $\psi_1$ in (7) are all negative valued.

[15] Equation (5) is a rather complicated relation between four variables: the water table depth $y$, the exfiltration rate $\psi(y)$, the depth $z$, and the matrix potential $\psi$ (or soil moisture level $s(z)$) at the depth $z$. Despite its complicated structure, equation (5) allows one to grasp some important features of the exfiltration process. For example, Figure 1 shows for some soil types how the exfiltration rate depends on the soil moisture content at the ground level, $s_{\text{surf}} = s_{z=0}$, and on the water table depth, $y$. It is here assumed that there is no limit in the capability of the atmosphere to evaporate moisture from the ground surface (this aspect will be considered in the next subsection). Two aspects emerge: (i) when the water table is shallow (i.e., $y$ is of the order of tens of centimeters), capillary rise is able to sustain high exfiltration rates, which are comparable to the typical rates of soil evaporation allowed by atmospheric conditions; (ii) apart from the case of sands, for the other types of soil the exfiltration velocity reaches its asymptotic value for relatively high values of $s_{\text{surf}}$.

[16] Other interesting considerations can be made if we determine the minimum water table depth at which the effect of capillary rise can be considered negligible for the purposes of soil water balance calculation. To this end, we set $z = 0$, $s_{\text{surf}} = s_{\text{fc}}$, with $s_0$ being the soil field capacity (that we take as the value of $s$ corresponding to $\psi = \psi_{s=0} = -330$ cm),
and \( v(y) = 0.05 \) cm/d as a sensible limit value for negligible exfiltration rates. Using equation (5) one finds a limit value for the water table depth in the sense that with shallower water tables the effect of exfiltration/capillary rise on the soil water balance is no longer negligible. Figure 2 shows how \( y \) depends on the soil properties, by using both the Rawls and Brakensiek [1989] and Cosby et al. [1984] parameterizations of the soil hydraulic properties. These results show that the exfiltration fluxes can be neglected only when the water table is deeper than 50–150 cm for a broad class of soil textures. Moreover, great differences (both quantitative and qualitative) emerge from the use of different parameterizations of the same functions of the hydraulic properties, demonstrating how the results are extremely sensitive to the soil water retention properties.

### 2.2. Evaporation Rate From the Water Table

[17] The analytical solution (5) allows one to determine the exfiltration rate, \( v(y) \), once soil parameters, water table depth, and soil moisture at the soil surface are given. However, it is important to recognize that this \( v(y) \) value represents a potential exfiltration rate, that can be different from the real one [e.g., Eagleson, 1978; Salvucci and Entekhabi, 1994]. In fact, the atmosphere may not be able to sustain large evaporation rates, in that the atmospheric evaporativity depends on solar irradiance, air humidity, air temperature, wind speed, and other atmospheric variables. Because we focus on timescales greater than the day, we neglect the diurnal fluctuations of these variables and indicate with \( E_{\text{atm}} \) the daily mean value of evaporativity typical of a given season. Exfiltration velocities larger than \( E_{\text{atm}} \) can be sustained only for short periods: in fact, if at a given instant \( v(y) > E_{\text{atm}} \), the surface soil layers become moister, and the exfiltration rate decreases until the condition \( v(y) = E_{\text{atm}} \) is reached.
Two regimes exist in the relation between water table depth and exfiltration rate, depending on whether evaporation is limited by the atmosphere (i.e., with shallow aquifers) or by the supply of moisture to the surface by capillary rise (i.e., with deep water tables). In the first case, the shape of the soil moisture profile in the unsaturated zone is dictated by the condition \( v(y) = E_{atm} \) and the value of \( s_{surf} \) is obtained from equation (5) by setting \( z = 0 \) and \( v = E_{atm} \) and then solving (numerically) for \( s \). As \( y \) decreases the soil moisture at the surface decreases because stronger gradients are needed to maintain the exfiltration rate equal to \( E_{atm} \).

