

JORDAN FORM CHEAT SHEET

Here is the basic recipe for finding the Jordan form of a matrix A .

- (1) Find the characteristic polynomial $p_A(\lambda)$.
- (2) Find the roots, with multiplicities, thereof. This tells you a ton: the multiplicity of a root r is the dimension of its corresponding generalized eigenspace (the biggest subspace of your vector space on which $A - rI$ is nilpotent). At this point, you can enumerate all of the possible Jordan forms: you write the various roots along the diagonal as many times as their multiplicities and then you figure out when you could have ones above the diagonal.
- (3) For each root r , compute the smallest value of k such that $\ker((A - rI)^k)$ stabilizes. This is the index of nilpotency for r , restricted to that biggest subspace on which $A - rI$ is nilpotent. That biggest subspace, in turn, is exactly $\ker((A - rI)^k)$, so we've actually found a lot. You can also find the dimension of $\ker(A - rI)$: this tells you the number of Jordan blocks for $A - rI$ on this subspace.
- (4) Now find the Jordan form of $A - rI$ restricted to $\ker((A - rI)^k)$. On this subspace, $A - rI$ is nilpotent. This is exactly the case we first considered, so we find a basis for $\ker((A - rI)^{k-1})$ and then extend that to a basis for $\ker((A - rI)^k)$ and these new vectors, together with their various images under $A - rI$ form the start of our Jordan basis. Then repeat.
- (5) Repeat with the other eigenvalues and then just concatenate the bases.

So maybe a small fake example: let's say that A is 4×4 . I'll just make up enough details as we go along so we can understand what happens. Here's the steps:

- (1) Find $p_A(\lambda)$. Let's say it's $\lambda^4 - 2\lambda^2 + 1$.
- (2) Find the roots: $p_A(\lambda) = (\lambda - 1)^2(\lambda + 1)^2$, so we have two roots: $\lambda = \pm 1$, each with multiplicity 2. So we have 4 possible Jordan forms:

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix},$$

corresponding to the various Jordan decompositions for each of the 2 dimensional generalized eigenspaces.

- (3) For the first root $\lambda = 1$, we find k such that $\ker((A - I)^k)$ stabilizes. Let's say that $\ker((A - I))$ is one dimensional, so there is only one Jordan block. Since we are looking at something that's nilpotent on a 2 dimensional space (corresponding to the multiplicity of 1 as a root), and we know that there is only one Jordan Block, we know that the index of nilpotency here has to be 2. So $\ker((A - I)^2) \neq \ker(A - I)$, but here it stabilizes. So we find a vector in $\ker((A - I)^2)$ that's not in $\ker(A - I)$, and then we apply $A - I$ to it to get our first two basis vectors.

- (4) We repeat this for $\lambda = -1$. Let's say in this case that $\ker(A + I)$ is 2 dimensional. That means that we have 2 Jordan blocks in a 2-dimensional space, so the index of nilpotency has to be 1, and they are 1×1 Jordan blocks. This helps us find the index of Nilpotency on this space, since we know that the index of Nilpotency here is the size of the largest block. We therefore just have to find a basis for $\ker(A + I)$, and we've finished with our Jordan basis. We also see that the Jordan form that fits this description is the 3rd one in the list above.

One more word on this. In general, $A - rI$ will *not* be nilpotent. It *will* be nilpotent on a subspace. The easiest way to check when $\ker((A - rI)^k)$ stabilizes, though, is to row reduce each power. The first time you have two powers with the same number of rows of all zeros, you are done. Take the kernel and you've got the subspace on which $A - rI$ is nilpotent *and* the index of nilpotency.