KAZHDAN GROUPS WHOSE FC-RADICAL IS NOT VIRTUALLY ABELIAN

MIKHAIL ERSHOV

Abstract. We construct examples of residually finite groups with Kazhdan’s property (T) whose FC-radical is not virtually abelian. This answers a question of Popa and Vaes about possible fundamental groups of II₁ factors arising from Kazhdan groups.

1. Introduction

If G is a group, the FC-radical of G, denoted by FC(G), is the set of all elements of G which centralize a finite index subgroup of G. Equivalently, FC(G) is the set of all elements of G with finite conjugacy class.

In [4] Popa and Vaes asked the following question:

Question 1.1. Does there exist a residually finite (discrete) group with Kazhdan’s property (T) whose FC-radical is not virtually abelian?

This question was motivated by [4, Theorem 6.4(b)], which asserts that any group satisfying these conditions admits a free ergodic profinite action whose associated II₁ factor has all positive real numbers in its fundamental group. No Kazhdan group with the latter property was previously known.

In this short note we give a positive answer to Question 1.1 using Golod-Shafarevich groups.

We shall prove the following theorem:

Theorem 1.2. Every Golod-Shafarevich group has a residually finite quotient whose FC-radical is not virtually abelian.

In [2] it was shown that there exist Golod-Shafarevich groups with property (T). Since property (T) is preserved by quotients, applying Theorem 1.2 to any Golod-Shafarevich group with (T), we obtain a group which settles Question 1.1.

Acknowledgments. I am grateful to Mark Sapir for telling me about Question 1.1.

2. Construction

Informally speaking, a finitely generated group G is Golod-Shafarevich if it has a presentation with a “small” set of relators, where relators are counted with suitable weights. The formal definition is given below.

Definition. Let G be a finitely generated group. Given a prime p let $F_p[G]$ be the $F_p$-group algebra of G and I the augmentation ideal of $F_p[G]$. Let $\{\omega_n G\}_{n \geq 1}$ be the Zassenhaus
Let $p$-filtration of $G$ defined by $\omega_n G = \{g \in G : g - 1 \in I^n\}$. For each $g \in G \setminus \bigcap_{n \in \mathbb{N}} \omega_n G$ we put $\deg_p (g)$ to be the largest $n$ such that $g \in \omega_n G$.

We will need the following well-known properties of the Zassenhaus filtrations. They have been originally established by Jennings [3]; see also [1, Ch.11.12]:

(A) The subgroups $\{\omega_n G\}$ are of finite index in $G$. Moreover, they form a base for the pro-$p$ topology on $G$, and thus $\bigcap_{n \in \mathbb{N}} \omega_n G$ is the kernel of the natural map from $G$ to its pro-$p$ completion $\hat{G}_p$. In particular, if $G$ is residually-$p$, then $\bigcap_{n \in \mathbb{N}} \omega_n G = \{1\}$, so $\deg_p (g)$ is defined for any $g \in G \setminus \{1\}$.

(B) $\omega_n G = \prod_{p^j \geq n} (\gamma_j G)^{p^j}$ where $\gamma_j G$ is the $j$th term of the lower central series of $G$ and $H^m$ is the subgroup generated by $m$th powers of a group $H$. Therefore, if $\varphi : G \to K$ is a group homomorphism, then $\omega_n (\varphi (G)) = \varphi (\omega_n G)$.

**Definition.** Fix a prime number $p$.

(a) A group presentation $\langle X | R \rangle$, with $X$ finite, is said to satisfy the Golod-Shafarevich (GS) condition (with respect to $p$), if there is a real number $t \in (0, 1)$ such that

$$1 - H_X (t) + H_R (t) < 0$$

where $H_X (t) = |X| t$ and $H_R (t) = \sum_{r \in R} t^{\deg_p (r)}$.

(b) A group $G$ is called Golod-Shafarevich if it has a presentation satisfying the Golod-Shafarevich condition.

Any Golod-Shafarevich group $G$ is infinite. In fact, its pro-$p$ completion $\hat{G}_p$ must be infinite and moreover satisfies a number of largeness properties (see, e.g., [6], [2] and references therein for precise statements). We shall only use a very weak statement about Golod-Shafarevich groups:

**Proposition 2.1.** If a group $G$ is Golod-Shafarevich with respect to $p$, then its pro-$p$ completion $\hat{G}_p$ is not virtually abelian.

Proposition 2.1 follows, for instance, from a theorem of Wilson [5] which asserts that every Golod-Shafarevich group has an infinite torsion quotient.

**Proof of Theorem 1.2.** Let $G$ be a Golod-Shafarevich group, so that $G$ has a presentation $\langle X | R \rangle$ with $1 - H_X (t) + H_R (t) < 0$ for some $t \in (0, 1)$, and let $\varepsilon = -(1 - H_X (t) + H_R (t))$.

Let $k_0 \in \mathbb{N}$ be such that $t^{k_0} < \frac{\varepsilon}{8}$. By Proposition 2.1 we can choose $x_1, y_1 \in \omega_{k_0} G$ whose images in $\hat{G}_p$ do not commute. Then there exists $k_1 > k_0$ such that $x_1$ and $y_1$ do not commute modulo $\omega_{k_1} G$. By making $k_1$ larger we can also assume that $t^{k_1} < \frac{\varepsilon}{8}$.