When \( s_{surf} \approx 0 \) (i.e., \( |v| \) is very large), a further decrease in water table elevation leads to a reduction of the gradient, hence of the exfiltration rate. In this second regime, the exfiltration rate (denoted with \( v_{lim}(y) \)) is no more limited by the atmosphere but by the water table depth and corresponds to the maximum water flux sustainable by capillary rise for a given water table depth \( y \). In conclusion, the actual exfiltration flux, \( v(y) \), is given by

\[
v(y) = \min[v_{lim}(y), E_{atm}].
\]

Figure 3 shows the behavior of \( v = v(y) \) for some soil types. The threshold depth, \( y_{lim} \), separating the two regimes is obtained by (5) with \( v = E_{atm} \) and \( \psi(z = 0) = -\infty \). With some algebra [Abramowitz and Stegun, 1965] one obtains

\[
y_{lim} = -\psi_1 F_1 \left\{ \frac{1}{2 + 3m}, 1, 1 + \frac{1}{2 + 3m} - \frac{E_{atm}}{K_1} \right\} + \psi_1 \Gamma \left\{ 1 + \frac{1}{2 + 3m} \right\} \Gamma \left\{ 1 - \frac{1}{2 + 3m} \right\} \left( \frac{K_1}{E_{atm}} \right)^{\frac{1}{3m}},
\]

where \( \Gamma [\cdot] \) is the Gamma function. It is useful to observe that once \( \psi(z = 0) \approx -330 \text{ cm} \) – i.e., soil moisture at the surface is equal to field capacity – a further decrease of \( \psi(z = 0) \) (i.e., a reduction of soil moisture) does not entail a significant increase of the exfiltration rate. This is consistent with Figure 1 and implies that the depth corresponding to \( y = E_{atm} \) and \( s_{surf} = s_c \) is very close to \( y_{lim} \). Figure 3 shows also the complexity of the link between soil texture, water table depth, and exfiltration rate: different soils can produce similar exfiltration rates. In fact, fine soil textures promote capillary rise but, on the other hand, they reduce the unsaturated hydraulic conductivity.

The relation between \( v_{lim} \) and \( y \) is obtained from (8), by introducing \( y \) and \( v_{lim} \) in place of \( y_{lim} \) and \( E_{atm} \), respectively,

\[
y = -\psi_1 F_1 \left[ \frac{1}{2 + 3m}, 1, 1 + \frac{1}{2 + 3m} - \frac{v_{lim}}{K_1} \right]
\]

and solving (numerically) for \( v_{lim} \).

Some explicit expressions for the relation between \( v_{lim} \) and \( y \) are found in the literature [e.g., Ripple et al., 1972; Warrick, 1988; Eagleson, 1978; Salvucci, 1993]. However, these relations are obtained using different parameterizations of the hydraulic relationships \( K = K(y) \) and \( \psi = \psi(s) \), which are not consistent with our solution (5). Moreover, these approximations have been found to lose accuracy when the water table is very shallow. A good approximation of the relation between \( y \) and \( v_{lim} \), which is suitable for our purposes, is

\[
v_{lim} = E_{atm} \left( \frac{y_{lim}}{y} \right)^a.
\]

The only free parameter is the exponent \( a \), which can be estimated as

\[
a = \frac{\ln 0.4}{\ln [y_{lim}] - \ln [y_0.4]},
\]

where \( y_{0.4} \) is the water table depth obtained from equation (9) with \( v_0.4 = 0.4 E_{atm} \). The choice of the value 0.4 is dictated by the optimization of equation (11) throughout different types of soil. This manner of estimating the \( a \) coefficient guarantees that the relations in equations (9) and (11) have a very similar shape, since they are forced to cross three common points: \( y_{lim} = E_{atm} \) for \( y = y_{lim} \), \( v_{lim} = 0.4 E_{atm} \) for \( y = y_{0.4} \), and \( v_{lim} \to 0 \) for \( y \to -\infty \). Figure 4 shows an example of how equations (11)–(12) may provide a good approximation of the relation \( v_{lim} = v_{lim}(y) \) obtained numerically from (9).

