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Fibonacci the Man

- Leonardo Pisano
- Lived from 1170 to 1250
- He was an Italian mathematician who has been called the most talented mathematician of the Middle Ages
- He is known for
  - Spreading the Hindu-Arabic numeral system in Europe
  - The Fibonacci numbers
The Fibonacci Numbers

0, 1, 1, 2, 3, 5, 8, 13, ...

The ratio of two any consecutive numbers, for example 13/8, approaches the Golden Section, \( G = 1.618033989... \).

The reciprocal of \( G \) is \( \frac{1}{G} = 0.618033989... \), so

\[
1 + G \approx 2.016071305
\]
The Fibonacci Numbers

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Background

- Fibonacci groups were first introduced by J. H. Conway, the creator of the Game of Life (a cellular automaton).
- Used largely to test various computational techniques.
Definitions

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- $k(m)$ - the Fibonacci length of a sequence mod $m$
Definitions Cont.

- **Fibonacci Sequence** - the first number of the sequence is 0, the second 1, and each subsequent number is equal to the sum of the previous two numbers
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Fibonacci Group - the Fibonacci group $F(n)$ is defined by $F(n) = \langle a_1, a_2, \ldots, a_n | a_ia_{i+1} = a_{i+2}, i = 1, \ldots, n \rangle$ where the subscripts are reduced mod $n$ to lie in the range 1,2,...n.
Golden Matrix Ring

The matrix $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$
can be used to derive an explicit formula for the Fibonacci Numbers in terms of the golden ratio, $\phi = (1 + \sqrt{5})/2$, and its conjugate.
Golden Matrix Ring

The matrix $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ can be used to derive an explicit formula for the Fibonacci Numbers in terms of the golden ratio, $\phi = (1 + \sqrt{5})/2$, and its conjugate. The simplest ring generated by $A$ is $\mathbb{Z}[A]$, the ring of polynomials in $A$ with integer coefficients.
G.M.R. Cont.

In \( \mathbb{Z}[A] \), \( A^2 - A - 1 = 0 \) because the characteristic polynomial of \( A \) is \( \det(XI - A) = X^2 - X - 1 \), which is also the characteristic polynomial of the Fibonacci recurrence relation \( F_{n+2} = F_{n+1} + F_n \).
G.M.R. Cont.

Since $\phi$ is the positive root of $\phi^2 - \phi - 1 = 0$, $\mathbb{Z}[A]$ and $\mathbb{Z}[\phi]$ are isomorphic under the eigenvalue map $\varepsilon: \mathbb{Z}[A] \rightarrow \mathbb{Z}[\phi]$ determined by $\varepsilon(A) = \phi$ and $\varepsilon(I) = 1$. 

G.M.R. Cont.

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- $x \mapsto \phi$
G.M.R. Cont.

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- So \( \text{Ker} = \langle x^2 - x - 1 \rangle \)
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This means \( \mathbb{Z}[\phi] \) is an integral domain.
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- Since $\phi$ is a unit, so is $\phi^n$
G.M.R. Cont.

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- So \(-\phi^n\) is also a unit
- So the units are \( \pm \phi^{\pm n} \)
By the isomorphism $\varepsilon$, $\mathbb{Z}[A]$ shares the same properties.

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$\mathbb{Z}[A]$ is called the Golden Matrix Ring
The Ring of Generalized Fibonacci Sequences

- Consider the set \( \mathbb{F} \) of all integer sequences \( \{G_n\}_{n=1}^{\infty} \) satisfying the recurrence relation \( G_{n+2} = G_{n+1} + G_n \), regardless of initial conditions.
The Ring of Generalized Fibonacci Sequences

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The Ring of Generalized Fibonacci Sequences

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- This is the generalized Fibonacci Sequence.
- $\mathbb{F}$ is an abelian group under the addition $\{G_n\} + \{H_n\} = \{G_n + H_n\}$. 
The Ring of Generalized Fibonacci Sequences

- Define the matrix map $\mathcal{M} : \mathbb{F} \to \mathbb{Z}[A]$
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The Ring of Generalized Fibonacci Sequences

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$$\mathcal{M} (\{G_n\}) = (G_1 - G_0)I + G_0 A.$$
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- Define the matrix map $\mathcal{M} : \mathbb{F} \rightarrow \mathbb{Z}[A]$ by $\mathcal{M}(\{G_n\}) = (G_1 - G_0)I + G_0A$.