Let $S_1$ be a finite generating set for $\omega_{k_1} G$, let $R_1 = \{[x_1, s], [y_1, s] : s \in S_1\}$ and $G_1 = G/\langle R_1 \rangle$. Note that if $\tilde{x}_1$ and $\tilde{y}_1$ are the images of $x_1$ and $y_1$ in $G_1$, then

(i) By property (B) above we have $G/\omega_{k_1} G \cong G_1/(\omega_{k_1} G/\langle R_1 \rangle G) = G_1/\omega_{k_1} G_1$, so $\tilde{x}_1$ and $\tilde{y}_1$ do not commute modulo $\omega_{k_1} G_1$;

(ii) $\tilde{x}_1$ and $\tilde{y}_1$ lie in the FC-radical of $G_1$ (and the same is true for any quotient of $G_1$).

The group $G_1$ need not be Golod-Shafarevich, but it surjects onto the group $\hat{G}_1 = G_1/\langle x_1, y_1 \rangle G_1$ which is Golod-Shafarevich by construction.

Thus, the pro-$p$ completion of $G_1$ is not virtually abelian, so we can find elements $x_2, y_2 \in \omega_{k_1} G$ and $k_2 > k_1$ such that the images of $x_2$ and $y_2$ in $G_1$ do not commute modulo $\omega_{k_2} G_1$ and $t^{k_2} < \frac{\varepsilon}{32}$. 

Let $S_2$ be a finite generating set for $\omega_{k_2}G$, let $R_2 = \{[x_2, s], [y_2, s] : s \in S_2\}$ and $G_2 = G/(R_1 \cup R_2)/G$. By construction we have $G_1/\omega_{k_2}G_1 \cong G_2/\omega_{k_2}G_2$, the images of $x_2$ and $y_2$ in $G_2$ lie in the FC-radical of $G_2$, and $G_2$ surjects onto a Golod-Shafarevich group.

Continuing this process indefinitely we obtain a sequence of groups $G = G_0 \to G_1 \to G_2 \to \ldots$, elements $(x_i, y_i)_{i \in \mathbb{N}}$ of $G$ and integers $k_0 < k_1 < k_2 < \ldots$ s.t.

(i) $G_{i+1}$ is a quotient of $G_i$ for all $i$.
(ii) $G_i$ surjects onto the group $G/\langle \bigcup_{j=1}^i \{x_j, y_j\}\rangle^G$.
(iii) $x_i$ and $y_i$ lie in $\omega_{k_{i-1}}G$, and $t^{k_i-1} < \frac{x_i}{y_i}$
(iv) The images of $x_i$ and $y_i$ in $G_{i-1}/\omega_{k_i}G_{i-1}$ do not commute.
(v) $G_{i-1}/\omega_{k_i}G_{i-1} \cong G_j/\omega_{k_j}G_j$ for all $j \geq i$.
(vi) The images of $x_i$ and $y_i$ in $G_i$ lie in the FC-radical of $G_i$.

Now let $G_\infty$ be the inductive limit of $\{G_i\}$; in other words, if $G_i = G/N_i$, we let $N_\infty = \bigcup_{i \in \mathbb{N}} N_i$ and $G_\infty = G/N_\infty$. Let $Q$ be the image of $G_\infty$ in its pro-$p$ completion, that is, $Q = G_\infty/\bigcap_{n \in \mathbb{N}} \omega_n G_\infty$.

Condition (ii) implies that $G_\infty$ surjects onto the group $G/\langle \bigcup_{j=1}^\infty \{x_j, y_j\}\rangle^G$ which is Golod-Shafarevich by (iii). Thus, by Proposition 2.1 the group $Q$ is infinite. Since $Q$ is a subset of $(G_\infty)_p$, it is also residually finite.

Let $\pi : G \to Q$ be the natural projection. By condition (vi) the FC-radical of $Q$ contains the subgroup $H$ generated by the elements $\{\pi(x_i), \pi(y_i)\}_{i \in \mathbb{N}}$. It remains to show that $H$ is not virtually abelian. Suppose not, so $H$ contains a finite index abelian subgroup $A$. Then there exists integers $i < j$ such that $\pi(x_i x_j^{-1}) \in A$ and $\pi(y_i y_j^{-1}) \in A$. Conditions (iv) and (v) imply that $\pi(x_i)$ and $\pi(y_i)$ do not commute modulo $\omega_{k_i}Q$. Thus, if $\varphi_i$ is the projection map $Q \to Q/\omega_{k_i}Q$, then $\varphi_i(\pi([x_i, y_i])) \neq 1$. On the other hand, by construction $x_j, y_j \in \omega_{k_j}G$, so $\varphi_i(\pi(x_j)) = \varphi_i(\pi(y_j)) = 1$. Therefore,

$$\varphi_i(\pi([x_i, y_i])) = \varphi_i(\pi([x_i x_j^{-1}, y_i y_j^{-1}])) \in \varphi_i([A, A]) = \{1\}.$$ 

The obtained contradiction shows that the FC-radical of $Q$ is not virtually abelian, which finishes the proof. 

\[\square\]

\textbf{References}


\textbf{University of Virginia, Department of Mathematics, P.O.Box 400137, Charlottesville, VA 22904}

\textit{E-mail address: ershov@virginia.edu}