### 2.3. Approximated \( s(z) \) Profile in the Zero Flux Case

In spite of the fact that the solution (5) is analytical, its mathematical complexity hampers some calculations in closed form. This fact prevents an entirely analytical description of the stochastic dynamics. However, these mathematical difficulties are significantly reduced if the vertical profile of soil moisture is approximated with the profile corresponding to zero-flux conditions (i.e., \( v = 0 \)), which is obtained from equations (3) and (7) as

\[
s = \left( \frac{\psi}{(y-z)+\psi} \right)^m.
\]

Figure 5 shows some comparisons between the vertical profile of soil moisture obtained with equation (5), with exfiltration velocity given by equation (8), and the approximated profile provided by (13). One observes that the differences between the two profiles increase when the depth \( y \) increases, the soil texture becomes finer, or the
ground surface is approached. In particular, the strong reduction in soil moisture content in the upper soil layers is due to the need of having a stronger gradient to sustain the flux $v(y)$ despite the reduction in hydraulic conductivity in dry soils. To evaluate quantitatively the difference between the two profiles, we have considered the error measure,

$$\text{Err} = \frac{\int_0^R_y s(z) \, dz}{\int_0^R_y s_v(z) \, dz},$$

where $s(z)$ is the soil moisture profile, obtained from (5) and $s_v(z)$ its zero-flow approximation. The $\text{Err}$ values remain smaller than 5–7% for a variety of soil textures, with maximum deviations in correspondence to $y = y_{\text{lim}}$. In fact, when $y = y_{\text{lim}}$, $s_{\text{surf}} \approx 0$ and strong differences exist with respect to the case $v = 0$.

With deeper water tables (i.e., $y < y_{\text{lim}}$), the exfiltration rate, $v(y)$, decreases and the difference between the two curves becomes smaller. These overall small error values justify the use of the approximated soil moisture profile obtained with (13) for the case $v = 0$.

### 2.4. Specific Yield

[24] One of the key points to investigate how capillary rise affects the water balance is to observe that the water volume contained in the unsaturated portion of the soil column depends on the position of the water table. This observation leads to the definition of the specific yield [Freeze and Cherry, 1979]

$$S_y = \frac{h_w}{\Delta},$$

which is the ratio between the volume of water released from storage per unit cross-sectional area of the aquifer, $h_w$, and the correspondent decline in the elevation of the water table, $\Delta$.

[25] In the evaluation of the specific yield, we do not take into account the time-dependence in the dynamics of the soil moisture profile [e.g., Nachabe, 2002; Hilberts et al., 2007], namely we assume that the profile evolves as a succession of instantaneous steady states described by equation (5). In the course of the exfiltration process this assumption is justified by the slowness of evaporation and by the fact that the process evolves with continuity, without

![Figure 4](image1.png)

**Figure 4.** Comparison between the relation, $v = v(y)$, expressed implicitly by equation (10) (dashed lines) and the approximation given by (11) with the exponent obtained from (12) (continuous lines). Upper and lower curves refer to loam and sand, respectively.

![Figure 5](image2.png)

**Figure 5.** Comparisons between the vertical profiles of soil moisture given by (5) (dashed lines) and by the zero-flux approximation (13) (continuous lines). Three water table depths ($y = −20$, $−40$, and $−100$ cm) are considered for a sandy soil (a) and a loamy soil (b).

![Figure 6](image3.png)

**Figure 6.** (a) Dependence of the specific yield $S_{0,y}$ on the water table depth for a sandy soil (upper lines) and a loamy soil (lower lines). Dashed lines are obtained with (5), while the continuous lines correspond to the zero-flux assumption (i.e., equation (16)). (b) Behavior of the function $\beta = \beta(y)$ according to the solution (5) (dashed lines) and the zero flux assumption (equation (19) (continuous lines) for a sandy soil (upper lines) and a loamy soil (lower lines).
abrupt jumps in the position of the water table. When the water table raises in response to precipitation, the use of the steady state profile of soil moisture after a rainfall event is instead justified by our choice to resolve the process at the daily timescale (see the next section).

