

## Problem Set #2

Due October 11<sup>th</sup>

The last two problems are optional. Feel free to try them, but not doing so will not hurt you.

In all that follows, fix a field  $\mathbb{F}$ , and let  $U$ ,  $V$  and  $W$  be vector spaces over it.

1. (a) Show that the natural projections  $\pi_V: V \oplus W \rightarrow V$  and  $\pi_W: V \oplus W \rightarrow W$ , defined by  $\pi_V(\mathbf{v}, \mathbf{w}) = \mathbf{v}$  and  $\pi_W(\mathbf{v}, \mathbf{w}) = \mathbf{w}$  are linear transformations.  
(b) Show that the natural inclusions  $\iota_V: V \rightarrow V \oplus W$  and  $\iota_W: W \rightarrow V \oplus W$ , defined by  $\iota_V(\mathbf{v}) = (\mathbf{v}, \mathbf{0})$  and  $\iota_W(\mathbf{w}) = (\mathbf{0}, \mathbf{w})$  are linear transformations.
2. (a) Show that given  $S \in \mathcal{L}(U, V)$  and  $T \in \mathcal{L}(U, W)$ , there is a natural linear transformation  $L = S \times T \in \mathcal{L}(U, V \oplus W)$  such that  $\pi_V \circ L = S$  and  $\pi_W \circ L = T$ . This shows that finite direct sums are the categorical product in the category **Vect**.  
(b) Show that given  $S \in \mathcal{L}(V, U)$  and  $T \in \mathcal{L}(W, U)$ , there is a natural linear transformation  $L = S \oplus T \in \mathcal{L}(V \oplus W, U)$  such that  $L \circ \iota_V = S$  and  $L \circ \iota_W = T$ . This shows that finite direct sums are also the categorical coproduct in the category **Vect**.
3. Book problems: Chapter #2, Exercises: 3, 16, 20, 22, 23
0. Show that the maps

$$\pi_i: \prod_{i \in I} V_i \rightarrow V_i, \quad f \mapsto f(i),$$

are linear transformations. Show also that given a vector space  $U$  and linear transformations  $L_i: U \rightarrow V_i$ , there is a unique linear transformation

$$L: U \rightarrow \prod_{i \in I} V_i \text{ such that } \pi_i \circ L = L_i.$$

This shows that **Vect** has arbitrary categorical products.

0. Show that the maps

$$\iota_i: V_i \rightarrow \bigoplus_{i \in I} V_i, \quad \iota_i(\mathbf{v}) = f_{\mathbf{v}},$$

where

$$f_{\mathbf{v}}(j) = \begin{cases} \mathbf{v} & i = j \\ 0 & \text{otherwise,} \end{cases}$$

are linear transformations. Show also that given a vector space  $U$  and linear transformations  $L_i: V_i \rightarrow U$ , there is a unique linear transformation

$$L: \bigoplus_{i \in I} V_i \rightarrow U \text{ such that } L \circ \iota_i = L_i.$$

This shows that **Vect** has arbitrary categorical coproducts.

In category theory, every notion has a kind of dual notion, denoted usually with the prefix “co”. If we write the morphisms in our category with arrows, then the “co” object satisfies the same properties as the object, but with all of the arrows reversed.