- $\mathcal{M}$ is a group homomorphism and by induction, using $A^2 = A + I$, we get
The Ring of Generalized Fibonacci Sequences

- Define the matrix map $\mathcal{M} : \mathbb{F} \rightarrow \mathbb{Z}[A]$ by $\mathcal{M}({G_n}) = (G_1 - G_0)I + G_0A$.
- $\mathcal{M}$ is a group homomorphism and by induction, using $A^2 = A + I$, we get $G_{n-1} + G_nA$.
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The Ring of Generalized Fibonacci Sequences

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- Here $a = G_{n-1}$ and $b = G_n$
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- Here \( a = G_{n-1} \) and \( b = G_n \)
- This gives you \( G_n = b = G(D) \).
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- $\mathcal{M}$ is a group homomorphism and by induction, using $A^2 = A + I$, we get $G_{n-1} + G_nA = A^n \mathcal{M}(\{G_n\}$ (Let this equal D).
- Here $a = G_{n-1}$ and $b = G_n$
- This gives you $G_n = b = G(D)$.
- Define the sequence map $\mathcal{L} : \mathbb{Z}[A] \rightarrow \mathbb{F}$
The Ring of Generalized Fibonacci Sequences

- Define the matrix map \( M : \mathbb{F} \to \mathbb{Z}[A] \) by \( M(\{G_n\}) = (G_1 - G_0)I + G_0 A \).
- \( M \) is a group homomorphism and by induction, using \( A^2 = A + I \), we get \( G_{n-1} + G_n A = A^n M(\{G_n\}) \) (Let this equal \( D \)).
- Here \( a = G_{n-1} \) and \( b = G_n \)
- This gives you \( G_n = b = G(D) \).
- Define the sequence map \( L : \mathbb{Z}[A] \to \mathbb{F} \) by \( L(a + bA) \).
The Ring of Generalized Fibonacci Sequences

- Define the matrix map $\mathcal{M} : \mathbb{F} \to \mathbb{Z}[A]$ by $\mathcal{M}(\{G_n\}) = (G_1 - G_0)I + G_0A$.
- $\mathcal{M}$ is a group homomorphism and by induction, using $A^2 = A + I$, we get $G_{n-1} + G_nA = A^n \cdot \mathcal{M}(\{G_n\})$ (Let this equal $D$).
- Here $a = G_{n-1}$ and $b = G_n$
- This gives you $G_n = b = G(D)$.
- Define the sequence map $\mathcal{L} : \mathbb{Z}[A] \to \mathbb{F}$ by $\mathcal{L}(a + bA) = \{G(A^n(a + bA))\}$. 
The Ring of Generalized Fibonacci Sequences

- Define the matrix map $\mathcal{M} : \mathbb{F} \to \mathbb{Z}[A]$ by $\mathcal{M}(\{G_n\}) = (G_1 - G_0)I + G_0A$.
- $\mathcal{M}$ is a group homomorphism and by induction, using $A^2 = A + I$, we get $G_{n-1} + G_nA = A^n \mathcal{M}(\{G_n\})$ (Let this equal D).
- Here $a = G_{n-1}$ and $b = G_n$
- This gives you $G_n = b = G(D)$.
- Define the sequence map $\mathcal{L} : \mathbb{Z}[A] \to \mathbb{F}$ by $\mathcal{L}(a + bA) = \{G(A^n(a + bA))\}$.
- Then $\mathcal{L}$ is a group homomorphism.
The Ring of Generalized Fibonacci Sequences

- $\mathcal{L}(\mathcal{M}([G_n])) = \{G_n\}$ and $\mathcal{M}(\mathcal{L}(a + bA)) = (a + bA)$. 

The Ring of Generalized Fibonacci Sequences

- \( \mathcal{L}(\mathcal{M}([G_n])) = \{G_n\} \) and \( \mathcal{M}(\mathcal{L}(a + bA)) = (a + bA) \).
- Thus, \( \mathcal{L} \) and \( \mathcal{M} \) form an inverse pair of group isomorphisms.
The Ring of Generalized Fibonacci Sequences

- \( \mathcal{L}(\mathcal{M}({G_n})) = \{G_n\} \) and \( \mathcal{M}(\mathcal{L}(a+bA)) = (a+bA) \).