[26] Several expressions for the specific yield have been formulated [e.g., Duke, 1972; Nachabe, 2002], but none of them is fully consistent with our modeling framework. We therefore evaluate $S_y$ according to its definition (14), using the soil moisture profile that can be obtained through the numerical solution of (5) with the exfiltration rate given by (8). Parallel to this rigorous (though numerical) approach, we also consider the specific yield calculated with (14) assuming that $v = 0$. In this case it is possible to obtain the analytical expression of the specific yield when the water table moves from a depth $y_1$ to a depth $y_2$.

$$S_y = \frac{n}{\Delta(m-1)} (m-1)\Delta - \psi_y \left[ (\psi_1 + y_1)^{1-m} - (\psi_1 + y_2)^{1-m} \right]$$

(15)

with $\Delta = [y_2 - y_1]$. Equation (15) coincides with the result by Nachabe [2002], who assumed the water table to be located at a depth where $\psi = 0$ (instead of $\psi = \psi_1$). In particular, if the initial position of the water table coincides with the soil surface (i.e., $y_1 = 0$), one obtains

$$S_y = \frac{n}{(1-m)y} \left[ (1-m)y - \psi_1 + \psi_y (y + \psi_1)^{1-m} \right].$$

(16)

[27] Figure 6a shows the behavior of $S_{y,v}$, normalized by the soil porosity, calculated using both the rigorous procedure and the zero-flux approximation (i.e., equation (16)). One observes that the specific yield is always smaller than the soil porosity, and that the difference between the two curves increases in the case of shallow water tables or fine textures. Moreover, Figure 6a shows (i) that the specific yield tends to $n$ only in the case of deep water tables, (ii) how small volumes of exfiltrating/infiltrating water correspond to relatively large fluctuations of shallow water tables [e.g., Gillham, 1984], and (iii) that the rigorous methodology and the zero-flux approximation give similar results, consistently with the outcome of section 2.3.

[28] Although these differences are usually irrelevant to the calculation of the specific yield, they may play a significant role in the modeling of the water table and soil moisture dynamics. In fact, modeling the water table dy-
namics requires that one calculates the variation of the water table depth, \( \text{dh} \), induced by the exfiltration (or infiltration) of an infinitesimal amount of water, \( \text{dh}_w \). The relationship between the two quantities is

\[
\beta(y) \text{dy} = \text{dh}_w \quad \rightarrow \quad \beta(y) = \frac{\text{dh}_w}{\text{dy}}
\]

and since \( h_w = S_0 \cdot y \), the link between \( \beta(y) \) and the specific yield is

\[
\beta(y) = S_0 \cdot y + y \frac{\partial S_0}{\partial y}
\]

The function \( \beta(y) \) can be evaluated either numerically through the solution (5) or analytically through \( S_0 \cdot y \) under the zero-flux assumption

\[
\beta_{\nu=0}(y) = \frac{\text{dh}_w}{\text{dy}} = (S_0 \cdot y)_{\nu=0} + y \frac{\partial(S_0 \cdot y)}{\partial y} = n \left[ 1 - \left( \frac{\psi_l}{y + \psi_l} \right)^m \right]
\]

where \( n = 1 - s_{\nu=0}(0) \).

Figure 6b shows some examples of the function \( \beta(y) \), evaluated both rigorously and by using equation (19). We observe that a discontinuity at \( y = y_{lim} \) appears in the curves obtained from the numerical solution of equation (5), due to the abrupt change in the derivative of \( \nu_{lim} \) at \( y = y_{lim} \) (see Figure 3). The differences between the two methods are stronger near \( y = y_{lim} \) or when the soil is finer, and they slowly decrease for \( y < y_{lim} \), as \( \nu \) tends to zero.

3. Stochastic Dynamics

The previous section has elucidated how the water content in the unsaturated soil is closely related to the position of the water table. For this reason, we first study the stochastic dynamics of the water table depth, \( y \), and then investigate the dynamics of soil moisture, \( s \). Finally, the probabilistic characteristics of evaporation, \( v(y) \), will be determined.

3.1. Water Table Probabilistic Dynamics

To study the dynamics of the water table we consider again the water balance equation (1). The terms \( \beta(y) \) and \( v(y) \) have been defined in the previous section. The remaining terms represent the input due to rainfall \( I(t) \), and the lateral flow \( J(y) \). The rainfall input, \( I(t) \), is modeled as a stochastic process assuming that: (i) rainfall occurrences are a series of point events in continuous time, while the finite duration of each event is neglected; (ii) the processes of infiltration and redistribution of rainwater [e.g., Jury et al., 1991] occur instantaneously. Both assumptions are justified by the fact that we model the soil water balance at the daily timescale [Laio et al., 2001; Laio, 2006] without resolving the infiltration and redistribution processes. Thus, we assume that, when a rainfall event occurs, the water table and the soil moisture profile adjust instantaneously to a new configuration given by the steady solution (5) corresponding to the water input imposed by the rainfall pulse.

