Problem Set 1

Due Tuesday, Sept 18

1. Find two distinct bases for $\mathbb{Q}^3$.
2. There are 3 bases for $\mathbb{F}_2^2$. Find them.
3. Let K and L be fields with K a subfield of L (so it is a subset and the multiplication, etc are those coming from L). The prototypical example is $\mathbb{Q} \subset \mathbb{R}$ or $\mathbb{R} \subset \mathbb{C}$. We saw in class that L is a vector space over K. If V is a finite dimensional vector space over L of dimension n and L is a finite dimensional vector space over K of dimension m, show that V is a finite dimensional vector space over K of dimension mn.
4. Let $a, b \in \mathbb{F}$, and let $\overrightarrow{u}, \overrightarrow{v}$ be elements of V a vector space over $\mathbb{F}$. Show that
$$(a+b)(\overrightarrow{u}+\overrightarrow{v}) = a\overrightarrow{u} + b\overrightarrow{u} + a\overrightarrow{v} + b\overrightarrow{v}.$$
5. Let $C(\mathbb{R})$ be the vector space of all continuous functions from $\mathbb{R}$ to itself. We say that a function f is even if $f(-x) = f(x)$ for all x. If $f(-x) = -f(x)$, then we say that f is odd.
   (a) Show that the collection C_+ of all even continuous functions is a subspace of $C(\mathbb{R})$.
   (b) Show that the same is true for the collection of odd continuous functions, C_-.
   (c) Show that $C(\mathbb{R}) = C_+ \oplus C_-$.
6. Show that the following are subspaces of V, the vector space of all (infinite) sequences of real numbers:
   (a) ℓ_1, the sequences $(a_1, \dots)$ for which $\sum |a_i| < \infty$.
   (b) c_0, the sequences that converge to 0.
   (c) c, the sequences that converge.
   (d) ℓ_∞, the collection of bounded sequences (that is, for a sequence $(a_1, \dots)$, there is an M such that $|a_i| < M$ for all i.
   (e) d, the sequences that are eventually 0.
7. Explain, in your own words and briefly, the proof of Theorem 1.12 in the book (all bases have the same cardinality).