Def A permutation group is any group of the form $\Sigma_{x}$:

$\{ f : x \mapsto x \mid f \text{ is } 1-1 \text{ or onto} \}$.

Thm 19 Any group is a subgroup of a permutation group.

Pf: $x = G$

Need an injective map $G \rightarrow S_{G}$

$\Rightarrow$ 1st isom. thm $\Rightarrow G \cong \text{Im} \leq S_{G}$

$\phi : G \rightarrow S_{G}$

$\phi (g)(h) = gh$

$\phi (g)$ is 1-1 $\Rightarrow$ onto:

$\phi (g)(h_{1}) = \phi (g)(h_{2})$

$gh_{1} = gh_{2} \Rightarrow h_{1} = h_{2}$ (left multiply by $g^{-1}$)

Given $h \in G$, $\phi (g)(g^{-1}h) = g(g^{-1}h) = e \cdot h = h$

$\phi$ is injective

$\phi (g) = \text{Id} : G \rightarrow G$, then $g = e$

If $gh = h$ $\forall h \in G$, then $g = e$. The blc e is unique.

$\Rightarrow \phi$ is injective $\Rightarrow G \cong \text{Im}(\phi)$. $\blacksquare$

If $|G| < \infty$, then $|S_{G}| = |G|!$

Any homomorphism $G \rightarrow S_{x}$ is called a representation of $G$.

If $G \rightarrow S_{x}$ is a homomorphism, $X$ is a $G$-set.

If $H \leq G$ is a subgroup, then $G$ acts on $G/H$

$\leftrightarrow$ have a homomorphism $G \rightarrow S_{G/H}$

$\phi_{H}(g)(aH) = gaH$
Prop 20: For any $H$, $\phi_H$ is a homomorphism.

**PF:**
\[ \phi_H(g \cdot h) = \phi_H(g) \cdot \phi_H(h) \]
\[ \text{mult in} \quad G \quad \text{mult in} \quad S_{G/H} \]
\[ \phi_H(g \cdot h)(aH) = (g \cdot h) \cdot aH = g \cdot (h \cdot aH) = \phi_H(g)(h \cdot aH) \]
\[ = \phi_H(g) \left( \phi_H(h)(aH) \right) = \phi_H(g) \cdot \phi_H(h)(aH) \]

Lemma 21: The kernel of $\phi_H$ is the largest normal s.g. of $G$ contained in $H$.

**PF:**
$\ker(\phi_H)$ is always normal.

1) $\ker(\phi_H) \subseteq H$

2) If $N$ is normal, $N \subseteq H$, then $N \subseteq \ker(\phi_H)$.

i) $g \in \ker(\phi_H) \iff \phi_H(g) = \text{Id} \iff \phi_H(g)(eH) = eH$
\[ \Rightarrow gH = H \Rightarrow g \in H. \]

ii) $n \in N \subseteq H \quad \phi_H(n)(gH) = n gH$
\[ N \triangleleft G \Rightarrow ng = gn' \Rightarrow \phi_H(n)(gH) = gn'H \]
\[ \exists g' : Ng = N \]
\[ \text{i.e.} \quad \phi_H(n) = \text{Id} \]
\[ \Rightarrow n \in \ker(\phi_H). \]

Assume $G$ is finite.

Cor 22: If $H$ is a s.g. of $G$ s.t. $|G| \nmid [G:H]$, then $G$ has a non-trivial normal s.g. contained in $H$.

**PF:**
\[ \phi_H : G \rightarrow S_{G/H}, \quad \text{Im}(\phi_H) \text{ is a s.g. of } S_{G/H} \]
\[ \Rightarrow |\text{Im}(\phi_H)| \text{ divides } |S_{G/H}| = [G:H]! \]
\[ |G|/|\ker(\phi_H)| = |G|/|\ker(\phi_H)| \]
\[ |G|/|\ker(\phi_H)| \mid [G:H]! \]
Cor 23: If \( H \) is a s.g. of \( G \) s.t. \( |H| \) and \((|G|/|H| - 1)!\) are relatively prime, then \( H \) is normal.