The precipitation input, \( I(t) \), is modeled as a marked Poisson process [e.g., Cox and Miller, 1965], i.e., as temporal sequence of rainfall events

\[
I(t) = \sum_i h_i \delta(t - \tau_i) \quad i = 1, 2, 3, \ldots
\]

where \( h_i \) are the random depths of the rainfall events, \( \delta(\cdot) \) is the Dirac delta function, and \( \tau_i \) are the arrival times of a stationary Poisson process with rate \( \lambda \) [Laio et al., 2001]. The probability density function of the rainfall depths, \( h_i \), is assumed to be exponential with mean \( \alpha \).

The lateral flow to or from an external water body is here modeled using a linear reservoir approximation

\[
f_i(y) = k_i(y_0 - y - \psi_i)
\]

where \( k_i \) is a parameter depending on the soil properties and the distance to the water body, and \( y_0 \) is the elevation of the water surface in the external water body. The bubbling pressure \( \psi_i \) in equation (21) accounts for the fact that the flux is zero when the stage in the external water body is the same as the elevation of the phreatic surface, which in our modeling framework is located at a depth \( |\psi_i| \) below the water table. Notice how the presence of the water body imposes a bound to the maximum depth of the water table. In fact, \( y \) has a minimum value, \( y_{min} \).
corresponding to the balance between later flow and exfiltration under no rainfall conditions

\[ f_l(y_{\text{min}}) = v_{\text{lim}}(y_{\text{min}}). \] \hspace{1cm} (22)

[35] Under the previous assumptions, the balance equation (1) is a stochastic differential equation driven by multiplicative Poisson noise [van Den Broeck, 1983], which can be written in the general form

\[ \frac{dy}{dt} = f(y) + g(y) \cdot \xi(t) \] \hspace{1cm} (23)

with

\[ f(y) = -v(y) + f_l(y) + \lambda \alpha \] \hspace{1cm} \[ g(y) = \frac{1}{\beta(y)} \] \hspace{1cm} \[ \xi(t) = I(t) - \lambda \alpha \] \hspace{1cm} (24)

We interpret the multiplicative noise in the Stratonovich sense [van Kampen, 1981], and obtain the steady state probability density function (pdf) [van Den Broeck, 1983]

\[ p(y) = \frac{N}{f(y) - \lambda \log(y)} \exp \left[ - \int f(y') \left( \frac{f(y')}{f(y) - \lambda \log(y')} \right) dy' \right] \] \hspace{1cm} (25)

where \( N \) is a normalization factor determined from the condition \( \int_{-\infty}^{0} p(y) \, dy = 1 \).

[36] To calculate \( p(y) \) with (25) we can either determine \( \beta(y) \) using the numerical rigorous procedure or rely on the exact results for the zero-flux approximation of the soil moisture profile above \( y \). In this case the function \( f(y) \) has the form

\[ f(y) = \frac{-E_{\text{atm}} \left( \frac{y_{\text{lim}}}{y} \right)^2 + k_l(y_0 - y - \psi) + \lambda \alpha}{n \left[ 1 - \left( \frac{\psi}{y + \psi} \right)^m \right]} \] \hspace{1cm} \text{if } y < y_{\text{lim}} \] \hspace{1cm} (26)

\[ f(y) = \frac{-E_{\text{atm}} + k_l(y_0 - y - \psi_1) + \lambda \alpha}{n \left[ 1 - \left( \frac{\psi_1}{y + \psi_1} \right)^m \right]} \] \hspace{1cm} \text{if } y > y_{\text{lim}} \]

where \( y_{\text{lim}} \) is found from equation (8), while the function \( g(y) \) reads

\[ g(y) = \frac{1}{n \left[ 1 - \left( \frac{\psi}{y + \psi} \right)^m \right]} \] \hspace{1cm} (27)

