Hilbert Spaces

A Math 551 Lecture

November 28, 2007

Outline

What Do We Want?

- Want a nicer notion of basis for the infinite dimensional case.
- Should fit well with finite dimensional subspaces.
- Should be useable for computation.

The Key Example: ℓ_2

- Recall: $\ell_2 = \{(x_1, \dots) | \sum |x_i|^2 < \infty\}$

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum x_i y_i.$$

- If $\mathbf{x} \in \ell_2$, then $\lim_{n \to \infty} x_n = 0$.
- If $\mathbf{x}_n = (x_1, \dots, x_n, 0)$, then $||\mathbf{x} - \mathbf{x}_n|| \to 0$.
- So with some idea of convergence, any $\mathbf{x}$ is a limit of things we understand.
- Any $\mathbf{x}$ is a limit of a sequence of elements in $\langle \mathbf{e}_1, \dots \rangle$.

A Warning

We must have a notion of *convergence* to do this.
A non-example:

- Let $V = \prod_{n\in\mathbb{N}} \mathbb{R}$.
- Inside V is $V' = \bigoplus_{n\in\mathbb{N}} \mathbb{R}$.
- If $\mathbf{x} \in V$, then we can let $\mathbf{x}_n = (x_1, \dots, x_n, 0, \dots)$ as before BUT
- we have no way to say that $\mathbf{x}_n$ converges to $\mathbf{x}$!

Moral: We have to have convergence.

Metric Spaces

- Convergence requires a notion of *closeness*
- This is summed up in a *topology* which comes from a metric.

Definition

A metric space is a set X together with a function $d\colon X \times X \to \mathbb{R}$ such that

1. $d(x, y) \geq 0$ *with equality only if* $x = y$*.*
2. $d(x, y) = d(y, x)$*.*
3. $d(x, y) \leq d(x, z) + d(z, y)$*.*

Examples

Example

Let V be an inner product space with $d(\mathbf{x}, \mathbf{y}) = ||\mathbf{x} - \mathbf{y}||$.Saw in class that this satisfies the axioms for a metric. This is our key example.

Example

Let X be any set, and let

$$d(x, y) = \begin{cases} 1 & x \neq y \\ 0 & x = y. \end{cases}$$

This is a metric space in which every point is equally far from every other point, so nothing is very close to anything else.

Sequences & Convergence

A sequence in a metric space is defined just as for $\mathbb{R}$: an ordered list of elements.
We define convergence / limits just as for $\mathbb{R}$:

Definition

Let $x_1, \ldots$ be a sequence in a metric space. This sequence converges to x if for every $\epsilon > 0$, there is an N such that $d(x_n, x) < \epsilon$ for all $n > N$.

In other words, $x_1, \ldots$ converges to x if the terms get closer and closer to x as n gets very large.

Theory Examples of Convergence

Two examples, using the two metrics from before:

Example

In an inner product space, a sequence $\mathbf{x}_n$ *converges to* $\mathbf{x}$ *if and only if*

$$\lim_{n\to\infty} ||\mathbf{x}_n - \mathbf{x}|| \to 0.$$

Put another way, we can see convergence in an inner product space by just checking convergence of the sequence $||\mathbf{x}_n - \mathbf{x}||$.

Theory Examples II

Example

With the other metric, a sequence converges iff it is eventually constant. In other words x_n converges to x if and only if there is some N such that for all $n > N$, $x_n = x$.
We can see this by choosing $\epsilon < 1$.
Since all points are of uniform distance 1, the only points of distance less than this ϵ are those equal to x.

Practical Examples of Convergence

Example

In $\mathbb{R}^n$ with the usual Euclidean distance, a sequence of vectors $\mathbf{x}_1, \dots$ converges to $\mathbf{x}$ iff each coordinate converges. This follows from the triangle inequality.
So in $\mathbb{R}^2$, the sequence $(3, 1)$, $(3.1, 1.4)$, $(3.14, 1.41)$, $(3.141, 1.414), \dots$ converges to $(\pi, \sqrt{2})$.
This is the old rule from multivariable calculus:

$$\lim_{n\to\infty} \big(f_1(n), \dots, f_k(n)\big) = \big(\lim_{n\to\infty} f_1(n), \dots, \lim_{n\to\infty} f_k(n)\big).$$

Example

In ℓ_2, a sequence $\mathbf{x}_1, \dots$ converges if it converges coordinate-wise. This makes it easy to check here too.

Cauchy Sequences

Convergence has a big draw-back: you have to know the limit of a sequence.

This is extra data. We want an intrinsic notion.

Definition

A sequence x_n in a metric space is Cauchy *if for all $\epsilon > 0$, there is an N such that for all $n, m > N$,*

$$d(x_n, x_m) < \epsilon.$$

Using the triangle inequality, we can see that all convergent sequences are Cauchy.

A Warning

In general, *NOT* all Cauchy sequences converge. This is a property of the space.