Thus, \( \mathcal{L} \) and \( \mathcal{M} \) form an inverse pair of group isomorphisms.

- We can transfer the multiplicative structure of \( \mathbb{Z}[A] \) to \( \mathbb{F} \) via \( \mathcal{L} \) and \( \mathcal{M} \).
The Ring of Generalized Fibonacci Sequences

- We define \( \{G_n\}\{H_n\} \)
The Ring of Generalized Fibonacci Sequences

- We define \( \{G_n\}\{H_n\} = \mathcal{L}(\mathcal{M}(\{G_n\})) \)
The Ring of Generalized Fibonacci Sequences

We define \( \{G_n\}\{H_n\} = \mathcal{L}(\mathcal{M}(\{G_n\}) \cdot \mathcal{M}(\{H_n\})) \)
The Ring of Generalized Fibonacci Sequences

We define \( \{G_n\}\{H_n\} = L(M(\{G_n\}) \cdot M(\{H_n\})) \) and denote it by \( \{(GH)_n\} \).
The Ring of Generalized Fibonacci Sequences

- We define \( \{G_n\}\{H_n\} = \mathcal{L}(\mathcal{M}(\{G_n\}) \cdot \mathcal{M}(\{H_n\})) \) and denote it by \( \{(GH)_n\} \).

- With this multiplication, \( \mathbb{F} \) becomes a ring, and the maps \( \mathcal{M}: \mathbb{F} \to \mathbb{Z}[A] \) and \( \mathcal{L}: \mathbb{Z}[A] \to \mathbb{F} \) are isomorphisms of rings.
Fibonacci Sequences and Groups

An ordered pair \((x_1, x_2)\) of elements of a group \(G\) determines a sequence in \(G\) by the rule \(x_n x_{n+1} = x_{n+2}, n \in \mathbb{N}\).
Fibonacci Sequences and Groups

- An ordered pair \((x_1, x_2)\) of elements of a group \(G\) determines a sequence in \(G\) by the rule \(x_nx_{n+1} = x_{n+2}, \ n \in \mathbb{N}\).
- When this sequence is periodic, its fundamental period is called the Fibonacci length of \((x_1, x_2)\) in \(G\).
An Example

For the group $\mathbb{Z}_2 \times \mathbb{Z}_2 = \langle a, b | a^2 = b^2 = 1 \rangle$ we obtain the sequence

$$a, b, ab, bab, b, ba, a, b, ...$$

showing that the infinite dihedral group has Fibonacci length 6
A Second Example

The values of $U_n \pmod{7}$ are

$$0, 1, 1, 2, 3, 5, 1, 6, 0, 6, 6, 5, 4, 2, 6, 1$$

and then repeat; so $k(7) = 16$. Note that $U_8 = 0 \pmod{7}$ so that the 16 terms in the period form two sets of 8 terms each.
Fibonacci in the $\mathbb{R}$ World
Fibonacci Spirals

Start with a rectangle
Spirals Cont.

Then you create a square and a rectangle within the original rectangle

You continue doing this until you reach...
Spirals Cont.
Then you draw a spiral within the rectangle...
Application to Nature

- The approach of a hawk to its prey: Their sharpest view is at an angle to their direction of flight; this angle is the same as the spiral’s pitch.
- The arms of tropical storms
- The arms of spiral galaxies; the Milky Way, is believed to have 4 major spiral arms, each a Fibonacci spiral with pitch of about 12 degrees
Turku Power Station, Finland
Fibonacci Numbers in Music

Lateratus by Tool: If you count between pauses, the syllables in the verses from the first several Fibonacci numbers:

- (1) Black,
- (1) then,
- (2) white are,
- (3) all I see,
- (5) in my infancy,
- (8) red and yellow then came to be,
- (5) reaching out to me,
- (3) lets me see.
- (2) There is,
- (1) so,
- (1) much,
- (2) more that
- (3) beckons me,
- (5) to look through to these,
- (8) infinite possibilities.
- (13) As below so above and beyond I imagine,
- (8) drawn outside the lines of reason,
- (5) push the envelope,
- (3) watch it bend.
The time signatures of the chorus change from 9/8 to 8/8 to 7/8, and the song’s original name was 9-8-7. 987 is the 17th step of the Fibonacci sequence.