[37] Figure 7 shows some examples of pdf’s of \( y \) for different soil types and rainfall regimes. As the average
The parameters $\lambda$ and $\alpha$) increases the mode of the pdf moves towards shallower depths. These shifts are more pronounced for finer textures. A typical feature of these pdf’s is that $p(y) \to 0$ as $y \to 0$. Because $\lim_{y\to 0} \beta(y) = 0$, as the water table depth, $y$, tends to zero, $f(y)$ tends to infinity with the effect of rapidly driving the system away from the condition $y = 0$. Thus, the probability that the water table is close to the ground surface is very small. This behavior is physically plausible: when a rainfall event is able to bring the water table up to the ground surface, evaporation immediately starts, thereby leading the water table away from the position $y = 0$. Moreover, Figure 7 shows the presence of the bound $y = y_{\text{min}}$ in the case of sandy soils, while the spike in the numerical pdfs is due to the discontinuity in $\beta(y)$. Another important result is that the pdf’s resulting from the zero-flux assumption capture very well the shape of the pdf obtained without making this approximation.

3.2. Soil Moisture Probabilistic Dynamics

[38] Once the steady pdf of the water table depth, $p(y)$, is known, it is straightforward to obtain the steady pdf’s of soil moisture at different depths, $p(s, z)$. The latter is found as a derived distribution of the probability density function $p(y)$ of the water table. In fact, in our modeling framework the variable $s(z)$ is related to $y$ through the deterministic function (5). Thus, the probability distribution of $s$ at a depth, $z$, can be expressed as

$$p_s[s, z] = p_y[y(z, s) \cdot \frac{dy}{ds}] \quad s < 1,$$  \hspace{1cm} (28)

$$p_s[s = 1, z] = \int_z^0 p_y(y)dy \quad s = 1,$$  \hspace{1cm} (29)

where the atom of probability at $s = 1$ corresponds to conditions in which the water table is shallower than $z$.

[39] Figure 8 shows the pdf’s corresponding to two different depths $z$ for given soil and rainfall conditions. One observes that the zero-flux approximation provides very accurate results also for the pdf of soil moisture. We note that the approximation has been used only in the calculation of the pdf of water table depth, $p(y)$, while we have used the exact expression (5) for the relation $y = y(z, s)$, rather than the profile of $s$ given by zero-flux assumption. Figure 9 reports some examples of soil moisture pdf’s for different soils and climate. As expected, the soil becomes drier in soil layers close to the ground surface and fine textured soils are on average moister than coarse soils.

3.3. Evaporation Probabilistic Dynamics

[40] The probability density functions of evaporation, $p(v)$, can be obtained following the same approach described for soil moisture. Because evaporation is also deterministically linked to the water table position (equation (8)), $p(v)$ can be obtained as a derived distribution

$$p_v[v] = p_y[y(E) \cdot \frac{dv}{dE}] \quad E < E_{\text{atm}}$$ \hspace{1cm} (30)

with the atom of probability at $v = E_{\text{atm}}$ equal to

$$p_v[E_{\text{atm}}] = \int_{0}^{E_{\text{atm}}} p_y(y)dy \quad v = E_{\text{atm}}.$$ \hspace{1cm} (31)

[41] The relation $v = v(y)$ in the regime limited by water table depth is given by equation (11) and one can follow the rigorous or the approximate approach depending on how $p(y)$ is calculated. Also in this case, the use of the zero-flux assumption (Figure 10) leads to a very good approximation of $p(y)$.

4. Conclusions

[42] This paper presented a new analytical framework for the probabilistic modeling of the soil water balance in groundwater-dependent environments, where the presence of relatively shallow water tables affects soil evaporation and the water content at the soil surface. The study concentrated on the case of bare soils and did not account for the effect of root uptake on the vertical profile of soil moisture. The main inputs of water are contributed either by (random) precipitation infiltrating into the ground or by lateral flow from a nearby water body, while the outflow is due to soil evaporation and soil drainage. Soil evaporation is related to the exfiltration process associated with capillary rise from the water table, while soil drainage contributes to the recharge of the underlying unconfined aquifer. The model is conceived for the study of the soil water balance at the daily timescale and does not resolve processes that typically occur at shorter timescales, such as diurnal fluctuations in atmospheric evaporativity and the processes of water infiltration, redistribution, and recharge.
[43] This model provides a theoretical framework for the investigation of groundwater-soil moisture interactions in areas with relatively shallow water tables, and paves the way to future studies addressing these interactions in vegetated soils.

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