Example

Let $X = \mathbb{Q}$*, together with the metric* $d(x, y) = |x - y|$*.*
The sequence $3, 3.1, 3.14, 3.141, \ldots$ *is Cauchy (It converges in* $\mathbb{R}$*), but it does not converge, since* $\pi \notin \mathbb{Q}$*.*
Another example: the sequence

$$x_n = \sum_{i=1}^{n} \frac{1}{10^{i!}}$$

is Cauchy. It is not convergent. In $\mathbb{R}$*, it converges to a number that is actually transcendental.*

Completeness

Definition

A metric space is complete if every Cauchy sequence converges.

Example

- $\mathbb{R}$ *and* $\mathbb{R}^n$ *are complete.*
- $\mathbb{C}$ *and* $\mathbb{C}^n$ *are complete.*
- ℓ_2 *is complete.*
- $\mathbb{Q}$ *is not complete.*

Completions

The only incomplete metric space we have seen in $\mathbb{Q}$, and this sits inside a complete metric space $\mathbb{R}$. This is true in general:

Theorem

- *Every metric space embeds in a complete metric space.*
- *There is a smallest (universal) complete metric space that contains any given metric space.*

This smallest complete metric space is called the completion. If V is an inner product space, then the completion of V is also an inner product space.

Definition

Definition

A Hilbert space is an inner product space that is complete with respect to the induced metric.

So basically, in a Hilbert space, we can easily tell if sequences converge.
We have several key examples:

Example

- $\mathbb{R}^n$ *and* $\mathbb{C}^n$
- ℓ_2.

So What?

From convergence of sequences, we get a notion of *infinite series*.

Definition

Let $\mathbf{x}_n$ be a sequence, and define

$$\mathbf{s}_k = \mathbf{x}_1 + \cdots + \mathbf{x}_k.$$

If the sequence $\mathbf{s}_n$ converges to $\mathbf{s}$, then we say that the series $\sum \mathbf{x}_n$ converges to $\mathbf{s}$.

Just as for sums in $\mathbb{R}$, this is relatively weak, and we might get a different answer by rearranging the sum.

Definition

A series $\sum \mathbf{x}_n$ is absolutely convergent if $\sum ||\mathbf{x}_n||$ converges.

Completeness and Absolute Convergence

Absolute convergence is again something we check only in $\mathbb{R}$! In general, absolute convergence only ensures that the sequence of partial sums is Cauchy. We have the following theorem:

Theorem

Let V be an inner product space. Then V is complete if and only if absolute convergence implies convergence.

Since we will focus on Hilbert spaces, we see that absolute convergence always implies convergence, and we can again do all of our checks in $\mathbb{R}$.

Hilbert Bases II

Theorem

The following notions are equivalent for an orthogonal set $\mathcal{O}$ of vectors in a Hilbert space H:

- *$\mathcal{O}$ is a Hilbert Basis*
- *If $\langle \mathbf{v}, \mathbf{u} \rangle = 0$ for all $\mathbf{u} \in \mathcal{O}$, then $\mathbf{v} = \mathbf{0}$.*
- *Any $\mathbf{v}$ is the limit of a sequence of vectors in the span of $\mathcal{O}$.*

The collection of all limits of all sequences of vectors in a set is called the *closure* of the set. Since the span is a subspace, the closure agrees with the completion of that subspace in the bigger, complete subspace.

Jubilation

In a Hilbert space, this shows that a Hilbert basis is *exactly* what we want:

- we have convergence, so we have infinite sums
- any vector can be written as an infinite [convergent] linear combination of vectors of our basis!

We can do even better: we can find the coefficients in this sum.

Theorem

If $\mathcal{O}$ is a Hilbert Basis of a Hilbert space H and if $\mathbf{v} \in H$, then

$$\mathbf{v} = \hat{\mathbf{v}} := \sum_{\mathbf{u} \in \mathcal{O}} \langle \mathbf{v}, \mathbf{u} \rangle \mathbf{u}.$$

This is the Fourier expansion of **v** with respect to $\mathcal{O}$.

Final Remarks

Theorem

Let $\mathcal{O}$ be an orthogonal set in a Hilbert space H, and for any $\mathbf{v}$, let $\hat{\mathbf{v}}$ be the Fourier expansion. Then the following are equivalent:

1. *$\mathcal{O}$ is a Hilbert basis.*
2. *If $\langle \mathbf{v}, \mathbf{u} \rangle = 0$ for all $\mathbf{u} \in \mathcal{O}$, then $\mathbf{v} = \mathbf{0}$.*
3. *Any $\mathbf{v}$ is the limit of a sequence of vectors in the span of $\mathcal{O}$.*
4. *$\mathbf{v} = \hat{\mathbf{v}}$ for all $\mathbf{v} \in H$.*
5. *$||\hat{\mathbf{v}}|| = ||\mathbf{v}||$.*
6. *$\langle \mathbf{v}, \mathbf{w} \rangle = \langle \hat{\mathbf{v}}, \hat{\mathbf{w}} \rangle$, i.e.*

$$\langle \mathbf{v}, \mathbf{w} \rangle = \sum_{\mathbf{u} \in \mathcal{O}} \langle \mathbf{v}, \mathbf{u} \rangle \overline{\langle \mathbf{w}, \mathbf{u} \rangle}.$$