Filtrations & Spectral Sequences

**Def.** A filtration is a functor from the poset category of the integers to another category.

The other category is normally built out of R-mod.

**Remark.** There is an issue of variance. A covariant functor gives an "increasing" filtration. A contravariant functor gives a "decreasing" filtration.

**Key Example:**

*Filtered R-modules:* \( R_{\text{-Mod, inj}} = \{ R\text{-modules } \} \) and injective homs

*Functor* \( \mathbb{Z} \to R_{\text{-Mod, inj}} \) \( \leftrightarrow \) a sequence of R-modules

\( F_{n} M \) with compatible inclusions

\( F_{n} M \hookrightarrow F_{n+1} M \).

\( \leftrightarrow \) a sequence of submodules

\( \cdots \subset F_{n} M \subset F_{n+1} M \subset \cdots M = \bigcup_{n \in \mathbb{Z}} F_{n} M = \lim_{\leftarrow} F_{n} M \)

**Remark.** This is the usual definition, and we will use the interchangeably.

**Ex:** \( R = M = \mathbb{Z} \)

\[ F_{n} M = \begin{cases} \mathbb{Z} & n \geq 0 \\ \mathbb{Z}/2^{n}\mathbb{Z} & n \leq -1 \end{cases} \]

This works more generally: If \( M \) is an abelian gp, then there is a filtration \( \cdots \subset m^{2} M \subset m M \subset M \subset M \subset \cdots \) for any \( m \in \mathbb{Z} \).

**Ex:** \( R = k[x] = M \)

\( p(x) \in R \)

\[ F_{n} M = \begin{cases} k[x] & n \geq 0 \\ p(x)^{n}k[x] & n \leq -1 \end{cases} \]

**Def.** A map of filtered something is a natural transformation of the functors.
In the filtered module context, this is the same thing as a homomorphism of $R$-modules $M \rightarrow N$ s.t.
$$f(F_n M) \subseteq F_n N.$$ 

Associated to a filtration (in an abelian category) is a graded object.

**Def** The associated graded $Gr(M)$ is defined by
$$Gr_n(M) = \frac{F_n M}{F_{n-1} M}.$$ 

Passage to associated graded is obviously functorial.

**Ex:** $R = \mathbb{Z} = M$, $F_n M = \left\{ \mathbb{Z}^n \mathbb{Z} \right\}$, then
$$Gr_n M = \begin{cases} \mathbb{Z} & n > 0 \\ \mathbb{Z}_2 & n \leq 0 \end{cases},$$ and this holds generally.

For us, the most important thing is a filtered DGM.

$\iff$ A sequence $(F_n M, d_n)$ of DGMs w/ compatible inclusions $F_n M \rightarrow F_{n-1} M$ s.t.
$$F_n M \rightarrow F_{n-1} M$$

commutes.

$\iff$ A DGM $(M, d)$ together with a filtration of $M$ s.t.
$$d(F_n M) \subseteq F_n M.$$ 

$\iff$ A filtered graded module w/ a map $d$ of filtered graded modules with $d^2 = 0$ and $d$ shifts degree by $\pm 1$.

So the associated graded of a DGM is a DGM!

If we filter things right, it will be easier to understand.

**Guiding Principle:**

Filter a DGM so that we can compute with the AG.
Guiding Question:

How can we recover $M$ from the AG?

**Ex:** $M_\ast = \mathbb{Z} \overset{2}{\to} \mathbb{Z}$

So $\text{Gr}_p(M_\ast) = 
\begin{bmatrix}
\beta^p & 1 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{bmatrix}$

$F_{n+1} M = \mathbb{Z} \overset{2}{\to} \mathbb{Z}$

$F_n M = a^p \mathbb{Z} \overset{2}{\to} \beta^p \mathbb{Z}$

Why is $\text{Gr}(d) = 0$? Check the base case $\mathbb{Z} \overset{2}{\to} \mathbb{Z}$

The differential on $\text{Gr}$ is induced by the composite $\mathbb{Z} \overset{2}{\to} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} = F_0/F_{-1}$

and this is zero.

So $\text{Gr}(M_\ast)$ is bigraded, but the differential is zero!

**Remark** The sketch above shows how to compute differentials/functions on the AG. Given $f: M \to N$, $\text{Gr}_f$ is defined by taking the composite $F_n M \overset{f}{\to} F_n N \to F_n N/F_{n-1} N$.

Since $f(F_{n-1} M) \subseteq F_{n-1} N$, $F_{n-1} M \overset{Gr_f}{\to} F_{n-1} N$.

So we get a canonical map $F_n M/F_{n-1} M \overset{Gr_f}{\to} F_n N/F_{n-1} N$.

The problem?

Associated graded doesn't commute with homology:

$\text{Gr}_\ast(H_\ast) \neq H_\ast(\text{Gr}_\ast)$.

This is obvious from the previous example.

The solution? Spectral sequences. Normally, we will have a SS $H_\ast(\text{Gr}_\ast) \Rightarrow \text{Gr}_\ast(H_\ast)$.

**Def** A cohomological, first quadrant spectral sequence is a sequence of bigraded DGMs $E^{p,q}_n$ s.t.
1. $d_n : E_n^{p,q} \rightarrow E_n^{p+n,q+1-n}$
2. $E_{n+1} = H^n(E_n)$.
3. $E_n^{p,q} = 0$ if $p < 0$ or $q < 0$

The first part just says that $d_n$ is "degree 1" in the total degree $r = p + q$. This seems very contrived, but we'll see how this plays out quite naturally.

This is an obvious homological version. The $d_n : E_n^{p,q} \rightarrow E_n^{p-n,q+n-1}$. We say that an element $x$ s.t. $d_n(x) = 0$ for all $n$ is a permanent cycle. Elements in $im(d_n)$ are boundaries.

The first quadrant condition shows that there are only finitely many possible non-zero differentials on any given class.

Generic pictures:

We'll actually have extra structure normally.

Def: $E^{*,*}$ is a spectral sequence of algebras if $E_n^{*,*}$ is a DGA and $E_n^{*,*} = H^{*,*}(E_n)$ as algebras.

Here we use the "total degree" in the Leibniz Rule.