**Proof:** \( \text{Im}(\phi_H) \) is a s.g. of \( S_{G/H} \), so if \( N \) is the kernel of \( \phi_H \), then \( |G|/|N| = |G/\langle \phi_H \rangle| = |\text{Im}(\phi_H)| = |[G:H]| = (|G|/|H|)! \). Therefore, \( |G|/|N| = |G|/|H| \cdot |H|/|N| \mid (|G|/|H|)! \).

\[ \Rightarrow \quad |H|/|N| \text{ divides } ([G:H]-1)! \]

\[ (|H|, ([G:H]-1)!) = 1 \iff \text{are relatively prime} \]

\[ \Rightarrow \quad \text{every factor of } |H| \text{ is a factor of } |N| \]

\[ \Rightarrow \quad |H| = |N| \Rightarrow H = N. \quad \Box \]

*(a, b) = greatest common divisor of \( a \) and \( b \).*

Cor 24: If \( p \) is the smallest prime dividing \( |G| \), then any subgroup of index \( p \) is normal.

**Proof:** \([G:H] = p\), and \( |H| \) divides \( |G| \); \( p \geq p \).

Cor 23: \( |H| \mid (p-1)! \) are relatively prime \( \Rightarrow H \) is normal.

**Def:** An \( G \)-action on a set \( X \) is a homomorphism
\[ G \rightarrow S_X \]

**Remark:** A functor \( \Phi: G \rightarrow \text{Sets} \)
\[ \begin{align*}
\text{obj} \ G & \rightarrow \text{Obj} \text{Sets} \\
\text{Mor} \ G & \rightarrow \text{Mor} \text{Sets} \\
\phi(g) \in \text{Hom}(x,x') & \in S_X
\end{align*} \]

**Ex:** \( G \) acts on itself:
\[ \Phi: G \rightarrow S_G, \quad R: G \rightarrow S_G \]
\[ R(gh)(a) = h g^{-1} a \text{ to make } R(gh) = R(g) \circ R(h) \]
\[ c: G \rightarrow S_G \]
\[ c(g)(h) = ghg^{-1} = \text{conjugation} \]

\[ c(g) \] is a group homomorphism for all \( g \).

\[\rightarrow \text{ an automorphism} \]

\[ c(g)(ab) = g(ab)g^{-1} = g(ae b)g^{-1} = g(a(g^{-1} b))g^{-1} = (gag^{-1})(gbg^{-1}) = (c(g)(a))(c(g)(b)) \]

\[ \text{Def} \quad \text{The kernel of } c \text{ is the center of } G: Z(G) \]

\[ \leftrightarrow \quad \exists g \in G \mid gag^{-1} = a \quad \forall a \in G \quad \mid ga = ag \quad \forall a \in G \]

\[ \text{ie } Z(G) \text{ is the set of elements that commute with everything.} \]

\[ \text{Def} \quad X \text{ is a } G \text{-set, } x \in X, \quad (\phi: G \rightarrow S_X) \]

\[ \text{the orbit of } x : Gx = \{ g \cdot x \mid g \in G \} \]

\[ \phi(g)(x) \]

\[ \text{the stabilize subgroup of } x : G(x) = \text{Stab}_G(x) = \{ g \in G \mid g \cdot x = x \} \]

Underlying the orbits is an equivalence relation:

\[ x \sim y \quad \text{if} \quad y = g \cdot x \quad \text{some } g \]

\[ [x] = Gx \]

\[ |Gx| = |G: \text{Stab}_G(x)| = |G/\text{Stab}_G(x)| \]

\[ g \cdot x \mapsto g \cdot \text{Stab}_G(x) \]

\[ g \cdot x = h \cdot x \quad \iff \quad g^{-1} h \cdot x = x \quad \iff \quad g^{-1} h \in \text{Stab}_G(x) \]

\[ \iff [g] = [h] \text{ in } G/\text{Stab}_G(x) \]

\[ \text{Lemma 25 : Class Equation:} \]

\[ |G| = |Z(G)| + \sum [G: \text{Stab}_G(x)] \]

G acts on itself by conjugation.

Hence \( \text{Stab}_G(x) = C(x) = \text{Cent}(x) = \{ g \mid g \cdot x = xg \} \).